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ENTROPY SOLUTIONS FOR NONLINEAR NONHOMOGENEOUS NEUMANN PROBLEMS INVOLVING THE GENERALIZED $P(X)$-LAPLACE OPERATOR

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Abstract In this work we investigate a class of nonlinear $p(x)$ Laplace problems with Neumann nonhomogeneous boundary conditions and L^1 data. The techniques of entropy solutions for elliptic equations are used to prove the existence of a solution.

Keywords Generalized Sobolev spaces, Neumann boundary conditions, Entropy solution, noncoercive operator.

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1. Introduction

The study of various mathematical problems with variable exponent has received considerable attention in recent years. These problems are interesting in applications (see [5,9,18]) and raise many difficult and interesting mathematical problems. Fo the interested reader, we refer to [1,2,6,7,10,11,13–15,17,19–22] for the advances and the references in this area.

In this paper, we consider the inhomogeneous and nonlinear Neumann boundary value problem:

$$\begin{cases} -\mathrm{div}(\Phi(\nabla u-\Theta(u)))+|u|^{p(x)-2}u+\alpha(u)=f & \text{in } \Omega \\ (\Phi(\nabla u-\Theta(u)).\eta+\gamma(u)=g & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

with

$$\Phi(\xi)=|\xi|^{p(x)-2}\xi, \quad \forall \xi\in I\!R^N,$$

where $\Omega\subseteq I\!R^N (N\geq 3)$ is a bounded open domain with Lipschitz boundary $\partial\Omega$, η is the outer unit normal vector on $\partial\Omega$, $\alpha,\ \gamma,\Theta$ are real functions defined on $I\!R$ or $I\!R^N$, $f\in L^1(\Omega)$ and $g\in L^1(\partial\Omega)$.

Neumann $p(x)$-problem with right-hand side in L^1 was considered in [13–15] to study the existence and uniqueness of entropy solutions. In these papers, they considered a Leray-Lions type operator, which permit them to exploit the growth

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condition, the coerciveness condition and the monotonicity condition of the operator to achieve their work. Unfortunately, in this paper, due to the term Θ in the operator, we don't have such Leray-Lions conditions and we can't use the main techniques as in [13–15] to prove the existence of solutions and we can't expect the uniqueness of the solution. To prove the existence of entropy solutions (we expect entropy solution since f and g are in L^1), we assume that Θ is a Lipschitz function with Lipschitz constant satisfies a condition connected to the exponent (see assumption (H_3) below). Note also that a partial result of uniqueness of entropy solution can be obtained under additionnal assumption on the operator $\Phi(\nabla u - \Theta(u))$ such as $[\Phi(\nabla u - \Theta(u)) - \Phi(\nabla v - \Theta(v))]\,.\,[u - v] \geq 0$ (see [3]). We proceed by approximation of the initial problem for which we prove that the associated operator is of type (M). Then, by some a priori estimates, we prove that the approximate sequence converges to an entropy solution of the initial problem.

Problem (1.1) is motivated by the homogenization in the particular case where $p(.) \equiv 2$ and for a perforated domain with Neumann condition on the boundary of the holes in the generalized case.

After the completion of this work the papers [4] and [16], have been brought to our attention. However, they deal with different problems, namely in [4] Andreu, Mazôn, Segura De León and Toledo, introduced the notion of entropy solutions for elliptic problems with Neumann boundary conditions in the classical case ($p(.) \equiv p$ (a constant)), in which $\Theta = 0$ and $(|u|^{p(x)-2}u + \alpha(u)) = u$, and in [16] the nonhomogeneous boundary value problems treated with Carathéodory functions are solved for an additive case $\gamma(u) = \lambda u$, with $\lambda > 0$.

This paper is divided into three sections, organized as follows: in section 2, we introduce some basic properties of the space $W^{1,p(x)}(\Omega)$, some useful lemmas and in section 3, we prove the existence of entropy solutions of (1.1).

