COMPOSITION CONDITIONS IN THE TRIGONOMETRIC ABEL EQUATION*

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Abstract   In this paper we deal with the center problem for the trigonometric Abel equation $d\rho/d\theta = a_1(\theta)\rho^2 + a_2(\theta)\rho^3$, where $a_1(\theta)$ and $a_2(\theta)$ are trigonometric polynomials in θ. This problem is closely connected with the classical Poincaré center problem for planar polynomial vector fields.

Keywords   Center problem, Abel differential equation, universal centers, polynomial differential equations.

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1. Introduction and statement of the main results

We consider the ordinary differential equation

$$\frac{d\rho}{d\theta} = a_1(\theta)\rho^2 + a_2(\theta)\rho^3, \tag{1.1}$$

where ρ is a real variable and $a_i(\theta)$ are trigonometric polynomials in θ for $i = 1, 2$. When $a_1(\theta)$ and $a_2(\theta)$ are identically zero, we say that (1.1) is a *trivial* center. We shall denote the derivative of ρ with respect θ by $d\rho/d\theta$ or ρ'. We can solve equation (1.1) by the Picard iteration and find a solution which is unique with the prescribed initial value $\rho(0) = \rho_0$. We say that equation (1.1) determines a *center* if for any sufficiently small initial values $\rho(0)$ the solution of (1.1) satisfies $\rho(0) = \rho(2\pi)$. The center problem for equation (1.1) is to find conditions on the coefficients a_i under which this equation determines a center.

The original center problem arises from the study of the planar analytic differential systems first studied by Poincaré [27] and later by Liapunov [26] and other authors, see [8, 20, 21, 24, 25]. In the case of a non-degenerate singular point the system can be written into the form

$$\dot{x} = -y + P(x, y), \quad \dot{y} = x + Q(x, y), \tag{1.2}$$

where P and Q are analytic functions without constant and linear terms i.e. $P(x,y) = \sum_{i=2}^{\infty} P_i(x,y)$ and $Q(x,y) = \sum_{i=2}^{\infty} Q_i(x,y)$, where P_i and Q_i are homogeneous polynomials of degree i. Poincaré proved that the origin of system (1.2) is a center if

and only if the coefficients of P and Q satisfy a certain infinite system of algebraic equations called the Poincaré-Liapunov constants. We note that taking polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ system (1.2) takes the form

$$\dot{r} = \sum_{s=2}^{\infty} f_s(\theta)r^s, \quad \dot{\theta} = 1 + \sum_{s=2}^{\infty} g_s(\theta)r^{s-1},\qquad (1.3)$$

where

$$f_i(\theta) = \cos\theta P_i(\cos\theta, \sin\theta) + \sin\theta Q_i(\cos\theta, \sin\theta),$$
$$g_i(\theta) = \cos\theta Q_i(\cos\theta, \sin\theta) - \sin\theta P_i(\cos\theta, \sin\theta).$$

We remark that f_i and g_i are homogeneous polynomials of degree $i+1$ in the variables $\cos\theta$ and $\sin\theta$. In the region $\mathcal{R} = \{(r,\theta) : \dot{\theta} > 0\}$ the differential system (1.3) is equivalent to the differential equation

$$\frac{dr}{d\theta} = \frac{\sum_{s=2}^{\infty} f_s(\theta)\, r^s}{1 + \sum_{s=2}^{\infty} g_s(\theta)\, r^{s-1}} = \sum_{i=1}^{\infty} a_i(\theta)\, r^{i+1},\qquad (1.4)$$

where, since P and Q are analytic functions, we have expanded as an analytic series in r to obtain equation (1.4) whose coefficients $a_i(\theta)$ are trigonometric polynomials. This reduces the center problem for the planar differential system (1.2) to the center problem for the class of equations (1.4).

In the particular case that P and Q are homogeneous polynomials of degree n then equation (1.4) takes the form

$$\frac{dr}{d\theta} = \frac{f(\theta)r^n}{1 + g(\theta)r^{n-1}},\qquad (1.5)$$

using the Cherkas transformation (see [13])

$$\rho = \frac{r^{n-1}}{1 + r^{n-1}g(\theta)}, \quad \text{whose inverse is} \quad r = \frac{\rho^{1/(n-1)}}{(1 - \rho g(\theta))^{1/(n-1)}},\qquad (1.6)$$

the differential equation (1.5) becomes the Abel differential equation

$$\frac{d\rho}{d\theta} = ((n-1)f(\theta) - g'(\theta))\, \rho^2 - (n-1)f(\theta)g(\theta)\rho^3,\qquad (1.7)$$

which corresponds to equation (1.1) with $a_1(\theta) = ((n-1)f(\theta) - g'(\theta))$ and $a_2(\theta) = -(n-1)f(\theta)g(\theta)$. Notice that in this case $a_1(\theta)$ and $a_2(\theta)$ are trigonometric polynomials of degree $n+1$ and $2(n+1)$ respectively. By the regularity of the Cherkas transformation and its inverse at $r = \rho = 0$, equation (1.5) has a center if and only if equation (1.7) has a center.

