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EXISTENCE AND UNIQUENESS OF A TRAVELING WAVE FRONT OF A MODEL EQUATION IN SYNAPTICALLY COUPLED NEURONAL NETWORKS

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Dedicated to Professor Yulin Zhou on the occasion of his 90th birthday!

Abstract Consider the model equation in synaptically coupled neuronal networks

$$\begin{aligned} & \frac{\partial u}{\partial t} + m(u-n) \\ = \quad & (\alpha - au)\int_0^\infty \xi(c)\left[\int_{\mathbb{R}} K(x-y)H\left(u\left(y,t-\tfrac{1}{c}|x-y|\right)-\theta\right)\mathrm{d}y\right]\mathrm{d}c \\ & +(\beta - bu)\int_0^\infty \eta(\tau)\left[\int_{\mathbb{R}} W(x-y)H(u(y,t-\tau)-\theta)\mathrm{d}y\right]\mathrm{d}\tau. \end{aligned}$$

In this model equation, $u = u(x,t)$ stands for the membrane potential of a neuron at position x and time t. The kernel functions $K \geq 0$ and $W \geq 0$ represent synaptic couplings between neurons in synaptically coupled neuronal networks. The Heaviside step function $H = H(u-\theta)$ represents the gain function and it is defined by $H(u-\theta) = 0$ for all $u < \theta$, $H(0) = \frac{1}{2}$ and $H(u-\theta) = 1$ for all $u > \theta$. The functions ξ and η represent probability density functions. The function $f(u) \equiv m(u-n)$ represents the sodium current, where $m > 0$ is a positive constant and n is a real constant. The constants $a \geq 0$, $b \geq 0$, $\alpha \geq 0$, $\beta \geq 0$ and $\theta > 0$ represent biological mechanisms. This model equation is motivated by previous models in synaptically coupled neuronal networks.

We will couple together intermediate value theorem, mean value theorem and many techniques in dynamical systems to prove the existence and uniqueness of a traveling wave front of this model equation. One of the most interesting and difficult parts is the proof of the existence and uniqueness of the wave speed. We will introduce several auxiliary functions and speed index functions to prove the existence and uniqueness of the front and the wave speed.

Keywords Synaptically coupled neuronal networks, traveling wave front, existence and uniqueness, speed index function, wave speed.

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1. Introduction

1.1. The Mathematical Model Equations

Consider the model equation in synaptically coupled neuronal networks

$$\begin{aligned} &\frac{\partial u}{\partial t} + m(u-n) \\ = \quad &(\alpha - au)\int_0^\infty \xi(c)\left[\int_{\mathbb{R}} K(x-y)H\left(u\left(y,t-\frac{1}{c}|x-y|\right)-\theta\right)\mathrm{d}y\right]\mathrm{d}c \\ &+(\beta - bu)\int_0^\infty \eta(\tau)\left[\int_{\mathbb{R}} W(x-y)H(u(y,t-\tau)-\theta)\mathrm{d}y\right]\mathrm{d}\tau. \end{aligned} \tag{1.1}$$

This model equation involves both nonlocal spatial temporal delay and nonlocal feedback delay in synaptically coupled neuronal networks. In this model equation, $u = u(x,t)$ stands for the membrane potential of a neuron at position x and time t. The kernel functions $K \geq 0$ and $W \geq 0$ represent synaptic couplings between neurons. The positive constant $\theta > 0$ represents the threshold for excitation. The gain function $H = H(u-\theta)$ denotes the output firing rate of a neuron. It is given by the Heaviside step function: $H(u-\theta) = 0$ for all $u < \theta$, $H(0) = \frac{1}{2}$ and $H(u-\theta) = 1$ for all $u > \theta$. The functions ξ and η represent probability density functions. The parameter $c > 0$ represents the finite propagation speed of an action potential along the axon and $\frac{1}{c}|x-y|$ denotes the spatial temporal delay. The parameter $\tau > 0$ represents a constant delay. The sodium channels are voltage gated channels, in other words, sodium conductance is a function of the membrane potential. The sodium current is derived by using Ohm's law and should be a nonlinear smooth function of u, just like the nonlinear function $f(u) = u(u-1)(u-a)$ in the Hodgkin-Huxley equations or in the Fitzhugh-Nagumo equations. The linear function $f(u) \equiv m(u-n)$ appeared in this paper stands for a good approximation of the sodium current, where $m > 0$ is a positive constant and n is a real constant. We may interpret the constant m as the sodium conductance and the constant n as the sodium reversal potential. The constants $a \geq 0$, $b \geq 0$, $\alpha \geq 0$, $\beta \geq 0$ and $\theta > 0$ represent biological mechanisms in synaptically coupled neuronal networks. The integrals represent nonlocal spatial temporal interactions between neurons.

This model equation is somehow similar to the equation

$$\begin{aligned} &\frac{\partial u}{\partial t} + m(u-n) = (\alpha - au)\int_0^\infty \xi(c)\int_0^\infty \eta(\tau) \\ \cdot \quad &\left[\int_{\mathbb{R}} K(x-y)H\left(u\left(y,t-\tau-\frac{1}{c}|x-y|\right)-\theta\right)\mathrm{d}y\right]\mathrm{d}\tau\mathrm{d}c. \end{aligned}$$

Here are some examples of synaptic couplings

$$K(x) = \frac{\rho}{2}\exp(-\rho|x|), \qquad W(x) = \frac{\sigma}{2}\exp(-\sigma|x|),$$

and

$$\begin{aligned} K(x) &= \frac{1}{2M}\sum_{k=1}^{M}\left[\delta(x-\rho_k)+\delta(x+\rho_k)\right], \\ W(x) &= \frac{1}{2N}\sum_{l=1}^{N}\left[\delta(x-\sigma_l)+\delta(x+\sigma_l)\right], \end{aligned}$$

where δ represents the Dirac delta impulse function, $M \geq 1$ and $N \geq 1$ are positive integers, $0 < \rho < \infty$, $0 < \rho_1 < \rho_2 < \cdots < \rho_M < \infty$, $0 < \sigma < \infty$, $0 < \sigma_1 < \sigma_2 < \cdots < \sigma_N < \infty$ are positive constants. Here are some examples of probability density functions

$$\xi(c) = \frac{1}{p}\sum_{k=1}^{p}\delta(c - c_k), \qquad \eta(\tau) = \frac{1}{q}\sum_{l=1}^{q}\delta(\tau - \tau_l),$$
$$\xi(c) = \frac{1}{c^2}H(c-1), \qquad \eta(\tau) = \exp(-\tau),$$

where $p \geq 1$ and $q \geq 1$ are positive integers, $0 < c_1 < c_2 < \cdots < c_p < \infty$ and $0 < \tau_1 < \tau_2 < \cdots < \tau_q < \infty$ are positive constants.

The model equation (1.1) contains many important equations in synaptically coupled neuronal networks as particular examples.

(I) If $a = 0$, $b = 0$, $m = 1$ and $n = 0$, then (1.1) reduces to the equation

$$\begin{aligned} \frac{\partial u}{\partial t} + u &= \alpha \int_0^\infty \xi(c)\left[\int_{\mathbb{R}} K(x-y)H\left(u\left(y, t - \frac{1}{c}|x-y|\right) - \theta\right)\mathrm{d}y\right]\mathrm{d}c \\ &\quad + \beta \int_0^\infty \eta(\tau)\left[\int_{\mathbb{R}} W(x-y)H(u(y,t-\tau)-\theta)\mathrm{d}y\right]\mathrm{d}\tau. \end{aligned}$$

(II) If $\xi(c) = \delta(c - c_0)$ and $\eta(\tau) = \delta(\tau - \tau_0)$, where $c_0 > 0$ and $\tau_0 > 0$ are positive constants, then (1.1) becomes the equation

$$\begin{aligned} \frac{\partial u}{\partial t} + m(u-n) &= (\alpha - au)\int_{\mathbb{R}} K(x-y)H\left(u\left(y, t - \frac{1}{c_0}|x-y|\right) - \theta\right)\mathrm{d}y \\ &\quad + (\beta - bu)\int_{\mathbb{R}} W(x-y)H(u(y,t-\tau_0)-\theta)\mathrm{d}y. \end{aligned}$$

(III) If $b = 0$ and $\beta = 0$, then (1.1) becomes the equation with only one kind of delay

$$\begin{aligned} &\frac{\partial u}{\partial t} + m(u-n) \\ = \ &(\alpha - au)\int_0^\infty \xi(c)\left[\int_{\mathbb{R}} K(x-y)H\left(u\left(y, t - \frac{1}{c}|x-y|\right) - \theta\right)\mathrm{d}y\right]\mathrm{d}c. \end{aligned}$$

(IV) If $a = 0$ and $\alpha = 0$, then (1.1) becomes the equation with another kind of delay

$$\frac{\partial u}{\partial t} + m(u-n) = (\beta - bu)\int_0^\infty \eta(\tau)\left[\int_{\mathbb{R}} W(x-y)H(u(y,t-\tau)-\theta)\mathrm{d}y\right]\mathrm{d}\tau.$$

(V) If $\xi(c) = \delta(c - \infty)$ and $\eta(\tau) = \delta(\tau)$, then (1.1) reduces to the equation without any delay

$$\begin{aligned} \frac{\partial u}{\partial t} + m(u-n) &= (\alpha - au)\int_{\mathbb{R}} K(x-y)H(u(y,t)-\theta)\mathrm{d}y \\ &\quad + (\beta - bu)\int_{\mathbb{R}} W(x-y)H(u(y,t)-\theta)\mathrm{d}y. \end{aligned}$$

(VI) If $a = 0$, $b = 0$, $\beta = 0$, $m = 1$, $n = 0$ and $\xi(c) = \delta(c - \infty)$, then (1.1) reduces to the equation

$$\frac{\partial u}{\partial t} + u = \alpha \int_{\mathbb{R}} K(x-y)H(u(y,t)-\theta)\mathrm{d}y.$$

These model equations occupy a central position in synaptically coupled neuronal networks. For each of these equations, under appropriate assumptions on the model equations, there exists a traveling wave front with a positive wave speed. Please see [1]- [30].

1.2. The Main Goal

In this paper, we will couple together intermediate value theorem, mean value theorem and many techniques in dynamical systems to prove the existence and uniqueness of a traveling wave front of the model equation (1.1). We will make use of several auxiliary functions and speed index functions and make use of rigorous mathematical analysis to prove the existence and uniqueness of the traveling wave front, under appropriate conditions on the constants and functions of the model equation (1.1). The proof of the existence and uniqueness of the wave speed is the most interesting and difficult part in this paper.