2. Preliminaries

As the exponent $p(x)$ appearing in (1.1) depends on the variable x, we must work with Lebesgue and Sobolev spaces with variable exponents, under the following assumptions on the data:

$$\begin{cases} p(.) : \overline{\Omega} \to I\!R \text{ is a continuous function such that} \\ 1 < p_- \leq p_+ < +\infty, \end{cases} \tag{2.1}$$

where $p_- := ess \inf\limits_{x\in\Omega} p(x)$ and $p_+ := ess \sup\limits_{x\in\Omega} p(x)$.

We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable function $u : \Omega \to I\!R$ for which the convex modular

$$\rho_{p(x)}(u) := \int_\Omega |u|^{p(x)} dx$$

is finite. If the exponent is bounded, i.e., if $p_+ < +\infty$, then the expression

$$||u||_{p(x)} = \inf\{\lambda > 0 : \rho_{p(x)}(u/\lambda) \leq 1\}$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxembourg norm. The space $(L^{p(x)}(\Omega), \|.\|_{p(x)})$ is a separable Banach space. Moreover, if $1 < p_- \leq p_+ < +\infty$, then $L^{p(x)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(x)}(\Omega)$, where $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$. Finally, we have the Hölder type inequality:

$$\left|\int_\Omega uv d\,x\right| \leq (\frac{1}{p_-} + \frac{1}{p'_-})\|u\|_{p(x)}\|v\|_{p'(x)} \tag{2.2}$$

for all $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$.

Let

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) \text{ and } |\nabla u| \in L^{p(x)}(\Omega)\},$$

which is a Banach space equipped with the following norm

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}.$$

The space $(W^{1,p(x)}(\Omega), ||.||_{1,p(x)})$ is a separable and reflexive Banach space.

An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by the modular $\rho_{p(x)}$ of the space $L^{p(x)}(\Omega)$. We have the following result :

Proposition 2.1. *(see [11, 23]) If $u_n, u \in L^{p(x)}(\Omega)$ and $p_+ < +\infty$, then the following properties hold true:*

(i) $\|u\|_{p(x)} > 1 \Rightarrow \|u\|_{p(x)}^{p_-} < \rho_{p(x)}(u) < \|u\|_{p(x)}^{p_+}$;

(ii) $\|u\|_{p(x)} < 1 \Rightarrow \|u\|_{p(x)}^{p_+} < \rho_{p(x)}(u) < \|u\|_{p(x)}^{p_-}$;

(iii) $\|u\|_{p(x)} < 1$ *(respectively* $= 1; > 1$*)* $\Leftrightarrow \rho_{p(x)}(u) < 1$ *(respectively* $= 1; > 1$*);*

(iv) $\|u_n\|_{p(x)} \to 0$ *(respectively* $\to +\infty$*)* $\Leftrightarrow \rho_{p(x)}(u_n) \to \infty$ *(respectively* $\to +\infty$*) ;*

(v) $\rho_{p(x)}(u/\|u\|_{p(x)}) = 1$.

For a measurable function $u : \Omega \to I\!R$, we introduce the following notation:

$$\rho_{1,p(x)}(u) = \int_\Omega |u|^{p(x)}\,dx + \int_\Omega |\nabla u|^{p(x)}\,dx.$$

Proposition 2.2. *(see [19, 20]) If $u \in W^{1,p(x)}(\Omega)$, then the following properties hold true:*

(i) $||u||_{1,p(x)} > 1 \Rightarrow ||u||_{1,p(x)}^{p_-} \leq \rho_{1,p(x)}(u) \leq ||u||_{1,p(x)}^{p_+}$;

(ii) $||u||_{1,p(x)} < 1 \Rightarrow ||u||_{1,p(x)}^{p_+} \leq \rho_{1,p(x)}(u) \leq ||u||_{1,p(x)}^{p_-}$;