In [11,12,22] it is studied the center problem for the analytic ordinary differential equation

$$\frac{d\rho}{d\theta} = \sum_{i=1}^{\infty} a_i(\theta)\rho^{i+1},\qquad (1.8)$$

on the cylinder $(\rho, \theta) \in \mathbb{R} \times \mathbb{S}^1$ in a neighborhood of $\rho = 0$ and where $a_i(\theta)$ are trigonometric polynomials in θ. An explicit expression for the first return map of

equation (1.8) is given in [11], see also [12]. The expression of the first return map is given in terms of the following iterated integrals of order k,

$$I_{i_1,\ldots,i_k}(a) := \int \cdots \int_{0 \le s_1 \le \cdots \le s_k \le 2\pi} a_{i_k}(s_k) \cdots a_{i_1}(s_1) \, ds_k \cdots ds_1,$$

where, by convention, for $k = 0$ we assume that this equals 1. Actually, iterated integrals appear historically in the study of Abel equations, see for instance [3,18,19]. Let $\rho(\theta; \rho_0; a)$, $\theta \in [0, 2\pi]$, be the solution of equation (1.8) corresponding to a with initial value $\rho(0; \rho_0; a) = \rho_0$. Then $P(a)(\rho_0) := \rho(2\pi; \rho_0; a)$ is the first return map of this equation and in [11, 12] it is proved the following result.

Theorem 1.1. *For sufficiently small initial values ρ_0 the first return map $P(a)$ is an absolute convergent power series $P(a)(\rho_0) = \rho_0 + \sum\limits_{n=1}^{\infty} c_n(a)\rho_0^{n+1}$, where*

$$c_n(a) = \sum_{i_1 + \cdots + i_k = n} c_{i_1,\ldots,i_k} I_{i_1,\ldots,i_k}(a), \quad and$$

$$c_{i_1,\ldots,i_k} = (n - i_1 + 1) \cdot (n - i_1 - i_2 + 1) \cdot (n - i_1 - i_2 - i_3 + 1) \cdots 1.$$

By Theorem 1.1 the center set $\mathcal{C}$ of equation (1.8) is determined by the system of polynomial equations $c_n(a) = 0$, for $n = 1, 2, \ldots$.

In [12] it is given the definition of *universal center* in terms of the monodromy group associated to equation (1.8). In fact we have a *universal center* when the monodromy group is trivial. Hence, the set $\mathcal{U}$ of universal centers is, in a sense, a stable part of the center set $\mathcal{C}$. It is also well-known that, in general, $\mathcal{U} \ne \mathcal{C}$, see for instance [22]. The following proposition establishes the characterization of the universal centers in terms of iterated integrals and it is also given in [12].

Proposition 1.1. *Equation* (1.8) *determines a universal center if and only if for all positive integers $i_1, \ldots, i_k$ with $k \ge 1$ the iterated integral $I_{i_1,\ldots,i_k}(a) = 0$.*

In [12] it is also considered the case when equation (1.8) has a finite number of terms, i.e.

$$\frac{dv}{d\theta} = \sum_{i=1}^{n} a_i(\theta)v^{i+1}. \tag{1.9}$$

It is proved that equation (1.9) with all a_i trigonometric polynomials has a universal center if and only if there are trigonometric polynomials q and polynomials $p_1, \ldots, p_n \in \mathbb{C}[z]$ such that

$$\tilde{a}_i = p_i \circ q, \qquad 1 \le i \le n, \qquad \tilde{a}_i(x) = \int_0^x a_i(s)ds. \tag{1.10}$$

Conditions (1.10) are called *composition conditions*. The vanishing of all iterated integrals $I_{i_1,\ldots,i_k}(a) = 0$ for all positive integers $i_1, \ldots, i_k$ with $k \ge 1$ is equivalent to composition conditions for equation (1.9), as it is proved in [12]. This result is generalized to equation (1.8) in [22] where the following theorem is established.

Theorem 1.2. *Any center of the differential equation* (1.8) *is universal if and only if equation* (1.8) *satisfies the composition condition.*

The composition conditions have been studied in several papers in the last years in different contexts, see for instance [1, 2, 4–6, 10, 14, 16] and references therein.