1.3. The Mathematical Assumptions

Suppose that the probability density functions $\xi \geq 0$ and $\eta \geq 0$ are defined on $(0, \infty)$. Suppose that there exists a positive constant $c_0 > 0$, such that $\xi = 0$ on $[0, c_0]$ and $\xi \geq 0$ on (c_0, ∞). Without loss of generality, let

$$c_0 = \sup\{c > 0 : \xi = 0 \text{ on } (0, c) \text{ and } \xi \geq 0 \text{ on } (c, \infty)\}.$$

Suppose that the synaptic couplings $K \geq 0$ and $W \geq 0$ are at least piecewise smooth functions defined on $\mathbb{R}$. Suppose that

$$n < \theta < \frac{\alpha + \beta + 2mn}{a + b + 2m}, \qquad an < \alpha, \qquad bn < \beta, \tag{1.2}$$

$$b\alpha - a\beta \leq 2m(\beta - bn), \tag{1.3}$$

$$\int_0^\infty \xi(c)\mathrm{d}c = 1, \qquad \int_0^\infty \eta(\tau)\mathrm{d}\tau = 1, \tag{1.4}$$

$$\int_0^\infty \frac{1}{c}\xi(c)\mathrm{d}c < \infty, \qquad \int_0^\infty \eta(\tau)\exp(m\tau)\mathrm{d}\tau < \infty, \tag{1.5}$$

$$\int_{\mathbb{R}} K(x)\mathrm{d}x = 1, \qquad \int_{\mathbb{R}} W(x)\mathrm{d}x = 1, \tag{1.6}$$

$$\int_{-\infty}^0 K(x)\mathrm{d}x = \frac{1}{2}, \qquad \int_{-\infty}^0 W(x)\mathrm{d}x = \frac{1}{2}, \tag{1.7}$$

$$|K(x)| + |W(x)| \leq C\exp(-\rho|x|), \qquad \text{on} \qquad \mathbb{R}, \tag{1.8}$$

$$\begin{aligned}
&(2m+a+b)^2-(2m+a)(2m+a+b)^2\frac{\theta-n}{\alpha+\beta-(a+b)n}\\
\leq\quad &2m(2m+a)\exp\left\{-\frac{2}{\alpha+\beta-(a+b)n}\frac{2m+a+b}{2c_0}\right. \qquad (1.9)\\
&\left.\cdot\left[(\alpha-an)\int_{-\infty}^{0}|x|K(x)\mathrm{d}x+(\beta-bn)\int_{-\infty}^{0}|x|W(x)\mathrm{d}x\right]\right\},
\end{aligned}$$

for two positive constants $C>0$ and $\rho>0$. Define the auxiliary function in $(0,c_0)\times(-\infty,0)$ by

$$\begin{aligned}
f(\mu,z)\equiv\quad &(a\beta-b\alpha)\left\{\int_0^\infty\eta(\tau)\left[\int_{-\infty}^{z-\mu\tau}W(x)\mathrm{d}x\right]\mathrm{d}\tau\right\}\\
&/\left\{\int_0^\infty\xi(c)\left[\int_{-\infty}^{cz/(c+s(z)\mu)}K(x)\mathrm{d}x\right]\mathrm{d}c\right\}.
\end{aligned}$$

Suppose that for each fixed $\mu\in(0,c_0)$, $f_1(z)\equiv f(\mu,z)$ is an increasing function of z in $(-\infty,0)$. Suppose that for each fixed $z<0$, $f_2(\mu)\equiv f(\mu,z)$ is a decreasing function of μ in $(0,c_0)$.

If $\alpha=0$ and $a=0$, or if $\beta=0$ and $b=0$, or if $a=0$ and $b=0$, then the assumptions on the monotonicity of f are automatically satisfied.

1.4. The Definitions of Several Auxiliary Functions

We define the sign function and several auxiliary functions to make the statements of the main results and the mathematical analysis as simple as possible. Define $s=s(z)$ by $s(z)=-1$ for all $z<0$, $s(0)=0$ and $s(z)=1$ for all $z>0$. Define $\kappa_1=\kappa_1(\mu,z)$, $\kappa_2=\kappa_2(\mu,z)$, $\kappa_3=\kappa_3(z)$, $\kappa_4=\kappa_4(z)$, $\kappa_5=\kappa_5(c,\mu,z)$, $\kappa_6=\kappa_6(\tau,\mu,z)$, $\kappa_7=\kappa_7(\mu,z)$ and $\kappa_8=\kappa_8(\mu,z)$ by

$$\begin{aligned}
\kappa_1(\mu,z)\quad&=\quad(\alpha-an)\int_0^\infty\xi(c)\left[\int_{-\infty}^{cz/(c+s(z)\mu)}K(x)\mathrm{d}x\right]\mathrm{d}c\\
&\qquad+(\beta-bn)\int_0^\infty\eta(\tau)\left[\int_{-\infty}^{z-\mu\tau}W(x)\mathrm{d}x\right]\mathrm{d}\tau,\\
\kappa_2(\mu,z)\quad&=\quad m+a\int_0^\infty\xi(c)\left[\int_{-\infty}^{cz/(c+s(z)\mu)}K(x)\mathrm{d}x\right]\mathrm{d}c\\
&\qquad+b\int_0^\infty\eta(\tau)\left[\int_{-\infty}^{z-\mu\tau}W(x)\mathrm{d}x\right]\mathrm{d}\tau,\\
\kappa_3(z)\quad&=\quad az\int_{-\infty}^{z}K(x)\mathrm{d}x-a\int_{-\infty}^{z}xK(x)\mathrm{d}x,\\
\kappa_4(z)\quad&=\quad bz\int_{-\infty}^{z}W(x)\mathrm{d}x-b\int_{-\infty}^{z}xW(x)\mathrm{d}x,\\
\kappa_5(c,\mu,z)\quad&=\quad a\frac{cz}{c+s(z)\mu}\int_{-\infty}^{cz/(c+s(z)\mu)}K(x)\mathrm{d}x-a\int_{-\infty}^{cz/(c+s(z)\mu)}xK(x)\mathrm{d}x,\\
\kappa_6(\tau,\mu,z)\quad&=\quad b(z-\mu\tau)\int_{-\infty}^{z-\mu\tau}W(x)\mathrm{d}x-b\int_{-\infty}^{z-\mu\tau}xW(x)\mathrm{d}x,\\
\kappa_7(\mu,z)\quad&=\quad\exp\left\{\frac{m}{\mu}z+\int_0^\infty\xi(c)\frac{c+s(z)\mu}{c\mu}\kappa_3\left(\frac{cz}{c+s(z)\mu}\right)\mathrm{d}c\right.\\
&\qquad\left.+\frac{1}{\mu}\int_0^\infty\eta(\tau)\kappa_4(z-\mu\tau)\mathrm{d}\tau\right\},
\end{aligned}$$

$$\begin{aligned}\kappa_8(\mu,z) &= -\frac{1}{\mu^2}\Big\{mz+\int_0^\infty \xi(c)\\ &\cdot\Big[\kappa_3\Big(\frac{cz}{c+s(z)\mu}\Big)-\kappa_3(0)+\frac{zs(z)\mu}{c+s(z)\mu}\frac{\partial\kappa_3}{\partial z}\Big(\frac{cz}{c+s(z)\mu}\Big)\Big]\mathrm{d}c\\ &+\int_0^\infty \eta(\tau)\big\{\mu\tau\big[\kappa_4'(z-\mu\tau)-\kappa_4'(-\mu\tau)\big]\\ &+\big[\kappa_4(z-\mu\tau)-\kappa_4(-\mu\tau)\big]\big\}\mathrm{d}\tau\Big\}.\end{aligned}$$

Note that the auxiliary functions κ_1, κ_2, κ_3, κ_4, κ_5, κ_6, κ_7 and κ_8 depend on the constants (a,b), (m,n), (α,β) and the functions (ξ,η), (K,W) and the parameter μ. These functions will help us study the dependence of the traveling wave front on the constants (a,b), (m,n), (α,β), θ, the probability density functions (ξ,η) and the kernel functions (K,W).

Let us investigate simple properties of the auxiliary functions. First of all, because $K\geq 0$ and $W\geq 0$ on $\mathbb{R}$, $\xi\geq 0$ and $\eta\geq 0$ on $(0,\infty)$, for all $0<\mu<c_0$ and for all z, we have

$$\begin{aligned}&0\leq \kappa_1(\mu,z)\leq \alpha+\beta-(a+b)n,\\ &m\leq \kappa_2(\mu,z)\leq m+a+b,\\ &\kappa_3(z)\geq 0,\qquad \kappa_4(z)\geq 0,\qquad \kappa_7(\mu,z)\geq 0,\qquad \kappa_8(\mu,z)\geq 0,\\ &\kappa_3\left(\frac{cz}{c+s(z)\mu}\right)-\kappa_3(0)\\ &=a\frac{cz}{c+s(z)\mu}\int_{-\infty}^{cz/(c+s(z)\mu)}K(x)\mathrm{d}x+a\int_{cz/(c+s(z)\mu)}^0 xK(x)\mathrm{d}x\leq 0,\\ &\kappa_3\left(\frac{cz}{c+s(z)\mu}\right)-\kappa_3(0)+\frac{zs(z)\mu}{c+s(z)\mu}\frac{\partial\kappa_3}{\partial z}\left(\frac{cz}{c+s(z)\mu}\right)\leq 0,\\ &\kappa_4(z-\mu\tau)-\kappa_4(-\mu\tau)\\ &=bz\int_{-\infty}^{z-\mu\tau}W(x)\mathrm{d}x-b\mu\tau\int_{-\mu\tau}^{z-\mu\tau}W(x)\mathrm{d}x-b\int_{-\mu\tau}^{z-\mu\tau}xW(x)\mathrm{d}x\leq 0,\\ &\kappa_4'(z-\mu\tau)-\kappa_4'(-\mu\tau)=b\int_{-\mu\tau}^{z-\mu\tau}W(x)\mathrm{d}x\leq 0,\\ &\kappa_5(c,\mu,z)=\kappa_3\left(\frac{cz}{c+s(z)\mu}\right)\geq 0,\qquad \kappa_6(\tau,\mu,z)=\kappa_4(z-\mu\tau)\geq 0,\\ &\frac{\kappa_7(\mu,z)}{\kappa_7(\mu,0)}=\exp\left\{\frac{m}{\mu}z+\int_0^\infty \xi(c)\frac{c+s(z)\mu}{c\mu}\left[\kappa_3\left(\frac{cz}{c+s(z)\mu}\right)-\kappa_3(0)\right]\mathrm{d}c\right.\\ &\qquad\left.+\frac{1}{\mu}\int_0^\infty \eta(\tau)\big[\kappa_4(z-\mu\tau)-\kappa_4(-\mu\tau)\big]\mathrm{d}\tau\right\}>0.\end{aligned}$$