(iii) $||u||_{1,p(x)} < 1$ *(respectively* $= 1; > 1$*)* $\Leftrightarrow \rho_{1,p(x)}(u) < 1$ *(respectively* $= 1; > 1$*).*

Put

$$p^\partial(x) := (p(x))^\partial := \begin{cases} \dfrac{(N-1)p(x)}{N-p(x)}, \text{ if } p(x) < N \\ \infty, \text{ if } p(x) \geq N. \end{cases}$$

Proposition 2.3. *(see [20])Let $p \in C(\bar{\Omega})$ and $p_- > 1$. If $q \in C(\partial\Omega)$ satisfies the condition*

$$1 \leq q(x) < p^\partial(x),\ \forall\, x \in \partial\Omega,$$

then, there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega)$.

In particular, there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\partial\Omega)$.

Let us introduce the following notation: given two bounded measurable functions $p(x), q(x) : \Omega \to I\!R$, we write

$$q(x) \ll p(x) \text{ if } ess \inf_{x\in\Omega}(p(x) - q(x)) > 0.$$

Lemma 2.1. *Let $\xi, \eta \in I\!R^N$ and let $1 < p < \infty$. We have*

$$\frac{1}{p}|\xi|^p - \frac{1}{p}|\eta|^p \leq |\xi|^{p-2}\xi.(\xi - \eta).$$

Proof. We consider the function $f : I\!R^+ \to I\!R$ defined by $x \mapsto x^p - px + (p-1)$.
We have

$$f(x) \geq \min_{y\in I\!R^+} f(y) = f(1) = 0 \text{ for all } x \in I\!R^+.$$

Therefore, we take $x = \frac{|\eta|}{|\xi|}$ (if $|\xi| = 0$, the result is obvious) in the inequality above to get the result of the lemma by using Cauchy-Schwarz inequality. □

In the sequel, we need the following two technical lemmas (see [12, 17]).

Lemma 2.2. *Let $(v_n)_{n\in I\!N}$ be a sequence of measurable functions in Ω. If v_n converges in measure to v and is uniformly bounded in $L^{p(.)}(\Omega)$ for some $1 \ll p(.) \in L^\infty(\Omega)$, then v_n strongly converges to v in $L^1(\Omega)$.*

The second technical lemma is a well known result in measure theory (see [12]):

Lemma 2.3. *Let $(X, \mathcal{M}, \mu)$ be a measure space such that $\mu(X) < +\infty$. Consider a measurable function $\gamma : X \longrightarrow [0, +\infty]$ such that*

$$\mu(\{x \in X : \gamma(x) = 0\}) = 0.$$

Then, for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\mu(A) < \epsilon \text{ for all } A \in \mathcal{M} \text{ with } \int_A \gamma d\mu < \delta.$$

3. Existence Result

In this part, we study the existence of an entropy solution of (1.1).

We begin by stating the following assumptions:

(H1) α and γ are continuous functions defined on $I\!R$ such that $\alpha(x).x \geq 0$ and $\gamma(x).x \geq 0$ for all $x \in I\!R$.

(H2) $f \in L^1(\Omega)$ and $g \in L^1(\partial\Omega)$.

(H3) $\Theta : I\!R \to I\!R^N$ is a continuous function such that $\Theta(0) = 0$ and $|\Theta(x) - \Theta(y)| \leq C|x - y|$, for all $x, y \in I\!R$, where C is a positive constant such that

$$C < \min\left(\left(\frac{p_-}{2}\right)^{\frac{1}{p_-}}, \left(\frac{p_-}{2}\right)^{\frac{1}{p_+}}\right).$$

(H4) $\Phi(\xi) = |\xi|^{p(x)-2}\xi$, for any $\xi \in I\!R^N$.

In this section we give the concept of entropy solution for the problem (1.1).

We first recall some notations.