Given an angle $\alpha \in [0, \pi)$, we say that the differential equation (1.8) is α-*symmetric* if its flow is symmetric with respect to the straight line $\theta = \alpha$. Obviously, this is equivalent to that equation (1.8) is invariant under the change of variables $\theta \mapsto 2\alpha - \theta$. Any differential equation (1.8) which is α-symmetric has a center, due to the symmetry.

We say that the differential equation (1.8) is of *separable variables* if the function on the right-hand side of equation (1.8) splits as product of two functions of one variable, one depending on ρ and the other on θ, that is, $\frac{d\rho}{d\theta} = a(\theta)\, b(\rho)$. In such a case there is only one center condition which is $\int_0^{2\pi} a(\theta)\, d\theta = 0$.

In [22] it is also proved the following result for equation (1.8).

Theorem 1.3. *If the differential equation* (1.8) *has a center which is either α-symmetric, or of separable variables, then it is universal.*

This result gives two big families of universal centers also for the Abel equation (1.1).

2. Universal centers of the Abel equation (1.1)

In this section we study the universal centers of equation (1.1). It is well-know that not all the centers of equation (1.1) are universal due to the following fact. Any quadratic system in the plane, i.e. system (1.2) with homogeneous P and Q of degree at most 2, can be transformed to an Abel equation of the form (1.7) where $a_1(\theta)$ and $a_2(\theta)$ are trigonometric polynomials of degree 3 and 6 respectively. Moreover in [22] it is proved that there are centers of the quadratic system (1.2) which are not universal (for instance the Darboux component except its intersection with the symmetric one). In [22] it is proved that these non-universal centers of the quadratic system (1.2) give non-universal centers of the associated Abel equation (1.7). In [15] there is another example of a center of an Abel equation which is not universal and where $a_1(\theta)$ and $a_2(\theta)$ are trigonometric polynomials of degree 3 and 6 respectively. Hence, the following open problem can be established:

Open problem: To determine the lowest degree of the trigonometric polynomials $a_1(\theta)$ and $a_2(\theta)$ such that the Abel equation (1.1) has a center which is not universal.

Blinov in [9] proved the following result which shows that the lowest possible degree such that an Abel equation can have a non-universal center is at least 3.

Proposition 2.1. *All the centers of equation* (1.1) *when $a_1(\theta)$ and $a_2(\theta)$ are trigonometric polynomials of degree 1 and 2 are universal centers and, in consequence, verify the composition condition.*

For sake of completeness we give a short proof of Proposition 2.1 in the appendix. The proof given in [9] and ours consist in solving the center problem for equation (1.1) with $a_1(\theta)$ and $a_2(\theta)$ of degree at most 2 and to check that all the center cases are universal. However, this procedure is unapproachable for higher degrees due to the cumbersome computations needed to solve the center problem.

In this paper we study the centers of equation (1.1) when $a_1(\theta)$ and $a_2(\theta)$ are

trigonometric polynomials of degree 3, i.e.,

$$
\begin{aligned}
a_1(\theta) &= b_{00} + b_{10}\cos\theta + b_{01}\sin\theta + b_{20}\cos(2\theta) + b_{02}\sin(2\theta) \\
&\quad + b_{30}\cos(3\theta) + b_{03}\sin(3\theta), \\
a_2(\theta) &= c_{00} + c_{10}\cos\theta + c_{01}\sin\theta + c_{20}\cos(2\theta) + c_{02}\sin(2\theta) \\
&\quad + c_{30}\cos(3\theta) + c_{03}\sin(3\theta),
\end{aligned}
\tag{2.1}
$$

where b_{ij} and c_{ij} are real constants. We remark that if $a_1(\theta)$ and $a_2(\theta)$ are both identically null, then we have a trivial center. If $a_1(\theta)$ or $a_2(\theta)$ is identically null, then all the centers are of separable variables and, consequently, all the centers are universal. Thus, we can assume that none $a_1(\theta)$ or $a_2(\theta)$ is identically null. Indeed, the first two center conditions $c_1(a) = 0$ and $c_2(a) = 0$ imply that $b_{00} = c_{00} = 0$, see Theorem 1.1. In order to make a systematic study of the problem for the Abel equation (1.1) with $a_1(\theta)$ and $a_2(\theta)$ of the form (2.1), we assume that the subdegree of $a_1(\theta)$ is either 1, 2 or 3. In each case, we can make an affine change of the variable θ and a rescalling of ρ such that $a_1(\theta)$ takes one of the following forms:

$$
\begin{aligned}
\text{Case I.} \quad & a_1(\theta) = \sin\theta + h.o.t., \\
\text{Case II.} \quad & a_1(\theta) = \sin(2\theta) + h.o.t., \\
\text{Case III.} \quad & a_1(\theta) = \sin(3\theta) + h.o.t.,
\end{aligned}
$$

where $h.o.t.$ means higher order terms. We have not been able to completely study Case I. Theorem 2.1 deals with Cases II and III.