If $z\neq 0$, then we have the derivatives

$$\begin{aligned}\frac{\partial\kappa_1}{\partial z}(\mu,z)=(\alpha-an)\int_0^\infty \xi(c)\left[\frac{c}{c+s(z)\mu}K\left(\frac{cz}{c+s(z)\mu}\right)\right]\mathrm{d}c\\ +(\beta-bn)\int_0^\infty \eta(\tau)W(z-\mu\tau)\mathrm{d}\tau,\end{aligned}$$

$$\frac{\partial \kappa_2}{\partial z}(\mu, z) = a\int_0^\infty \xi(c)\left[\frac{c}{c+s(z)\mu}K\left(\frac{cz}{c+s(z)\mu}\right)\right]\mathrm{d}c \\ + b\int_0^\infty \eta(\tau)W(z-\mu\tau)\mathrm{d}\tau,$$

$$\frac{\partial \kappa_3}{\partial z}(z) = a\int_{-\infty}^z K(x)\mathrm{d}x,$$

$$\frac{\partial \kappa_4}{\partial z}(z) = b\int_{-\infty}^z W(x)\mathrm{d}x,$$

$$\frac{\partial \kappa_5}{\partial z}(c, \mu, z) = a\frac{c}{c+s(z)\mu}\int_{-\infty}^{cz/(c+s(z)\mu)} K(x)\mathrm{d}x,$$

$$\frac{\partial \kappa_6}{\partial z}(\tau, \mu, z) = b\int_{-\infty}^{z-\mu\tau} W(x)\mathrm{d}x,$$

$$\frac{\partial \kappa_7}{\partial z}(\mu, z) = \frac{1}{\mu}\kappa_2(\mu, z)\kappa_7(\mu, z),$$

$$\frac{\partial}{\partial \mu}\left[\frac{\kappa_7(\mu, z)}{\kappa_7(\mu, 0)}\right] = \frac{\kappa_7(\mu, z)}{\kappa_7(\mu, 0)}\kappa_8(\mu, z).$$

Moreover, we have the limits

$$\lim_{z\to-\infty}\kappa_1(\mu, z) = 0, \qquad \lim_{z\to\infty}\kappa_1(\mu, z) = \alpha+\beta-(a+b)n,$$
$$\lim_{z\to-\infty}\kappa_2(\mu, z) = m, \qquad \lim_{z\to\infty}\kappa_2(\mu, z) = m+a+b,$$
$$\lim_{z\to-\infty}\kappa_3(\mu, z) = 0, \qquad \lim_{z\to-\infty}\kappa_4(\mu, z) = 0,$$
$$\lim_{z\to-\infty}\kappa_5(\mu, z) = 0, \qquad \lim_{z\to-\infty}\kappa_6(\mu, z) = 0,$$
$$\lim_{z\to-\infty}\kappa_7(\mu, z) = 0, \qquad \lim_{z\to-\infty}\frac{\kappa_7(\mu, z)}{\kappa_7(\mu, 0)} = 0.$$

These auxiliary functions may be simplified for special cases.

(I) For the model equation

$$\frac{\partial u}{\partial t} + m(u-n) \\ = (\alpha - au)\int_0^\infty \xi(c)\left[\int_{\mathbb{R}} K(x-y)H\left(u\left(y, t-\frac{1}{c}|x-y|\right)-\theta\right)\mathrm{d}y\right]\mathrm{d}c,$$

we have $\beta = 0$, $b = 0$ and the auxiliary functions become

$$\kappa_1(\mu, z) = (\alpha - an)\int_0^\infty \xi(c)\left[\int_{-\infty}^{cz/(c+s(z)\mu)} K(x)\mathrm{d}x\right]\mathrm{d}c,$$

$$\kappa_2(\mu, z) = m + a\int_0^\infty \xi(c)\left[\int_{-\infty}^{cz/(c+s(z)\mu)} K(x)\mathrm{d}x\right]\mathrm{d}c$$

$$\kappa_3(z) = az\int_{-\infty}^z K(x)\mathrm{d}x - a\int_{-\infty}^z xK(x)\mathrm{d}x, \qquad \kappa_4(z) = 0,$$

$$\kappa_7(\mu, z) = \exp\left\{\frac{m}{\mu}z + \int_0^\infty \xi(c)\frac{c+s(z)\mu}{c\mu}\kappa_3\left(\frac{cz}{c+s(z)\mu}\right)\mathrm{d}c\right\},$$

$$\kappa_8(\mu,z) = -\frac{1}{\mu^2}\Big\{mz + \int_0^\infty \xi(c) \cdot \Big[\kappa_3\Big(\frac{cz}{c+s(z)\mu}\Big) - \kappa_3(0) + \frac{zs(z)\mu}{c+s(z)\mu}\frac{\partial\kappa_3}{\partial z}\Big(\frac{cz}{c+s(z)\mu}\Big)\Big]\mathrm{d}c\Big\}.$$

(II) For the model equation

$$\frac{\partial u}{\partial t} + m(u-n) = (\beta - bu)\int_0^\infty \eta(\tau)\left[\int_{\mathbb{R}} W(x-y)H(u(y,t-\tau)-\theta)\mathrm{d}y\right]\mathrm{d}\tau,$$

we have $\alpha = 0$, $a = 0$ and the auxiliary functions become

$$\kappa_1(\mu,z) = (\beta - bn)\int_0^\infty \eta(\tau)\left[\int_{-\infty}^{z-\mu\tau} W(x)\mathrm{d}x\right]\mathrm{d}\tau,$$
$$\kappa_2(\mu,z) = m + b\int_0^\infty \eta(\tau)\left[\int_{-\infty}^{z-\mu\tau} W(x)\mathrm{d}x\right]\mathrm{d}\tau,$$
$$\kappa_3(z) = 0, \qquad \kappa_4(z) = bz\int_{-\infty}^{z} W(x)\mathrm{d}x - b\int_{-\infty}^{z} xW(x)\mathrm{d}x,$$
$$\kappa_7(\mu,z) = \exp\left\{\frac{m}{\mu}z + \frac{1}{\mu}\int_0^\infty \eta(\tau)\kappa_4(z-\mu\tau)\mathrm{d}\tau\right\},$$
$$\kappa_8(\mu,z) = -\frac{1}{\mu^2}\Big\{mz + \int_0^\infty \eta(\tau) \cdot\big\{\mu\tau\big[\kappa_4'(z-\mu\tau) - \kappa_4'(-\mu\tau)\big] + \big[\kappa_4(z-\mu\tau) - \kappa_4(-\mu\tau)\big]\big\}\mathrm{d}\tau\Big\}.$$

1.5. The Speed Index Function

Let us define the speed index function in $(0, c_0)$ by

$$\phi(\mu) = \int_{-\infty}^{0} \frac{\kappa_7(\mu,z)}{\kappa_7(\mu,0)}\frac{\partial}{\partial z}\left[\frac{\kappa_1(\mu,z)}{\kappa_2(\mu,z)}\right]\mathrm{d}z - \frac{\kappa_1(\mu,0)}{\kappa_2(\mu,0)}.$$

It is easy to see that $\phi = \phi(\mu)$ is a continuous function of μ even if K and W are piecewise continuous on $\mathbb{R}$. Also it is easy to find that

$$\lim_{\mu\to 0^+}\frac{\kappa_7(\mu,z)}{\kappa_7(\mu,0)} = 0, \lim_{\mu\to 0^+}\phi(\mu) = -\frac{\alpha+\beta-(a+b)n}{2m+a+b}, \lim_{\mu\to c_0}\phi(\mu) = \phi(c_0) \ge 0.$$

The speed function is a very important concept in mathematical neuroscience and will play very important role in proving the existence and uniqueness of the wave speed of the traveling wave front of (1.1).

As before, the speed index function may be simplified for special cases.

(I) For the model equation

$$\frac{\partial u}{\partial t} + m(u-n) = (\alpha - au)\int_0^\infty \xi(c)\left[\int_{\mathbb{R}} K(x-y)H\left(u\left(y, t - \frac{1}{c}|x-y|\right) - \theta\right)\mathrm{d}y\right]\mathrm{d}c,$$

we have $\beta = 0$, $b = 0$ and

$$\begin{aligned}
\frac{\kappa_1(\mu,0)}{\kappa_2(\mu,0)} &= \frac{\alpha - an}{2m+a},\\
\frac{\kappa_1(\mu,z)}{\kappa_2(\mu,z)} &= (\alpha - an)\int_0^\infty \xi(c)\left[\int_{-\infty}^{cz/(c+s(z)\mu)} K(x)\mathrm{d}x\right]\mathrm{d}c\\
&\quad /\left\{m + a\int_0^\infty \xi(c)\left[\int_{-\infty}^{cz/(c+s(z)\mu)} K(x)\mathrm{d}x\right]\mathrm{d}c\right\},\\
\frac{\partial}{\partial z}\left[\frac{\kappa_1(\mu,z)}{\kappa_2(\mu,z)}\right] &= m(\alpha - an)\int_0^\infty \xi(c)\frac{c}{c+s(z)\mu}K\left(\frac{cz}{c+s(z)\mu}\right)\mathrm{d}c\\
&\quad /\left\{m + a\int_0^\infty \xi(c)\left[\int_{-\infty}^{cz/(c+s(z)\mu)} K(x)\mathrm{d}x\right]\mathrm{d}c\right\}^2,\\
\frac{\kappa_7(\mu,z)}{\kappa_7(\mu,0)} &= \exp\left\{\frac{m}{\mu}z + \int_0^\infty \xi(c)\frac{c+s(z)\mu}{c\mu}\Big[\kappa_3\big(\frac{cz}{c+s(z)\mu}\big) - \kappa_3(0)\Big]\mathrm{d}c\right\}.
\end{aligned}$$

Now the speed index function becomes

$$\begin{aligned}
&\phi_1(\mu) = \int_{-\infty}^0 \frac{\kappa_7(\mu,z)}{\kappa_7(\mu,0)}\frac{\partial}{\partial z}\left[\frac{\kappa_1(\mu,z)}{\kappa_2(\mu,z)}\right]\mathrm{d}z - \frac{\kappa_1(\mu,0)}{\kappa_2(\mu,0)}\\
= \;& m(\alpha - an)\int_{-\infty}^0 \exp\left\{\frac{m}{\mu}z + \int_0^\infty \xi(c)\frac{c+s(z)\mu}{c\mu}\Big[\kappa_3\left(\frac{cz}{c+s(z)\mu}\right)\right.\\
&\left. -\kappa_3(0)\Big]\mathrm{d}c\right\}\cdot\left\{\int_0^\infty \xi(c)\frac{c}{c+s(z)\mu}K\left(\frac{cz}{c+s(z)\mu}\right)\mathrm{d}c\right\}\\
&/\left\{m + a\int_0^\infty \xi(c)\left[\int_{-\infty}^{cz/(c+s(z)\mu)} K(x)\mathrm{d}x\right]\mathrm{d}c\right\}^2\mathrm{d}z - \frac{\alpha - an}{2m+a}.
\end{aligned}$$