For any $k > 0$, we define the truncation function T_k by $T_k(s) := \max\{-k, min\{k, s\}\}$. For any $u \in W^{1,p(x)}(\Omega)$, we denote by $\tau(u)$ the trace of u on $\partial\Omega$ in the usual sense. In the sequel, we will identify at the boundary u and $\tau(u)$.

Set

$$\mathcal{T}^{1,p(x)}(\Omega) = \{ \quad u : \Omega \to I\!R, \text{ measurable such that } T_k(u) \in W^{1,p(x)}(\Omega), \text{ for any } k > 0\}.$$

As in [8], we can prove the following result.

Proposition 3.1. *Let $u \in \mathcal{T}^{1,p(x)}(\Omega)$. Then there exists a unique measurable function $v : \Omega \to I\!R^N$ such that $\nabla T_k(u) = v\chi_{\{|u|<k\}}$, for all $k > 0$. The function v is denoted by ∇u.*

Moreover if $u \in W^{1,p(x)}(\Omega)$ then $v \in (L^{p(x)}(\Omega))^N$ and $v = \nabla u$ in the usual sense.

Following [3, 4, 13–15], we define $\mathcal{T}_{tr}^{1,p(x)}(\Omega)$ as the the set of functions $u \in \mathcal{T}^{1,p(x)}(\Omega)$ such that there exists a sequence $(u_n)_{n\in I\!N} \subset W^{1,p(x)}(\Omega)$ satisfying the following conditions:

(C_1) $u_n \to u$ a.e. in Ω.

(C_2) $\nabla T_k(u_n) \to \nabla T_k(u)$ in $(L^1(\Omega))^N$ for any $k > 0$.

(C_3) There exists a measurable function v on $\partial\Omega$, such that $u_n \to v$ a.e. in $\partial\Omega$.

The function v is the trace of u in the generalized sense introduced in [3,4]. In the sequel the trace of $u \in \mathcal{T}_{tr}^{1,p(x)}(\Omega)$ on $\partial\Omega$ will be denoted by $tr(u)$. If $u \in W^{1,p(x)}(\Omega)$, $tr(u)$ coincides with $\tau(u)$ in the usual sense. Moreover, for $u \in \mathcal{T}_{tr}^{1,p(x)}(\Omega)$ and for every $k > 0$, $\tau(T_k(u)) = T_k(tr(u))$ and if $\varphi \in W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ then $(u - \varphi) \in \mathcal{T}_{tr}^{1,p(x)}(\Omega)$ and $tr(u - \varphi) = tr(u) - tr(\varphi)$.

We can now introduce the notion of entropy solution of (1.1).

Definition 3.1. A measurable function $u : \Omega \to I\!R$ is called an entropy solution of the elliptic problem (1.1) if $u \in \mathcal{T}_{tr}^{1,p(x)}(\Omega)$, $|u|^{p(x)-2}u \in L^1(\Omega)$, $\alpha(u) \in L^1(\Omega)$, $\gamma(u) \in L^1(\partial\Omega)$ and

$$\begin{aligned}
&\int_\Omega \Phi(\nabla u - \Theta(u))\nabla T_k(u-\varphi)dx + \int_\Omega |u|^{p(x)-2}uT_k(u-\varphi)dx \\
&+ \int_\Omega \alpha(u)T_k(u-\varphi)dx + \int_{\partial\Omega} \gamma(u)T_k(u-\varphi)d\sigma \qquad (3.1)\\
&\le \int_\Omega fT_k(u-\varphi)\,dx + \int_{\partial\Omega} gT_k(u-\varphi)d\sigma,
\end{aligned}$$

for every $\varphi \in W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ and for every $k > 0$.