The procedure is to compute a set of necessary conditions $c_n(a) = 0$ for $n = \overline{3, M}$, with M large, which are the coefficients of the first return map, see Theorem 1.1. In general, these necessary conditions are very long. Therefore, it is computationally very difficult to determine the irreducible components of the variety $V := V(\langle c_3, c_4, \ldots, c_M \rangle)$. We are using the classical notation of computational algebra given for instance in the textbook [17]. If the center conditions are smaller, as for instance in the proof of Proposition 2.1 given in the appendix, one can use resultants between polynomials of several variables to find the points of this variety. When this computations cannot be overcome, we look for the irreducible decomposition of the variety V. This is an extremely difficult computational problem. We have followed the algorithm described in [28] which makes use of modular arithmetics. The last step of this algorithm has not been verified. This step ensures that all the points of the variety V have been found. That is, we know that all the encountered points belong to the decomposition of V but we do not know whether the given decomposition is complete. We remark that, nevertheless, it is practically sure that the given list is complete, see for instance [7,28]. Therefore, in the following we provide sufficient conditions to have a center, which are practically necessary. We denote this situation by the expression *with probability close to* 1.

Theorem 2.1. *All the centers of equation* (1.1) *when* $a_1(\theta)$ *and* $a_2(\theta)$ *are trigonometric polynomials of degree* 3 *of the form* (2.1) *with either*

- $b_{10} = b_{01} = b_{20} = 0$ *and* $b_{02} = 1$ (Case II), *with probability close to* 1, *or;*
- $b_{10} = b_{01} = b_{20} = b_{02} = b_{30} = 0$ *and* $b_{03} = 1$ (Case III)

are universal centers and, consequently, verify the composition condition.

Proof. To proof this result we have computed eleven necessary conditions $c_n(a) = 0$ for $n = \overline{3, 13}$. These necessary conditions are very long, so we do not present them here. However, one can check our computations with the help of any available computer algebra system. In this case, in order to obtain the families of centers we look for the irreducible decomposition of the variety $V(I)$ of the ideal $I = \langle c_3, c_4, \ldots, c_{13} \rangle$. We have used the routine $minAssGTZ$ of the computer algebra system Singular [23] and we have found the irreducible decomposition of the variety of the ideal I over the field of rational numbers for (Case III) and over the finite field $\mathbb{Z}/(p)$, with $p = 32003$, for (Case II).

The obtained decomposition for the case $b_{10} = b_{01} = b_{20} = 0$ and $b_{02} = 1$ (Case II) consists of 3 components defined by the following ideals

1) $\langle b_{03}, c_{03}, c_{20}, c_{01} \rangle$;

2) $\langle b_{30}, c_{30}, c_{20}, c_{10} \rangle$;

3) $\langle c_{20}, c_{01}, c_{10}, b_{03}c_{02} - c_{03}, c_{02}b_{30} - c_{30} \rangle$.

In the first case 1) we have $a_1(\theta) = \sin(2\theta) + b_{30}\cos(3\theta)$ and $a_2(\theta) = c_{10}\cos\theta + c_{02}\sin(2\theta) + c_{30}\cos(3\theta)$. Therefore equation (1.1) is invariant under the change of variables $\theta \mapsto \pi - \theta$ and the differential equation (1.1) is α-symmetric with $\alpha = \pi/2$ and, thus, it is universal by Theorem 1.3.

In the second case 2) we have $a_1(\theta) = \sin(2\theta) + b_{03}\sin(3\theta)$ and $a_2(\theta) = c_{01}\sin\theta + c_{02}\sin(2\theta) + c_{03}\sin(3\theta)$. Therefore equation (1.1) is invariant under the change of variables $\theta \mapsto -\theta$ and the differential equation (1.1) is also α-symmetric with $\alpha = 0$.

The third case 3) corresponds to a particular case studied in Theorem 2.2 given by $b_{10} = b_{01} = c_{10} = c_{01} = 0$.

Finally, we take the eleven necessary conditions $c_n(a) = 0$ for $n = \overline{3, 13}$ and we impose the case $b_{10} = b_{01} = b_{20} = b_{02} = b_{30} = 0$ and $b_{03} = 1$ (Case III). Here we can obtain the irreducible decomposition of the variety $V(I)$ over the field $\mathbb{Q}$. To show that all the obtained families are universal centers for equation (1.1) we refer to the case studied in Theorem 2.2 given by $b_{10} = b_{01} = b_{30} = 0$. □

Moreover, although we cannot completely solve Case I, we present the following result.