(II) For the model equation

$$\begin{aligned}
&\frac{\partial u}{\partial t} + m(u-n)\\
= \;& (\beta - bu)\int_0^\infty \eta(\tau)\left[\int_{\mathbb{R}} W(x-y)H(u(y,t-\tau)-\theta)\mathrm{d}y\right]\mathrm{d}\tau,
\end{aligned}$$

we have $\alpha = 0$, $a = 0$ and

$$\begin{aligned}
\frac{\kappa_1(\mu,0)}{\kappa_2(\mu,0)} &= (\beta - bn)\int_0^\infty \eta(\tau)\left[\int_{-\infty}^{-\mu\tau} W(x)\mathrm{d}x\right]\mathrm{d}\tau\\
&\quad /\left\{m + b\int_0^\infty \eta(\tau)\left[\int_{-\infty}^{-\mu\tau} W(x)\mathrm{d}x\right]\mathrm{d}\tau\right\},\\
\frac{\kappa_1(\mu,z)}{\kappa_2(\mu,z)} &= (\beta - bn)\int_0^\infty \eta(\tau)\left[\int_{-\infty}^{z-\mu\tau} W(x)\mathrm{d}x\right]\mathrm{d}\tau\\
&\quad /\left\{m + b\int_0^\infty \eta(\tau)\left[\int_{-\infty}^{z-\mu\tau} W(x)\mathrm{d}x\right]\mathrm{d}\tau\right\},
\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial z}\left[\frac{\kappa_1(\mu,z)}{\kappa_2(\mu,z)}\right] &= m(\beta-bn)\int_0^\infty \eta(\tau)W(z-\mu\tau)\mathrm{d}\tau \\ &\qquad / \left\{m+b\int_0^\infty \eta(\tau)\left[\int_{-\infty}^{z-\mu\tau} W(x)\mathrm{d}x\right]\mathrm{d}\tau\right\}^2, \\ \frac{\kappa_7(\mu,z)}{\kappa_7(\mu,0)} &= \exp\left\{\frac{m}{\mu}z+\frac{1}{\mu}\int_0^\infty \eta(\tau)\big[\kappa_4(z-\mu\tau)-\kappa_4(-\mu\tau)\big]\mathrm{d}\tau\right\}.\end{aligned}$$

Now the speed index function reduces to

$$\begin{aligned}&\phi_2(\mu) = \int_{-\infty}^0 \frac{\kappa_7(\mu,z)}{\kappa_7(\mu,0)}\frac{\partial}{\partial z}\left[\frac{\kappa_1(\mu,z)}{\kappa_2(\mu,z)}\right]\mathrm{d}z - \frac{\kappa_1(\mu,0)}{\kappa_2(\mu,0)} \\ = \;& m(\beta-bn)\int_{-\infty}^0 \exp\left\{\frac{m}{\mu}z+\frac{1}{\mu}\int_0^\infty \eta(\tau)\big[\kappa_4(z-\mu\tau)-\kappa_4(-\mu\tau)\big]\mathrm{d}\tau\right\} \\ \cdot \;& \left\{\int_0^\infty \eta(\tau)W(z-\mu\tau)\mathrm{d}\tau\right\} / \left\{m+b\int_0^\infty \eta(\tau)\left[\int_{-\infty}^{z-\mu\tau} W(x)\mathrm{d}x\right]\mathrm{d}\tau\right\}^2 \mathrm{d}z \\ - \;& (\beta-bn)\int_0^\infty \eta(\tau)\left[\int_{-\infty}^{-\mu\tau} W(x)\mathrm{d}x\right]\mathrm{d}\tau \\ / \;& \left\{m+b\int_0^\infty \eta(\tau)\left[\int_{-\infty}^{-\mu\tau} W(x)\mathrm{d}x\right]\mathrm{d}\tau\right\}.\end{aligned}$$

1.6. The Main Result

Theorem 1.1. *There exists a unique wave speed μ_0 and there exists a unique traveling wave front $u(x,t) = U(x+\mu_0 t)$ to the model equation (1.1). The traveling wave front crosses the threshold θ exactly once, where μ_0 represents the wave speed and $z = x + \mu_0 t$ represents the moving coordinate. The front is the unique solution of the boundary value problem*

$$\begin{aligned}&\mu_0 U' + m(U-n) \\ = \;& (\alpha - aU)\int_0^\infty \xi(c)\left[\int_{\mathbb{R}} K(z-y)H\left(U\left(y-\frac{\mu_0}{c}|z-y|\right)-\theta\right)\mathrm{d}y\right]\mathrm{d}c \\ & +(\beta-bU)\int_0^\infty \eta(\tau)\left[\int_{\mathbb{R}} W(z-y)H(U(y-\mu_0\tau)-\theta)\mathrm{d}y\right]\mathrm{d}\tau, \\ & \lim_{z\to-\infty} U(z) = n, \qquad \lim_{z\to\infty} U(z) = \frac{\alpha+\beta+mn}{a+b+m}, \qquad \lim_{z\to\infty} U'(z) = 0.\end{aligned}$$

The wave speed μ_0 is the unique solution of the speed equation

$$\int_{-\infty}^0 \frac{\kappa_7(\mu,z)}{\kappa_7(\mu,0)}\frac{\partial}{\partial z}\left[\frac{\kappa_1(\mu,z)}{\kappa_2(\mu,z)}\right]\mathrm{d}z - \frac{\kappa_1(\mu,0)}{\kappa_2(\mu,0)} \quad = \quad n-\theta.$$

2. The Mathematical Analysis - the Proof of the Main Result

The main goal of this section is to prove the existence and uniqueness of the traveling wave front and the existence and uniqueness of the wave speed.

Suppose that $u(x,t) = U(x+\mu t)$ is the traveling wave front of (1.1), where μ represents the wave speed and $z = x+\mu t$ represents a moving coordinate. Then the traveling wave front and the wave speed μ satisfy the integral differential equation

$$\begin{aligned} & \mu U' + m(U-n) \\ = \ & (\alpha - aU)\int_0^\infty \xi(c)\left[\int_{\mathbb{R}} K(z-y)H\left(U\left(y-\frac{\mu}{c}|z-y|\right)-\theta\right)\mathrm{d}y\right]\mathrm{d}c \\ & +(\beta - bU)\int_0^\infty \eta(\tau)\left[\int_{\mathbb{R}} W(z-y)H(U(y-\mu\tau)-\theta)\mathrm{d}y\right]\mathrm{d}\tau. \end{aligned}$$

First of all, we will derive formal but explicit representation of the traveling wave front of the model equation. Then, we will prove the existence and uniqueness of the wave speed. Finally, we will prove the existence and uniqueness of the traveling wave front. We will use the auxiliary functions and the speed index function introduced earlier to achieve our main goal.

2.1. The Representation of the Traveling Wave Front

The main goal of this subsection is to derive the representation of the traveling wave front of (1.1).

Recall that there exists a maximum positive constant $c_0 > 0$, such that $\xi = 0$ on $[0, c_0]$ and $\xi \geq 0$ on (c_0, ∞). That is

$$c_0 = \sup\{c > 0 : \xi = 0 \text{ on } (0,c) \text{ and } \xi \geq 0 \text{ on } (c,\infty)\}.$$

Suppose that the wave speed μ satisfies the condition $0 < \mu < c_0$. For the first integral in the last equation, making the change of variable

$$\omega = y - \frac{\mu}{c}|z-y|,$$

then

$$\begin{aligned} z-\omega &= z-y+\frac{\mu}{c}|z-y| \\ &= (z-y)\left[1+\frac{\mu}{c}s(z-y)\right] \\ &= (z-y)\left[1+\frac{\mu}{c}s(z-\omega)\right], \end{aligned}$$

and then we find that

$$\begin{aligned} z-y &= \frac{c}{c+s(z-\omega)\mu}(z-\omega), \\ \mathrm{d}y &= \frac{c}{c+s(z-\omega)\mu}\mathrm{d}\omega - \frac{c\mu}{[c+s(z-\omega)\mu]^2}(z-\omega)s'(z-\omega)\mathrm{d}\omega. \end{aligned}$$

Now the above integral differential equation becomes

$$\begin{aligned} & \mu U' + m(U-n) \\ = \ & (\alpha - aU)\int_0^\infty \xi(c)\left[\int_{\mathbb{R}} \frac{c}{c+s(z-\omega)\mu}K\left(\frac{c(z-\omega)}{c+s(z-\omega)\mu}\right)H(U(\omega)-\theta)\mathrm{d}\omega\right]\mathrm{d}c \\ & +(\beta - bU)\int_0^\infty \eta(\tau)\left[\int_{\mathbb{R}} W(z-\mu\tau-y)H(U(y)-\theta)\mathrm{d}y\right]\mathrm{d}\tau. \end{aligned}$$

The traveling wave front is translation invariant. Suppose that the traveling wave front satisfies the conditions $U < \theta$ on $(-\infty, 0)$, $U(0) = \theta$, $U'(0) > 0$ and $U > \theta$ on $(0, \infty)$. Then the traveling wave equation becomes

$$\begin{aligned}
& \mu U' + m(U-n) \\
= \; & (\alpha - aU)\int_0^\infty \xi(c)\left[\int_0^\infty \frac{c}{c+s(z-\omega)\mu} K\left(\frac{c(z-\omega)}{c+s(z-\omega)\mu}\right)\mathrm{d}\omega\right]\mathrm{d}c \\
& +(\beta - bU)\int_0^\infty \eta(\tau)\left[\int_0^\infty W(z-\mu\tau-y)\mathrm{d}y\right]\mathrm{d}\tau \\
= \; & (\alpha - aU)\int_0^\infty \xi(c)\left[\int_{-\infty}^{cz/(c+s(z)\mu)} K(x)\mathrm{d}x\right]\mathrm{d}c \\
& +(\beta - bU)\int_0^\infty \eta(\tau)\left[\int_{-\infty}^{z-\mu\tau} W(x)\mathrm{d}x\right]\mathrm{d}\tau,
\end{aligned}$$

where in the first integral of the right hand side, we let

$$\begin{aligned}
x &= \frac{c}{c+s(z-\omega)\mu}(z-\omega), \\
\mathrm{d}x &= -\frac{c}{c+s(z-\omega)\mu}\mathrm{d}\omega + \frac{c\mu}{[c+s(z-\omega)\mu]^2}(z-\omega)s'(z-\omega)\mathrm{d}\omega.
\end{aligned}$$