Remark 3.1. Notice that each integral in the above definition is well defined. Indeed, since $\varphi \in W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ then $u - \varphi \in \mathcal{T}_{tr}^{1,p(x)}(\Omega)$, hence $T_k(u - \varphi) \in W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ and consequently the first, the second, the third and the fifth integral in (3.1) are well defined. Moreover, in the fourth and the sixth integral, we can use the fact that the trace of $g \in W^{1,p(x)}(\Omega)$ on $\partial\Omega$ is well defined in $L^{p(x)}(\partial\Omega)$.

Now we announce the main result of this section

Theorem 3.1. *Let assumptions (H1)-(H4) hold true. Then there exists at least one entropy solution of the problem (1.1).*

Proof. The proof of Theorem 3.1 consists into two steps. □

STEP 1. The approximate problem

We consider the sequence of approximate problems:

$$\begin{aligned}
&\int_\Omega \Phi(\nabla u_n - \Theta(u_n))\nabla v\, dx + \int_\Omega |u_n|^{p(x)-2}u_n v dx + \int_\Omega T_n(\alpha(u_n))v\, dx \\
&+ \int_{\partial\Omega} T_n(\gamma(u_n))v\, d\sigma \\
= \; & \int_\Omega T_n(f)v\, dx + \int_{\partial\Omega} T_n(g)v\, d\sigma.
\end{aligned} \tag{3.2}$$

We define the following reflexive space

$$E = W^{1,p(x)}(\Omega) \times L^{p(x)}(\partial\Omega).$$

Let X_0 be the subspace of E defined by

$$X_0 = \{(u,v) \in E : v = \tau(u)\}.$$

In the sequel, we will identify an element $(u,v) \in X_0$ with is representative $u \in W^{1,p(x)}(\Omega)$.

We define the operator A_n by

$$\langle A_n u, v\rangle = \langle Au, v\rangle + \int_\Omega T_n(\alpha(u))v\, dx + \int_{\partial\Omega} T_n(\gamma(u))v\, d\sigma \qquad \forall\, u, v \in X_0,$$

where

$$\langle Au, v\rangle = \int_\Omega \Phi(\nabla u - \Theta(u))\nabla v\, dx + \int_\Omega |u|^{p(x)-2}uv\, dx.$$

Assertion 1. The operator A_n is of type (M).

Indeed, let $(u_k)_{k\in\mathbb{N}}$ be a sequence in X_0 such that

$$\begin{cases} u_k \rightharpoonup u \quad \text{weakly in } \; X_0, \\ A_n u_k \rightharpoonup \chi \text{ weakly in } \; X_0', \\ \limsup\limits_{k\to\infty}\langle A_n u_k, u_k\rangle \le \langle \chi, u\rangle. \end{cases} \tag{3.3}$$

By (H1), we obtain

$$T_n(\alpha(u))u \ge 0 \text{ and } T_n(\gamma(u))u \ge 0.$$

We use the Fatou's lemma to obtain that

$$\begin{aligned}
&\liminf_{k\to\infty}\left(\int_\Omega T_n(\alpha(u_k))u_k\, dx + \int_{\partial\Omega} T_n(\gamma(u_k))u_k\, d\sigma\right) \\
\ge \; & \int_\Omega T_n(\alpha(u))u\, dx + \int_{\partial\Omega} T_n(\gamma(u))u\, d\sigma.
\end{aligned}$$

On the other hand, thanks to the Lebesgue dominated convergence theorem, we have

$$\begin{aligned} & \lim_{k\to\infty}\left(\int_\Omega T_n(\alpha(u_k))v\,dx+\int_{\partial\Omega}T_n(\gamma(u_k))v\,d\sigma\right)\\ = & \int_\Omega T_n(\alpha(u))vdx+\int_{\partial\Omega}T_n(\gamma(u))v\,d\sigma, \forall v\in X_0. \end{aligned}$$

Consequently,

$$T_n(\alpha(u_k))+T_n(\gamma(u_k))\rightharpoonup T_n(\alpha(u))+T_n(\gamma(u)) \text{ weakly in } X_0'.$$

Thus, it follows that

$$Au_k\rightharpoonup \chi-(T_n(\alpha(u))+T_n(\gamma(u))) \text{ weakly in } X_0'.$$

As the operator A is of type (M), so we have immediately

$$Au=\chi-(T_n(\alpha(u))+T_n(\gamma(u))).$$

Therefore we deduce that $A_n u=\chi$.