Theorem 2.2. *All the centers (with probability close to* 1*) of equation* (1.1) *when* $a_1(\theta)$ *and* $a_2(\theta)$ *are trigonometric polynomials of degree* 3 *of the form* (2.1) *with either*

- $b_{10} = b_{01} = c_{10} = c_{01} = 0$ *or;*
- $b_{20} = b_{02} = c_{20} = c_{02} = 0$ *or;*
- $b_{10} = b_{01} = b_{30} = 0$ *or;*
- $b_{10} = b_{01} = b_{03} = 0$

are universal centers and, consequently, verify the composition condition.

Proof. To proof this result we have followed the same computations than in the previous theorem to obtain eleven center conditions $c_n(a)$ for $n = \overline{3, 13}$, and we have proceeded analogously.

The obtained decomposition for the case $b_{10} = b_{01} = c_{10} = c_{01} = 0$ consists of 3 components defined by the following ideals

1) $\langle c_{30}, c_{03}, b_{20}, b_{02}, b_{30}, b_{03} \rangle$;

2) $\langle b_{03}c_{30} - b_{30}c_{03}, b_{02}c_{20} - b_{20}c_{02}, b_{02}c_{30} - b_{30}c_{02}\rangle$;

3) $\langle b_{03}c_{30} - b_{30}c_{03}, b_{02}c_{20} - b_{20}c_{02}, -3b_{02}^2b_{03}^2b_{20} + b_{03}^2b_{20}^3 + 2b_{02}^3b_{03}b_{30} - 6b_{02}b_{03}b_{20}^2b_{30} + 3b_{02}^2b_{20}b_{30}^2 - b_{20}^3b_{30}^2\rangle$.

We now show that equation (1.1) has a universal center under these conditions. In the first case 1) we have that $a_1(\theta) = 0$ and $a_2(\theta) = c_{20}\sin(2\theta) + c_{02}\cos(2\theta)$. Therefore equation (1.1) is in this case of separable variables and by Theorem 1.3 it has a universal center. In the second case we have $b_{20}a_2(\theta) = c_{20}a_1(\theta)$. Hence we have composition condition and equation (1.1) has a universal center. In the third case 3) we take $b_{20} = r_0\sin\beta$ and $b_{02} = r_0\cos\beta$ and it is easy to see that equation (1.1) is invariant under the change of variables $\theta \mapsto \pi - \beta - \theta$. Hence the differential equation (1.1) is α-symmetric with $\alpha = (\pi - \beta)/2$ and by Theorem 1.3 it has a universal center.

The obtained decomposition for the case $b_{20} = b_{02} = c_{20} = c_{02} = 0$ consists of 4 components defined by the following ideals

1) $\langle b_{03}c_{30} - b_{30}c_{03}, b_{01}c_{10} - b_{10}c_{01}, b_{01}c_{03} - b_{03}c_{01}\rangle$;

2) $\langle b_{03}c_{30} - b_{30}c_{03}, b_{01}c_{10} - b_{10}c_{01}, -3b_{01}^2b_{03}b_{10} + b_{03}b_{10}^3 + b_{01}^3b_{30} - 3b_{01}b_{10}^2b_{30}\rangle$;

3) $\langle b_{03}c_{30} - b_{30}c_{03}, b_{10}, b_{01}, -b_{30}c_{01}^3 + 3b_{03}c_{01}^2c_{10} + 3b_{30}c_{01}c_{10}^2 - b_{03}c_{10}^3\rangle$;

4) $\langle b_{01}c_{10} - b_{10}c_{01}, b_{30}, b_{03}, -3b_{01}^2b_{10}c_{03} + b_{10}^3c_{03} + b_{01}^3c_{30} - 3b_{01}b_{10}^2c_{30}\rangle$.

We now show that equation (1.1) has a universal center under these conditions. In the first case 1) we have that $b_{01}a_2(\theta) = c_{01}a_1(\theta)$. Hence we have composition condition and equation (1.1) has a universal center. In the second case 2) and fourth case 4) we take $b_{10} = r_1\sin\beta_1$ and $b_{01} = r_1\cos\beta_1$ and it is easy to see that equation (1.1) is invariant under the change of variables $\theta \mapsto -2\beta_1 - \theta$. Hence the differential equation (1.1) is α-symmetric with $\alpha = -\beta_1$ and by Theorem 1.3 it has a universal center. In the third case 3), if we take $c_{10} = r_2\sin\beta_2$ and $c_{01} = r_2\cos\beta_2$, it is easy to see that equation (1.1) is invariant under the change of variables $\theta \mapsto -2\beta_2 - \theta$. Therefore the differential equation (1.1) is also α-symmetric with $\alpha = -\beta_2$ and by Theorem 1.3 it has a universal center.