Rewriting this equation as a first order nonhomogeneous linear differential equation, we have

$$\begin{aligned}
& \mu(U-n)' + \left\{m + a\int_0^\infty \xi(c)\left[\int_{-\infty}^{cz/(c+s(z)\mu)} K(x)\mathrm{d}x\right]\mathrm{d}c\right. \\
& \left. + \, b\int_0^\infty \eta(\tau)\left[\int_{-\infty}^{z-\mu\tau} W(x)\mathrm{d}x\right]\mathrm{d}\tau\right\}(U-n) \\
= \; & (\alpha - an)\int_0^\infty \xi(c)\left[\int_{-\infty}^{cz/(c+s(z)\mu)} K(x)\mathrm{d}x\right]\mathrm{d}c \\
& +(\beta - bn)\int_0^\infty \eta(\tau)\left[\int_{-\infty}^{z-\mu\tau} W(x)\mathrm{d}x\right]\mathrm{d}\tau.
\end{aligned}$$

Now let us use the auxiliary functions introduced in Subsection 1.4 to rewrite this equation in a simple way. We get

$$\mu\big[U(z) - n\big]' + \kappa_2(\mu, z)\big[U(z) - n\big] = \kappa_1(\mu, z).$$

Multiplying the differential equation by the integrating factor $\kappa_7(\mu, z)$ and integrating the result with respect to z, we get

$$\begin{aligned}
\big\{\mu\kappa_7(\mu, z)\big[U(z) - n\big]\big\}' &= \kappa_1(\mu, z)\kappa_7(\mu, z), \\
\mu\kappa_7(\mu, z)\big[U(z) - n\big] &= \int_{-\infty}^z \kappa_1(\mu, x)\kappa_7(\mu, x)\mathrm{d}x.
\end{aligned}$$

Therefore, the traveling wave front may be represented as

$$U(z) = n + \frac{1}{\mu\kappa_7(\mu, z)}\int_{-\infty}^z \kappa_1(\mu, x)\kappa_7(\mu, x)\mathrm{d}x.$$

Note that, by using integration by parts, we get

$$\begin{aligned}
& \frac{1}{\mu\kappa_7(\mu,z)}\int_{-\infty}^{z}\kappa_1(\mu,x)\kappa_7(\mu,x)\mathrm{d}x \\
= \; & \frac{1}{\kappa_7(\mu,z)}\int_{-\infty}^{z}\frac{\kappa_1(\mu,x)}{\kappa_2(\mu,x)}\left[\frac{\kappa_2(\mu,x)\kappa_7(\mu,x)}{\mu}\right]\mathrm{d}x \\
= \; & \frac{1}{\kappa_7(\mu,z)}\int_{-\infty}^{z}\frac{\kappa_1(\mu,x)}{\kappa_2(\mu,x)}\frac{\partial\kappa_7}{\partial x}(\mu,x)\mathrm{d}x \\
= \; & \frac{\kappa_1(\mu,z)}{\kappa_2(\mu,z)}-\frac{1}{\kappa_7(\mu,z)}\int_{-\infty}^{z}\kappa_7(\mu,x)\frac{\partial}{\partial x}\left[\frac{\kappa_1(\mu,x)}{\kappa_2(\mu,x)}\right]\mathrm{d}x.
\end{aligned}$$

Therefore, we obtain the formal representation of the traveling wave front

$$U(z)=n+\frac{\kappa_1(\mu,z)}{\kappa_2(\mu,z)}-\int_{-\infty}^{z}\frac{\kappa_7(\mu,x)}{\kappa_7(\mu,z)}\frac{\partial}{\partial x}\left[\frac{\kappa_1(\mu,x)}{\kappa_2(\mu,x)}\right]\mathrm{d}x.$$

By using the limits

$$\begin{aligned}
&\lim_{z\to-\infty}\kappa_1(\mu,z)=0, && \lim_{z\to-\infty}\kappa_2(\mu,z)=m,\\
&\lim_{z\to\infty}\kappa_1(\mu,z)=\alpha+\beta-(a+b)n, && \lim_{z\to\infty}\kappa_2(\mu,z)=m+a+b,
\end{aligned}$$

and by using L'Hospital's rule, we find that

$$\lim_{z\to-\infty}U(z)=n, \qquad \lim_{z\to\infty}U(z)=\frac{\alpha+\beta+mn}{a+b+m}, \qquad \lim_{z\to\pm\infty}U'(z)=0.$$

2.2. The Existence and Uniqueness of the Wave Speed

The main goal of this subsection is to prove the existence and uniqueness of the wave speed μ_0 of the traveling wave front of (1.1).

Letting $z=0$ in the solution representation, we have

$$U(0)=n+\frac{\kappa_1(\mu,0)}{\kappa_2(\mu,0)}-\int_{-\infty}^{0}\frac{\kappa_7(\mu,z)}{\kappa_7(\mu,0)}\frac{\partial}{\partial z}\left[\frac{\kappa_1(\mu,z)}{\kappa_2(\mu,z)}\right]\mathrm{d}z.$$

The wave speed μ_0 should be the solution of the speed equation $U(0)=\theta$, that is

$$\phi(\mu)=\int_{-\infty}^{0}\frac{\kappa_7(\mu,z)}{\kappa_7(\mu,0)}\frac{\partial}{\partial z}\left[\frac{\kappa_1(\mu,z)}{\kappa_2(\mu,z)}\right]\mathrm{d}z-\frac{\kappa_1(\mu,0)}{\kappa_2(\mu,0)}=n-\theta.$$

Now let us use intermediate value theorem to prove the existence of the wave speed μ_0, which is a solution of this above equation. Recall that $\kappa_1(\mu,z)$, $\kappa_2(\mu,z)$ and

$\kappa_7(\mu,z)$ are defined by

$$\begin{aligned}
\kappa_1(\mu,z) &= (\alpha-an)\int_0^\infty \xi(c)\left[\int_{-\infty}^{cz/(c+s(z)\mu)} K(x)\mathrm{d}x\right]\mathrm{d}c \\
&\quad +(\beta-bn)\int_0^\infty \eta(\tau)\left[\int_{-\infty}^{z-\mu\tau} W(x)\mathrm{d}x\right]\mathrm{d}\tau, \\
\kappa_2(\mu,z) &= m+a\int_0^\infty \xi(c)\left[\int_{-\infty}^{cz/(c+s(z)\mu)} K(x)\mathrm{d}x\right]\mathrm{d}c \\
&\quad +b\int_0^\infty \eta(\tau)\left[\int_{-\infty}^{z-\mu\tau} W(x)\mathrm{d}x\right]\mathrm{d}\tau, \\
\kappa_7(\mu,z) &= \exp\left\{\frac{m}{\mu}z+\int_0^\infty \xi(c)\frac{c+s(z)\mu}{c\mu}\kappa_3\left(\frac{cz}{c+s(z)\mu}\right)\mathrm{d}c\right. \\
&\quad \left.+\frac{1}{\mu}\int_0^\infty \eta(\tau)\kappa_4(z-\mu\tau)\mathrm{d}\tau\right\}, \\
\frac{\kappa_7(\mu,z)}{\kappa_7(\mu,0)} &= \exp\left\{\frac{m}{\mu}z+\int_0^\infty \xi(c)\frac{c+s(z)\mu}{c\mu}\left[\kappa_3\left(\frac{cz}{c+s(z)\mu}\right)-\kappa_3(0)\right]\mathrm{d}c\right. \\
&\quad \left.+\frac{1}{\mu}\int_0^\infty \eta(\tau)\big[\kappa_4(z-\mu\tau)-\kappa_4(-\mu\tau)\big]\mathrm{d}\tau\right\}.
\end{aligned}$$

It is easy to see that

$$\lim_{z\to-\infty}\kappa_1(\mu,z)=0, \qquad \lim_{z\to-\infty}\kappa_2(\mu,z)=m.$$

Let us derive lower bounds for the functions $\frac{\kappa_7(\mu,z)}{\kappa_7(\mu,0)}$ and $\frac{\partial}{\partial z}\left[\frac{\kappa_1(\mu,z)}{\kappa_2(\mu,z)}\right]$. Recall that the probability density functions $\xi\ge 0$ and $\eta\ge 0$ on $(0,\infty)$, and the synaptic couplings $K\ge 0$ and $W\ge 0$ on $\mathbb{R}$. By mean value theorem, we find that for all $z<0$,

$$\begin{aligned}
&\frac{1}{z}\int_0^\infty \xi(c)\frac{c+s(z)\mu}{c\mu}\left[\kappa_3\left(\frac{cz}{c+s(z)\mu}\right)-\kappa_3(0)\right]\mathrm{d}c \le \frac{a}{\mu}\int_{-\infty}^0 K(x)\mathrm{d}x=\frac{a}{2\mu}, \\
&\frac{1}{\mu z}\int_0^\infty \eta(\tau)\big[\kappa_4(z-\mu\tau)-\kappa_4(-\mu\tau)\big]\mathrm{d}\tau \le \frac{b}{\mu}\int_{-\infty}^0 W(x)\mathrm{d}x=\frac{b}{2\mu}.
\end{aligned}$$

Hence, by using the assumptions on the monotonicity of f_1, for all $z<0$, we get

$$\begin{aligned}
\frac{\kappa_7(\mu,z)}{\kappa_7(\mu,0)} &\ge \exp\left\{\frac{2m+a+b}{2\mu}z\right\}, \\
\frac{\partial}{\partial z}\left[\frac{\kappa_1(\mu,z)}{\kappa_2(\mu,z)}\right] &\ge \frac{4m}{(2m+a+b)^2}\frac{\partial\kappa_1}{\partial z}(\mu,z).
\end{aligned}$$