The decomposition for the case $b_{10} = b_{01} = b_{30} = 0$ consists of 4 components defined by the following ideals

1) $\langle b_{03}, b_{02}, b_{20}\rangle$;

2) $\langle b_{03}, b_{02}c_{20} - b_{20}c_{02}, -3c_{01}^2c_{03}c_{10} + c_{03}c_{10}^3 + c_{01}^3c_{30} - 3c_{01}c_{10}^2c_{30}\rangle$;

3) $\langle c_{30}, c_{01}, c_{10}, b_{02}c_{20} - b_{20}c_{02}, c_{20}b_{03} - b_{20}c_{03}\rangle$;

4) $\langle c_{30}, c_{20}, c_{10}, b_{20}\rangle$.

In the first case 1) we have that $a_1(\theta) = 0$. Therefore equation (1.1) is of separable variables and by Theorem 1.3 it has a universal center. In the second case 2) we take $c_{10} = r_4\sin\beta_4$ and $c_{01} = r_4\cos\beta_4$ and it is easy to see that equation (1.1) is invariant under the change of variables $\theta \mapsto -2\beta_4 - \theta$. Hence the differential equation (1.1) is α-symmetric with $\alpha = -\beta_4$ and by Theorem 1.3 it has a universal center. The third case 3) correspond to case 2) of the decomposition studied in the case $b_{10} = b_{01} = c_{10} = c_{01} = 0$ (first paragraph of this proof). In the last case 4) we have $a_1(\theta) = b_{02}\sin(2\theta) + b_{03}\sin(3\theta)$ and $a_2(\theta) = c_{01}\sin\theta + c_{02}\sin(2\theta) + c_{03}\sin(3\theta)$. Therefore equation (1.1) is invariant under the change of variables $\theta \mapsto -\theta$ and the differential equation (1.1) is also α-symmetric with $\alpha = 0$.

The decomposition for the case $b_{10} = b_{01} = b_{03} = 0$ also consists of 4 components defined by the following ideals

1) $\langle b_{30}, b_{02}, b_{20} \rangle$;

2) $\langle b_{30}, b_{02}c_{20} - b_{20}c_{02}, -3c_{01}^2 c_{03}c_{10} + c_{03}c_{10}^3 + c_{01}^3 c_{30} - 3c_{01}c_{10}^2 c_{30} \rangle$;

3) $\langle c_{03}, c_{01}, c_{10}, b_{02}c_{20} - b_{20}c_{02}, c_{20}b_{03} - b_{20}c_{03} \rangle$;

4) $\langle c_{03}, c_{20}, c_{01}, b_{20} \rangle$.

The first case 1) and the second case 2) are studied in the decomposition of the case $b_{10} = b_{01} = b_{30} = 0$ (previous paragraph of this proof). The third case 3) corresponds to case 2) of the decomposition studied in the case $b_{10} = b_{01} = c_{10} = c_{01} = 0$ (first paragraph of this proof). In the last case 4) we have $a_1(\theta) = b_{02}\sin(2\theta) + b_{30}\cos(3\theta)$ and $a_2(\theta) = c_{10}\cos\theta + c_{02}\sin(2\theta) + c_{30}\cos(3\theta)$. Therefore equation (1.1) is invariant under the change of variables $\theta \mapsto -\theta - \pi$ and the differential equation (1.1) is again α-symmetric with $\alpha = -\pi/2$. □

Acknowledgements

Appendix

Proof of Proposition 2.1. First we study the case when $a_1(\theta)$ and $a_2(\theta)$ are trigonometric polynomials of degree 1, therefore we have

$$
\begin{aligned}
a_1(\theta) &= b_{00} + b_{10}\cos\theta + b_{01}\sin\theta, \\
a_2(\theta) &= c_{00} + c_{10}\cos\theta + c_{01}\sin\theta,
\end{aligned}
$$

where b_{ij} and c_{ij} are real constants. We recall that the first two center conditions imply that $b_{00} = 0$ and $c_{00} = 0$. The next center condition is $c_3(a) = 0$ with $c_3(a) = b_{01}c_{10} - b_{10}c_{01}$. We take $b_{10} = kc_{10}$ and $c_{01} = kc_{01}$, with $k \in \mathbb{R}$, and some of the next center conditions are zero. In this case equation (1.1) takes the form

$$
\dot{r} = r^2(k+r)(c_{10}\cos\theta + c_{01}\sin\theta). \tag{2.2}
$$

Equation (2.2) is of separable variables and by Theorem 1.3 has a universal center.

Second, in the case where $a_1(\theta)$ and $a_2(\theta)$ are trigonometric polynomials of degree 2 we have

$$
\begin{aligned}
a_1(\theta) &= b_{00} + b_{10}\cos\theta + b_{01}\sin\theta + b_{20}\cos(2\theta) + b_{02}\sin(2\theta), \\
a_2(\theta) &= c_{00} + c_{10}\cos\theta + c_{01}\sin\theta + c_{20}\cos(2\theta) + c_{02}\sin(2\theta),
\end{aligned}
$$

where b_{ij} and c_{ij} are real constants. The first conditions to have a center are, as before, that $b_{00} = 0$ and $c_{00} = 0$. Applying a rotation and a rescaling we can divide the study in two separate cases:

$$
\text{(i) } b_{10} = 1 \text{ and } b_{01} = 0 \quad \text{and} \quad \text{(ii) } b_{10} = b_{01} = 0.
$$

We begin to study case (i). In this case the next center condition is $c_3(a) = 0$ with $c_3(a) = -2c_{01} - b_{20}c_{02} + b_{02}c_{20} = 0$. From this condition we isolate $c_{01} = (b_{02}c_{20} - b_{20}c_{02})/2$. The next center conditions take the form

$$
\begin{aligned}
c_4(a) \;=\;& b_{02}b_{20}c_{02} - 2b_{20}c_{10} + 2c_{20} - b_{02}^2 c_{20}, \\
c_5(a) \;=\;& -12b_{20}c_{02} - 5b_{02}^2 b_{20}c_{02} + 3b_{20}^3 c_{02} - 4b_{20}c_{02}^2 + 2b_{20}^3 c_{02}^2 + 16b_{02}b_{20}c_{10} \\
& +8b_{02}b_{20}c_{02}c_{10} - 8b_{20}c_{10}^2 - 4b_{02}c_{20} + 5b_{02}^3 c_{20} - 3b_{02}b_{20}^2 c_{20} \\
& +4b_{02}c_{02}c_{20} - 4b_{02}b_{20}^2 c_{02}c_{20} + 8c_{10}c_{20} - 8b_{02}^2 c_{10}c_{20} + 2b_{02}^2 b_{20}c_{20}^2.
\end{aligned}
$$

The resultant between these two polynomials with respect to c_{10} gives the following result

$$
\mathrm{res}(c_4(a), c_5(a), c_{10}) = b_{20}(b_{20}c_{02} - b_{02}c_{20})C_{56},
$$

where

$$
\begin{aligned}
C_{56} \;=\;& -12b_{20} + 3b_{02}^2 b_{20} + 3b_{20}^3 - 4b_{20}c_{02} + 2b_{02}^2 b_{20}c_{02} + 2b_{20}^3 c_{02} + 4b_{02}c_{20} \\
& -2b_{02}^3 c_{20} - 2b_{02}b_{20}^2 c_{20}.
\end{aligned}
$$

a) Case $b_{20} = 0$. In this case the condition $c_4(a) = 0$ with $c_4(a) = (b_{02}^2 - 2)c_{20}$. The cases $b_{02} = \pm\sqrt{2}$ do not satisfy the next center conditions, so they do not give rise to centers. In the case $c_{20} = 0$ equation (1.1) takes the form

$$
\dot{r} = r^2 \cos\theta (1 + c_{10}r + 2(b_{02} + c_{02}r)\sin\theta). \tag{2.3}
$$

System (2.3) has an α-symmetric center, with $\alpha = \pi/2$, because it has the symmetry $\theta \to \pi - \theta$ and in virtue of Theorem 1.3 it is a universal center.

b) Case $b_{20}c_{02} - b_{02}c_{20} = 0$ and $b_{20} \neq 0$. In this case we take $b_{20} = c_{20}k$ and $b_{02} = c_{02}k$ and the next center condition is $c_4(a) = 0$ with $c_4(a) = c_{20}(c_{10}k - 1)$. The case $c_{02} = 0$ implies $b_{20} = 0$ which is out of our assumptions in this case. Hence we must take $c_{10} = 1/k$. In this case equation (1.1) has the form

$$
\dot{r} = r^2(k + r)(\cos\theta + c_{20}k\cos 2\theta + c_{02}k\sin 2\theta)
$$

which is of separable variables and by Theorem 1.3 it has a universal center.