Let us derive lower bound and upper bound for the functions $\kappa_1(\mu,z)$, $\kappa_2(\mu,z)$ on $(0,c_0)\times(-\infty,0)$ and the integral $\int_{-\infty}^0 \kappa_1(\mu,z)\mathrm{d}z$ on $(0,c_0)$. By using the definitions, we have

$$\begin{aligned}
0 &< \kappa_1(\mu,z)\\
&= (\alpha-an)\int_0^\infty \xi(c)\left[\int_{-\infty}^{cz/(c+s(z)\mu)} K(x)\mathrm{d}x\right]\mathrm{d}c\\
&\quad +(\beta-bn)\int_0^\infty \eta(\tau)\left[\int_{-\infty}^{z-\mu\tau} W(x)\mathrm{d}x\right]\mathrm{d}\tau\\
&< \frac{\alpha+\beta-(a+b)n}{2},\\
\kappa_2(\mu,z) &= m+a\int_0^\infty \xi(c)\left[\int_{-\infty}^{cz/(c+s(z)\mu)} K(x)\mathrm{d}x\right]\mathrm{d}c\\
&\quad +b\int_0^\infty \eta(\tau)\left[\int_{-\infty}^{z-\mu\tau} W(x)\mathrm{d}x\right]\mathrm{d}\tau > m+\frac{a}{2},\\
0 &< \int_{-\infty}^0 \kappa_1(\mu,z)\mathrm{d}z\\
&= z\kappa_1(\mu,z)|_{-\infty}^0\\
&\quad +(\alpha-an)\int_0^\infty \xi(c)\left[\int_{-\infty}^0 |z|\frac{c}{c+s(z)\mu}K\left(\frac{cz}{c+s(z)\mu)}\right)\mathrm{d}z\right]\mathrm{d}c\\
&\quad +(\beta-bn)\int_0^\infty \eta(\tau)\left[\int_{-\infty}^0 |z|W(z-\mu\tau)\mathrm{d}z\right]\mathrm{d}\tau\\
&= (\alpha-an)\int_0^\infty \xi(c)\frac{c-\mu}{c}\left[\int_{-\infty}^0 |x|K(x)\mathrm{d}x\right]\mathrm{d}c\\
&\quad +(\beta-bn)\int_0^\infty \eta(\tau)\left[\int_{-\infty}^0 |x|W(x-\mu\tau)\mathrm{d}x\right]\mathrm{d}\tau\\
&\le (\alpha-an)\int_{-\infty}^0 |x|K(x)\mathrm{d}x+(\beta-bn)\int_{-\infty}^0 |x|W(x)\mathrm{d}x.
\end{aligned}$$

Now, by using these lower bounds and upper bounds for $\kappa_1(\mu,z)$, $\kappa_2(\mu,z)$ and $\int_{-\infty}^0 \kappa_1(\mu,z)\mathrm{d}z$, we get the following valuable lower bound

$$\begin{aligned}
&\int_{-\infty}^0 \frac{\kappa_7(\mu,z)}{\kappa_7(\mu,0)}\frac{\partial}{\partial z}\left[\frac{\kappa_1(\mu,z)}{\kappa_2(\mu,z)}\right]\mathrm{d}z\\
&\ge \frac{4m}{(2m+a+b)^2}\int_{-\infty}^0 \exp\left\{\frac{2m+a+b}{2\mu}z\right\}\frac{\partial\kappa_1}{\partial z}(\mu,z)\mathrm{d}z\\
&\ge \frac{4m}{(2m+a+b)^2}\int_{-\infty}^0 \frac{\partial\kappa_1}{\partial z}(\mu,z)\mathrm{d}z\\
&\quad\cdot\exp\left\{-\frac{2m+a+b}{2\mu}\int_{-\infty}^0 |z|\frac{\partial\kappa_1}{\partial z}\mathrm{d}z\Big/\int_{-\infty}^0 \frac{\partial\kappa_1}{\partial z}\mathrm{d}z\right\}\\
&= \frac{4m}{(2m+a+b)^2}\kappa_1(\mu,0)\exp\left\{-\frac{2m+a+b}{2\mu}\int_{-\infty}^0 \frac{\kappa_1(\mu,z)}{\kappa_1(\mu,0)}\mathrm{d}z\right\}.
\end{aligned}$$

As a function of $\kappa_1(\mu,0)$,

$$\frac{4m}{(2m+a+b)^2}\kappa_1(\mu,0)\exp\left\{-\frac{2m+a+b}{2\mu}\int_{-\infty}^0\frac{\kappa_1(\mu,z)}{\kappa_1(\mu,0)}\mathrm{d}z\right\}-\frac{\kappa_1(\mu,0)}{\kappa_2(\mu,0)}$$

is a decreasing function on $\big(0,\frac{\alpha+\beta-(a+b)n}{2}\big)$. Furthermore, we have the following important lower bound on $(0,c_0)$ which is closely related to the speed index function $\phi(\mu)$

$$\begin{aligned}
& \frac{4m}{(2m+a+b)^2}\kappa_1(\mu,0)\exp\left\{-\frac{2m+a+b}{2\mu}\int_{-\infty}^0\frac{\kappa_1(\mu,z)}{\kappa_1(\mu,0)}\mathrm{d}z\right\}-\frac{\kappa_1(\mu,0)}{\kappa_2(\mu,0)}\\
\geq\ & \frac{4m}{(2m+a+b)^2}\frac{\alpha+\beta-(a+b)n}{2}\\
& \cdot\exp\left\{-\frac{2m+a+b}{2\mu}\frac{2}{\alpha+\beta-(a+b)n}\int_{-\infty}^0\kappa_1(\mu,z)\mathrm{d}z\right\}-\frac{\alpha+\beta-(a+b)n}{2\kappa_2(\mu,0)}\\
\geq\ & \frac{4m}{(2m+a+b)^2}\frac{\alpha+\beta-(a+b)n}{2}\\
& \cdot\exp\left\{-\frac{2m+a+b}{2\mu}\frac{2}{\alpha+\beta-(a+b)n}\right.\\
& \left.\cdot\left[(\alpha-an)\int_{-\infty}^0|x|K(x)\mathrm{d}x+(\beta-bn)\int_{-\infty}^0|x|W(x)\mathrm{d}x\right]\right\}-\frac{\alpha+\beta-(a+b)n}{2m+a}\\
\geq\ & n-\theta.
\end{aligned}$$

The last inequality is due to assumption (1.9). Thus, we obtain the desired estimates

$$\begin{aligned}
& \lim_{\mu\to0^+}\phi(\mu)=-\frac{\alpha+\beta-(a+b)n}{2m+a+b}<n-\theta,\\
& \lim_{\mu\to c_0}\phi(\mu)=\phi(c_0)\\
& \geq\frac{4m}{(2m+a+b)^2}\kappa_1(c_0,0)\exp\left\{-\frac{2m+a+b}{2c_0}\int_{-\infty}^0\frac{\kappa_1(c_0,z)}{\kappa_1(c_0,0)}\mathrm{d}z\right\}\\
& \quad-\frac{\kappa_1(c_0,0)}{\kappa_2(c_0,0)}\geq n-\theta.
\end{aligned}$$

Therefore, by using intermediate value theorem, the existence of the wave speed μ_0 is easy to prove.

Now we will use mean value theorem to prove the uniqueness of the wave speed. It is not difficult to use condition (1.2) to find that

$$\begin{aligned}
\frac{\kappa_1(\mu,0)}{\kappa_2(\mu,0)} &= \left\{\alpha-an+2(\beta-bn)\int_0^\infty\eta(\tau)\left[\int_{-\infty}^{-\mu\tau}W(x)\mathrm{d}x\right]\mathrm{d}\tau\right\}\\
&\quad/\left\{2m+a+2b\int_0^\infty\eta(\tau)\left[\int_{-\infty}^{-\mu\tau}W(x)\mathrm{d}x\right]\mathrm{d}\tau\right\}>0,
\end{aligned}$$

and then use condition (1.3) to estimate the derivative

$$\begin{aligned}
\frac{\partial}{\partial\mu}\left[\frac{\kappa_1(\mu,0)}{\kappa_2(\mu,0)}\right] &= 2\big[b\alpha-a\beta-2m(\beta-bn)\big]\left\{\int_0^\infty\tau\eta(\tau)W(-\mu\tau)\mathrm{d}\tau\right\}\\
&\quad/\left\{2m+a+2b\int_0^\infty\eta(\tau)\left[\int_{-\infty}^{-\mu\tau}W(x)\mathrm{d}x\right]\mathrm{d}\tau\right\}^2\leq0.
\end{aligned}$$

Additionally, by using condition (1.2) and the monotonicity of f_2, we have

$$
\begin{aligned}
& \frac{\partial}{\partial\mu}\left[\frac{\kappa_1(\mu,z)}{\kappa_2(\mu,z)}\right] \\
= \; & \Bigg\{-m(\alpha-an)\left\{\int_0^\infty \xi(c)\left[\frac{czs(z)}{(c+s(z)\mu)^2}K\left(\frac{cz}{c+s(z)\mu}\right)\right]\mathrm{d}c\right\} \\
& -m(\beta-bn)\left\{\int_0^\infty \tau\eta(\tau)W(z-\mu\tau)\mathrm{d}\tau\right\} \\
& -(a\beta-b\alpha)\left\{\int_0^\infty \xi(c)\left[\int_{-\infty}^{cz/(c+s(z)\mu)}K(x)\mathrm{d}x\right]\mathrm{d}c\right\} \\
& \cdot\left\{\int_0^\infty \tau\eta(\tau)W(z-\mu\tau)\mathrm{d}\tau\right\} \\
& +(a\beta-b\alpha)\left\{\int_0^\infty \eta(\tau)\left[\int_{-\infty}^{z-\mu\tau}W(x)\mathrm{d}x\right]\mathrm{d}\tau\right\} \\
& \cdot\left\{\int_0^\infty \xi(c)\left[\frac{czs(z)}{(c+s(z)\mu)^2}K\left(\frac{cz}{c+s(z)\mu}\right)\right]\mathrm{d}c\right\}\Bigg\} \\
& /\Bigg\{m+a\int_0^\infty \xi(c)\left[\int_{-\infty}^{cz/(c+s(z)\mu)}K(x)\mathrm{d}x\right]\mathrm{d}c \\
& +b\int_0^\infty \eta(\tau)\left[\int_{-\infty}^{z-\mu\tau}W(x)\mathrm{d}x\right]\mathrm{d}\tau\Bigg\}^2 \le 0,
\end{aligned}
$$

in $(0,c_0)\times(-\infty,0)$. Similarly, we have

$$
\begin{aligned}
\frac{\kappa_7(\mu,z)}{\kappa_7(\mu,0)} \; = \; & \exp\Bigg\{\frac{m}{\mu}z+\int_0^\infty \xi(c)\frac{c+s(z)\mu}{c\mu}\left[\kappa_3\left(\frac{cz}{c+s(z)\mu}\right)-\kappa_3(0)\right]\mathrm{d}c \\
& +\frac{1}{\mu}\int_0^\infty \eta(\tau)\big[\kappa_4(z-\mu\tau)-\kappa_4(-\mu\tau)\big]\mathrm{d}\tau\Bigg\} > 0,
\end{aligned}
$$

and

$$
\begin{aligned}
& \frac{\partial}{\partial\mu}\left[\frac{\kappa_7(\mu,z)}{\kappa_7(\mu,0)}\right] = \frac{\kappa_7(\mu,z)}{\kappa_7(\mu,0)}\kappa_8(\mu,z) \\
= \; & -\frac{1}{\mu^2}\Bigg\{mz+\int_0^\infty \xi(c)\Big[\kappa_3\left(\frac{cz}{c+s(z)\mu}\right)-\kappa_3(0) \\
& +\frac{zs(z)\mu}{c+s(z)\mu}\frac{\partial\kappa_3}{\partial z}\left(\frac{cz}{c+s(z)\mu}\right)\Big]\mathrm{d}c \\
& +\int_0^\infty \eta(\tau)\big\{\mu\tau\big[\kappa_4'(z-\mu\tau)-\kappa_4'(-\mu\tau)\big]+\big[\kappa_4(z-\mu\tau)-\kappa_4(-\mu\tau)\big]\big\}\mathrm{d}\tau\Bigg\} \\
& \cdot\frac{\kappa_7(\mu,z)}{\kappa_7(\mu,0)} \ge 0.
\end{aligned}
$$