c) Case $C_{56} = 0$ with $b_{20}c_{02} - b_{02}c_{20} \neq 0$ and $b_{20} \neq 0$. In this case we compute the following resultants:

$$
\begin{aligned}
\mathrm{res}(c_4(a), c_6(a), c_{10}) &= b_{20}(-b_{20}c_{02} + b_{02}c_{20})C_{57}, \\
\mathrm{res}(c_4(a), c_7(a), c_{10}) &= b_{20}(-b_{20}c_{02} + b_{02}c_{20})C_{58}, \\
\mathrm{res}(c_4(a), c_8(a), c_{10}) &= b_{20}(-b_{20}c_{02} + b_{02}c_{20})C_{59}, \\
\mathrm{res}(c_4(a), c_9(a), c_{10}) &= b_{20}(-b_{20}c_{02} + b_{02}c_{20})C_{510},
\end{aligned}
$$

where C_{57}, C_{58}, C_{59} and C_{510} are polynomials in the variables b_{20}, b_{02}, c_{20} and c_{02}. The next step is to make the following resultants with respect to c_{02}.

$$
\begin{aligned}
\mathrm{res}(C_{56}, C_{57}, c_{02}) &= b_{20}C_{67}, \quad \mathrm{res}(C_{56}, C_{58}, c_{02}) = b_{20}^2 C_{68}, \\
\mathrm{res}(C_{56}, C_{59}, c_{02}) &= b_{20}^2 C_{69}, \quad \mathrm{res}(C_{56}, C_{510}, c_{02}) = b_{20}^3 C_{610},
\end{aligned}
$$

where C_{67}, C_{68}, C_{69} and C_{610} are polynomials in the variables b_{20}, b_{02} and c_{20}. Now we perform the following resultants with respect to b_{02}.

$$\begin{aligned}
\operatorname{res}(C_{67}, C_{68}, b_{02}) &= b_{20}^{10} C_{78}, \\
\operatorname{res}(C_{67}, C_{69}, b_{02}) &= b_{20}^{8}(b_{20}^{2} - 2)c_{20} C_{79}, \\
\operatorname{res}(C_{67}, C_{610}, b_{02}) &= b_{20}^{12} C_{710},
\end{aligned}$$

where C_{78}, C_{79} and C_{710} are polynomials in the variables b_{20} and c_{20}. The cases $b_{20}^{2} - 2 = 0$ and $c_{20} = 0$ give no common root. Hence, we make the following resultants with respect to c_{20}.

$$\begin{aligned}
\operatorname{res}(C_{78}, C_{79}, c_{02}) &= b_{20}^{48}(b_{20} + 2)^{2}(b_{20} - 2)^{2} C_{89}, \\
\operatorname{res}(C_{78}, C_{710}, c_{02}) &= b_{20}^{72}(b_{20} + 2)^{2}(b_{20} - 2)^{2} C_{810},
\end{aligned}$$

where C_{89}, and C_{810} are polynomials uniquely in the variable b_{20}. The cases $b_{20}^{2} - 4 = 0$ give no common root. Therefore we make the last resultant with respect to b_{20} which gives the result

$$\operatorname{res}(C_{89}, C_{810}, b_{20}) \neq 0.$$

Therefore, there is no common root and consequently there are no more cases.

Now we study the case (ii) $b_{10} = b_{01} = 0$. In this case the first center condition has the form $c_3(a) = 0$ with $c_3(a) = b_{02}c_{20} - b_{20}c_{02}$. We take $b_{20} = c_{20}k$ and $b_{02} = c_{02}k$ and the next center conditions are $c_4(a) = 0$ and $c_5(a) = k(-2c_{01}c_{02}c_{10} + c_{01}^{2}c_{20} - c_{10}^{2}c_{20})$.

a) Case $k = 0$. In this case $a_2(\theta) = 0$ and (1.1) is of separable variables and by Theorem 1.3 it has a universal center.

b) Case $-2c_{01}c_{02}c_{10} + c_{01}^{2}c_{20} - c_{10}^{2}c_{20} = 0$. We take $c_{20} = 2c_{01}c_{10}m$ and $c_{02} = (c_{01}^{2} - c_{10}^{2})m$, with $m \in \mathbb{R}$, and equation (1.1) takes the form

$$\dot{r} = r^{2}\psi(\theta)(r + 2\psi'(\theta)m(k + r)), \tag{2.4}$$

where $\psi(\theta) = c_{10}\sin\theta + c_{01}\cos\theta$. In this case equation (2.4) has an α-symmetric center, with $\alpha = -\tau$, because it has the symmetry $\theta \to -2\tau - \theta$ where $\tau = \arctan(c_{01}/c_{10})$. Hence, by Theorem 1.3 it is also a universal center.

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