Additional to all assumptions made in Subsection 1.3, let us make the last assumption

$$
\kappa_8(\mu,z)\frac{\partial}{\partial z}\left[\frac{\kappa_1(\mu,z)}{\kappa_2(\mu,z)}\right]+\frac{\partial^2}{\partial\mu\partial z}\left[\frac{\kappa_1(\mu,z)}{\kappa_2(\mu,z)}\right] \ge 0,
$$

in $(0,c_0)\times(-\infty,0)$. This condition is reasonable because $\kappa_8(\mu,z)\geq\frac{m|z|}{\mu^2}$ in $(0,c_0)\times(-\infty,0)$. Now, it is not difficult to see that

$$\begin{aligned}&\frac{\partial}{\partial\mu}\left\{\int_{-\infty}^{0}\frac{\kappa_7(\mu,z)}{\kappa_7(\mu,0)}\frac{\partial}{\partial z}\left[\frac{\kappa_1(\mu,z)}{\kappa_2(\mu,z)}\right]\mathrm{d}z\right\}\\ =&\int_{-\infty}^{0}\left\{\frac{\partial}{\partial\mu}\left[\frac{\kappa_7(\mu,z)}{\kappa_7(\mu,0)}\right]\frac{\partial}{\partial z}\left[\frac{\kappa_1(\mu,z)}{\kappa_2(\mu,z)}\right]+\frac{\kappa_7(\mu,z)}{\kappa_7(\mu,0)}\frac{\partial^2}{\partial\mu\partial z}\left[\frac{\kappa_1(\mu,z)}{\kappa_2(\mu,z)}\right]\right\}\mathrm{d}z\\ =&\int_{-\infty}^{0}\frac{\kappa_7(\mu,z)}{\kappa_7(\mu,0)}\left\{\kappa_8(\mu,z)\frac{\partial}{\partial z}\left[\frac{\kappa_1(\mu,z)}{\kappa_2(\mu,z)}\right]+\frac{\partial^2}{\partial\mu\partial z}\left[\frac{\kappa_1(\mu,z)}{\kappa_2(\mu,z)}\right]\right\}\mathrm{d}z\geq 0.\end{aligned}$$

Therefore, we obtain

$$\phi'(\mu)=\int_{-\infty}^{0}\frac{\partial}{\partial\mu}\left\{\frac{\kappa_7(\mu,z)}{\kappa_7(\mu,0)}\frac{\partial}{\partial z}\left[\frac{\kappa_1(\mu,z)}{\kappa_2(\mu,z)}\right]\right\}\mathrm{d}z-\frac{\partial}{\partial\mu}\left[\frac{\kappa_1(\mu,0)}{\kappa_2(\mu,0)}\right]>0.$$

for all $0<\mu<c_0$. By using intermediate value theorem and mean value theorem, we know that there exists a unique wave speed μ_0, such that $\phi(\mu_0)=n-\theta$.

2.3. The Existence and Uniqueness of the Traveling Wave Front

The main goal of this subsection is to prove the existence and uniqueness of the traveling wave front of the model equation (1.1).

Suppose that the kernel functions are nonnegative synaptic couplings defined on $\mathbb{R}$. By the assumption on the monotonicity of f_1, we have

$$\begin{aligned}&\frac{\partial}{\partial z}\left[\frac{\kappa_1(\mu,z)}{\kappa_2(\mu,z)}\right]\\ =&\left\{m(\alpha-an)\left\{\int_0^\infty\xi(c)\left[\frac{c}{c+s(z)\mu}K\left(\frac{cz}{c+s(z)\mu}\right)\right]\mathrm{d}c\right\}\right.\\ &+m(\beta-bn)\left\{\int_0^\infty\eta(\tau)W(z-\mu\tau)\mathrm{d}\tau\right\}\\ &+(a\beta-b\alpha)\left\{\int_0^\infty\xi(c)\left[\int_{-\infty}^{cz/(c+s(z)\mu)}K(x)\mathrm{d}x\right]\mathrm{d}c\right\}\\ &\cdot\left\{\int_0^\infty\eta(\tau)W(z-\mu\tau)\mathrm{d}\tau\right\}\\ &-(a\beta-b\alpha)\left\{\int_0^\infty\eta(\tau)\left[\int_{-\infty}^{z-\mu\tau}W(x)\mathrm{d}x\right]\mathrm{d}\tau\right\}\\ &\left.\cdot\left\{\int_0^\infty\xi(c)\left[\frac{c}{c+s(z)\mu}K\left(\frac{cz}{c+s(z)\mu}\right)\right]\mathrm{d}c\right\}\right\}\\ &/\left\{m+a\int_0^\infty\xi(c)\left[\int_{-\infty}^{cz/(c+s(z)\mu)}K(x)\mathrm{d}x\right]\mathrm{d}c\right.\\ &\left.+b\int_0^\infty\eta(\tau)\left[\int_{-\infty}^{z-\mu\tau}W(x)\mathrm{d}x\right]\mathrm{d}\tau\right\}^2\geq 0,\end{aligned}$$

in $(0, c_0) \times (-\infty, 0)$. Therefore, if we let $\mu = \mu_0$ in the formal representation of the traveling wave front, then we find that

$$U'(z) = \frac{\kappa_2(\mu_0, z)}{\mu_0} \int_{-\infty}^{z} \frac{\kappa_7(\mu_0, x)}{\kappa_7(\mu_0, z)} \frac{\partial}{\partial x} \left[\frac{\kappa_1(\mu_0, x)}{\kappa_2(\mu_0, x)} \right] \mathrm{d}x \geq 0, \qquad \text{on } \mathbb{R}.$$

In particular, we have $U'(0) > 0$. Thus, the traveling wave front $u(x,t) = U(x+\mu_0 t)$ satisfies the desired conditions $U < \theta$ on $(-\infty, 0)$, $U(0) = \theta$, $U'(0) > 0$ and $U > \theta$ on $(0, \infty)$. This means that the traveling wave front is a solution of the model equation (1.1).

The proof of Theorem (1.1) is completely finished.

2.4. Some Remarks

In this paper, assuming that both kernel functions K and W are nonnegative on $\mathbb{R}$, we have proved the existence and uniqueness of the wave speed and the existence and uniqueness of the traveling wave front of the model equation (1.1). Nevertheless, we did not prove the same or similar results when the kernel functions represent lateral inhibitions or lateral excitations in synaptically coupled neuronal networks. We hope that these results are also true.

The stability of the traveling wave front is very important/interesting in mathematical neuroscience, see [9, 18, 23–28]. This is worth of rigorous mathematical analysis.

3. Conclusion

3.1. Summary

We have coupled together intermediate value theorem, mean value theorem and many techniques in dynamical systems to prove the existence and uniqueness of the wave speed μ_0 and the existence and uniqueness of the traveling wave front $u(x,t) = U(x+\mu_0 t)$ of the model equation in synaptically coupled neuronal networks

$$\begin{aligned} & \frac{\partial u}{\partial t} + m(u-n) \\ = \ & (\alpha - au) \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(x-y) H\left(u\left(y, t - \frac{1}{c}|x-y| \right) - \theta \right) \mathrm{d}y \right] \mathrm{d}c \\ & + (\beta - bu) \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(x-y) H(u(y, t-\tau) - \theta) \mathrm{d}y \right] \mathrm{d}\tau. \end{aligned}$$

The wave speed μ_0 is the unique solution of the speed equation

$$\begin{aligned} \phi(\mu_0) \ = \ & \int_{-\infty}^{0} \frac{\kappa_7(\mu_0, z)}{\kappa_7(\mu_0, 0)} \frac{\partial}{\partial z} \left[\frac{\kappa_1(\mu_0, z)}{\kappa_2(\mu_0, z)} \right] \mathrm{d}z - \frac{\kappa_1(\mu_0, 0)}{\kappa_2(\mu_0, 0)} \\ = \ & \int_{-\infty}^{0} \exp\left\{ \frac{m}{\mu_0} z + \int_0^\infty \xi(c) \frac{c + s(z)\mu_0}{c\mu_0} \left[\kappa_3\left(\frac{cz}{c + s(z)\mu_0} \right) - \kappa_3(0) \right] \mathrm{d}c \right. \\ & \left. + \frac{1}{\mu_0} \int_0^\infty \eta(\tau) \big[\kappa_4(z - \mu_0 \tau) - \kappa_4(-\mu_0 \tau) \big] \mathrm{d}\tau \right\} \frac{\partial}{\partial z} \left[\frac{\kappa_1(\mu_0, z)}{\kappa_2(\mu_0, z)} \right] \mathrm{d}z \\ & - \left\{ \alpha - an + 2(\beta - bn) \int_0^\infty \eta(\tau) \left[\int_{-\infty}^{-\mu_0 \tau} W(x) \mathrm{d}x \right] \mathrm{d}\tau \right\} \end{aligned}$$

$$
\begin{aligned}
& /\left\{2m + a + 2b\int_0^\infty \eta(\tau)\left[\int_{-\infty}^{-\mu_0\tau} W(x)\mathrm{d}x\right]\mathrm{d}\tau\right\} \\
= \;& n - \theta.
\end{aligned}
$$

A very important step in establishing the uniqueness of the wave speed μ_0 is to show that

$$
\begin{aligned}
\phi'(\mu) &= \int_{-\infty}^{0} \frac{\partial}{\partial\mu}\left\{\frac{\kappa_7(\mu,z)}{\kappa_7(\mu,0)}\frac{\partial}{\partial z}\left[\frac{\kappa_1(\mu,z)}{\kappa_2(\mu,z)}\right]\right\}\mathrm{d}z - \frac{\partial}{\partial\mu}\left[\frac{\kappa_1(\mu,0)}{\kappa_2(\mu,0)}\right] \\
&= \int_{-\infty}^{0} \frac{\kappa_7(\mu,z)}{\kappa_7(\mu,0)}\left\{\kappa_8(\mu,z)\frac{\partial}{\partial z}\left[\frac{\kappa_1(\mu,z)}{\kappa_2(\mu,z)}\right] + \frac{\partial^2}{\partial\mu\partial z}\left[\frac{\kappa_1(\mu,z)}{\kappa_2(\mu,z)}\right]\right\}\mathrm{d}z \\
&\quad - \frac{\partial}{\partial\mu}\left[\frac{\kappa_1(\mu,0)}{\kappa_2(\mu,0)}\right] > 0.
\end{aligned}
$$

In the above model equation, $u = u(x,t)$ stands for the membrane potential of a neuron at position x and time t in synaptically coupled neuronal networks. The kernel functions $K \geq 0$ and $W \geq 0$ represent synaptic couplings between neurons. The Heaviside step function H represents the gain function and it is defined by $H(u-\theta) = 0$ for all $u < \theta$, $H(0) = \frac{1}{2}$ and $H(u-\theta) = 1$ for all $u > \theta$. The functions ξ and η are probability density functions. The function $f(u) = m(u-n)$ represents a sodium current, where $m > 0$ is a positive constant and n is a real constant. The constants $a \geq 0$, $b \geq 0$, $\alpha \geq 0$, $\beta \geq 0$, $\theta > 0$ represent biological mechanisms. This model equation is motivated by previous models in synaptically coupled neuronal networks, see [1, 3, 5, 7, 9, 17, 18, 21, 23–28].

Overall, there exists a unique traveling wave front $u(x,t) = U(x+\mu_0 t)$ to this model equation, where $\mu_0 = \mu_0(a,b,m,n,\alpha,\beta,\xi,\eta,K,W,\theta)$ represents the wave speed and $z = x + \mu_0 t$ represent the moving coordinate. The traveling wave front $u(x,t) = U(x+\mu_0 t)$ and the wave speed μ_0 satisfy the traveling wave equation

$$
\begin{aligned}
& \mu_0 U' + m(U-n) \\
= \;& (\alpha - aU)\int_0^\infty \xi(c)\left[\int_{\mathbb{R}} K(z-y)H\left(U\left(y - \frac{\mu_0}{c}|z-y|\right) - \theta\right)\mathrm{d}y\right]\mathrm{d}c \\
& +(\beta - bU)\int_0^\infty \eta(\tau)\left[\int_{\mathbb{R}} W(z-y)H(U(y-\mu_0\tau)-\theta)\mathrm{d}y\right]\mathrm{d}\tau,
\end{aligned}
$$

and some boundary conditions at $z = \pm\infty$.

3.2. Open Problems

Consider the nonlinear singularly perturbed system of integral differential equations in synaptically coupled neuronal networks

$$
\begin{aligned}
& \frac{\partial u}{\partial t} + m(u-n) + w \\
= \;& (\alpha - au)\int_0^\infty \xi(c)\left[\int_{\mathbb{R}} K(x-y)H\left(u\left(y, t - \frac{1}{c}|x-y|\right) - \theta\right)\mathrm{d}y\right]\mathrm{d}c \\
& +(\beta - bu)\int_0^\infty \eta(\tau)\left[\int_{\mathbb{R}} W(x-y)H(u(y,t-\tau)-\theta)\mathrm{d}y\right]\mathrm{d}\tau, \\
& \frac{\partial w}{\partial t} = \varepsilon(\gamma u - w),
\end{aligned}
$$

where $u = u(x,t)$ represents the membrane potential of a neuron at position x and time t, $w = w(x,t)$ represents the leaking current in synaptically coupled neuronal networks. The kernel functions K and W represent synaptic couplings between neurons in synaptically coupled neuronal networks. The gain function is given by the Heaviside step function: $H(u-\theta) = 0$ for all $u < \theta$, $H(0) = \frac{1}{2}$, and $H(u-\theta) = 1$ for all $u > \theta$. The probability density functions ξ and η are defined on $(0,\infty)$.

The traveling pulse solutions $(u(x,t), w(x,t)) = \big(U(x+\mu(\varepsilon)t), W(x+\mu(\varepsilon)t)\big)$ and their wave speeds $\mu = \mu(\varepsilon)$ would satisfy

$$\begin{aligned}
& \mu U' + m(U-n) + W \\
= \quad & (\alpha - aU)\int_0^\infty \xi(c)\left[\int_{\mathbb{R}} K(z-y)H\left(U\left(y - \frac{\mu}{c}|z-y|\right) - \theta\right)\mathrm{d}y\right]\mathrm{d}c \\
& +(\beta - bU)\int_0^\infty \eta(\tau)\left[\int_{\mathbb{R}} W(z-y)H(U(y-\mu\tau)-\theta)\mathrm{d}y\right]\mathrm{d}\tau, \\
& \mu W' = \varepsilon(U - \gamma W),
\end{aligned}$$

if they exist, where $z = x + \mu(\varepsilon)t$ represents a moving coordinate. However, the existence and stability/instability of fast/slow traveling pulse solutions of the nonlinear singularly perturbed system of integral differential equations have not been proved yet. They are worth of rigorous mathematical investigation.

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References

[1] S. Amari, *Dynamics of pattern formation in lateral-inhibition type neural fields.* Biological Cybernetics, 27(1977), 77-87.

[2] F.M. Atay and A. Hutt, *Stability and bifurcations in neural fields with finite propagation speed and general connectivity*, SIAM Journal on Applied Mathematics, 65(2004/2005), 644-666.

[3] F.M. Atay and A. Hutt, *Neural fields with distributed transmission speeds and long-range feedback delays*, SIAM Journal on Applied Dynamical Systems, 5(2006), 670-698.

[4] P.C. Bressloff, *Weakly interacting pulses in synaptically coupled neural media*, SIAM Journal Applied Mathematics, 66(2005), 57-81.

[5] P.C. Bressloff and S.E. Folias, *Front bifurcations in an excitatory neural network*, SIAM Journal on Applied Mathematics, 65(2004), 131-151.

[6] P.C. Bressloff, S.E. Folias, A. Prat and Y. Li, *Oscillatory waves in inhomogeneous neural media*, Physical Review Letters, 91(2003), 178101-1 to 178101-4.

[7] S. Coombes, *Waves, bumps, and patterns in neural field theories*, Biological Cybernetics, 93(2005), 91-108.

[8] S. Coombes, G.J. Lord and M.R. Owen, Waves and bumps in neuronal networks with axo-dendritic synaptic interactions, Physica D, 178(2003), 219-241.

[9] S. Coombes and M.R. Owen, *Evans functions for integral neural field equations with Heaviside firing rate function*, SIAM Journal on Applied Dynamical Systems, 3(2004), 574-600.

[10] G.B. Ermentrout, *Neural networks as spatio-temporal pattern-forming systems*, Report on Progress of Physics, 61(1998), 353-430.

[11] G.B. Ermentrout and J.B. McLeod, *Existence and uniqueness of travelling waves for a neural network*, Proceedings of the Royal Society of Edinburgh, Section A, 123(1993), 461-478.

[12] G.B. Ermentrout and D. Terman, *Foundations of mathematical neuroscience,* Interdisciplinary Applied Mathematics 35. Springer. New York, London, 2010, ISBN 978-0387-87707-5.

[13] S.E. Folias and P.C. Bressloff, *Breathing pulses in an excitatory neural network.* SIAM Journal on Applied Dynamical Systems, 3(2004), 378-407.

[14] S.E. Folias and P.C. Bressloff, *Stimulus-locked traveling waves and breathers in an excitatory neural network*, SIAM Journal on Applied Mathematics, 65(2005), 2067-2092.

[15] Y. Guo and C.C. Chow, *Existence and stability of standing pulses in neural networks: I. Existence. II Stability*, SIAM Journal on Applied Dynamical Systems, 4(2005), I: 217-248, II 249-281.

[16] C.R. Laing and W.C. Troy, *PDE methods for nonlocal models.* SIAM Journal on Applied Dynamical Systems, 2(2003), 487-516.

[17] D.J. Pinto and G.B. Ermentrout, *Spatially structured activity in synaptically coupled neuronal networks. I. traveling fronts and pulses, II. Lateral inhibition and standing pulses*, SIAM Journal on Applied Mathematics, 62(2001), I. 206-225, II. 226-243.

[18] D.J. Pinto, R.K. Jackson and C.E. Wayne, *Existence and stability of traveling pulses in a continuous neuronal network*, SIAM Journal on Applied Dynamical Systems, 4(2005), 954-984.

[19] K.A. Richardson, S.J. Schiff and B.J. Gluckman, *Control of traveling waves in the mammalian cortex*, Physical Review Letters, 94(2005), 028103-1 to 028103-4.

[20] D.H. Terman, G.B. Ermentrout and A.C. Yew, *Propagating activity patterns in thalamic neuronal networks*, SIAM Journal on Applied Mathematics, 61(2001), 1578-1604.

[21] H.R. Wilson and J.D. Cowan, *Excitatory and inhibitory interactions in localized populations of model neurons*, Biophysical Journal, 12(1972), 1-24.

[22] H.R. Wilson and J.D. Cowan, *A mathematical theory of the functional dynamics of cortical and thalamic nervous tissue*, Kybernetic, 13(1973), 55-80.

[23] L. Zhang, *On stability of traveling wave solutions in synaptically coupled neuronal networks*, Differential and Integral Equations, 16(2003), 513-536.

[24] L. Zhang, *Existence, uniqueness and exponential stability of traveling wave solutions of some integral differential equations arising from neuronal networks*, Journal of Differential Equations, 197(2004), 162-196.

[25] L. Zhang, *Traveling waves of a singularly perturbed system of integral-differential equations arising from neuronal networks*, Journal of Dynamics and Differential Equations, 17(2005), 489-522.

[26] L. Zhang, *Dynamics of neuronal waves*, Mathematische Zeitschrift, 255(2007), 283-321.

[27] L. Zhang, *How do synaptic coupling and spatial temporal delay influence traveling waves in nonlinear nonlocal neuronal networks?*, SIAM Journal on Applied Dynamical Systems, 6(2007), 597-644.

[28] L. Zhang, *Traveling waves arising from synaptically coupled neuronal networks*, in "*Advances in Mathematics Research*". Editor-in-Chief: Albert R. Baswell. Nova Science Publishers, INC. New York., 10(2010), 53-204.

[29] L. Zhang, P. Wu and M.A. Stoner, *Influence of sodium currents on speeds of traveling wave fronts in synaptically coupled neuronal networks*, Physica D, 239(2010), 9-32.

[30] L. Zhang, P. Wu and M.A. Stoner, *Influence of neurobiological mechanisms on speeds of traveling wave fronts in mathematical neuroscience*, Discrete and Continuous Dynamical Systems, Series B, 16(2011), 1003-1037.