Hence, the operator A_n is of type (M).

Assertion 2. The operator A_n is coercive.

By (H1), we have

$$\int_\Omega T_n(\alpha(u))u\,dx+\int_{\partial\Omega}T_n(\gamma(u))u\,d\sigma\geq 0,$$

then $\langle A_n u,u\rangle\geq\langle Au,u\rangle$.

On the other hand, using lemma 2.1, we obtain

$$\begin{aligned} \langle Au,u\rangle &= \int_\Omega \Phi(\nabla u-\Theta(u))\nabla u dx+\int_\Omega |u|^{p(x)}dx\\ &= \int_\Omega |\nabla u-\Theta(u)|^{p(x)-2}(\nabla u-\Theta(u))\nabla u dx+\int_\Omega |u|^{p(x)}dx\\ &\geq \int_\Omega \frac{1}{p(x)}|\nabla u-\Theta(u)|^{p(x)}dx-\int_\Omega\frac{1}{p(x)}|\Theta(u)|^{p(x)}dx+\int_\Omega|u|^{p(x)}dx. \end{aligned} \tag{3.4}$$

Since

$$(a+b)^p\leq 2^{p-1}(|a|^p+|b|^p),$$

we have

$$\begin{aligned} \frac{1}{2^{p_+-1}}|\nabla u|^{p(x)} &= \frac{1}{2^{p_+-1}}|\nabla u-\Theta(u)+\Theta(u)|^{p(x)}\\ &\leq |\nabla u-\Theta(u)|^{p(x)}+|\Theta(u)|^{p(x)}, \end{aligned} \tag{3.5}$$

then

$$\frac{1}{2^{p_+-1}}|\nabla u|^{p(x)}-|\Theta(u)|^{p(x)}\leq|\nabla u-\Theta(u)|^{p(x)}.$$

Consequently

$$\begin{aligned}\langle Au,u\rangle \;&\geq \int_\Omega \frac{1}{p(x)}\left[\frac{1}{2^{p_+-1}}|\nabla u|^{p(x)} - |\Theta(u)|^{p(x)}\right]dx - \int_\Omega \frac{1}{p(x)}|\Theta(u)|^{p(x)}dx \\ &\quad + \int_\Omega |u|^{p(x)}dx \\ &\geq \int_\Omega \frac{1}{p(x)}\frac{1}{2^{p_+-1}}|\nabla u|^{p(x)}dx - \int_\Omega \frac{2}{p(x)}|\Theta(u)|^{p(x)}dx + \int_\Omega |u|^{p(x)}dx \\ &\geq \int_\Omega \frac{1}{p(x)}\frac{1}{2^{p_+-1}}|\nabla u|^{p(x)}dx - \int_\Omega \frac{2}{p(x)}C^{p(x)}|u|^{p(x)}dx + \int_\Omega |u|^{p(x)}dx \\ &\geq \int_\Omega \frac{1}{p(x)}\frac{1}{2^{p_+-1}}|\nabla u|^{p(x)}dx + \int_\Omega \left(1-\frac{2}{p(x)}C^{p(x)}\right)|u|^{p(x)}dx \\ &\geq \int_\Omega \frac{1}{p_+}\frac{1}{2^{p_+-1}}|\nabla u|^{p(x)}dx + \int_\Omega \left(1-\frac{2}{p_-}C^{p(x)}\right)|u|^{p(x)}dx.\end{aligned} \tag{3.6}$$

So the choice of the constant C in (H3) gives the existence of a positive constant C_0 such that

$$\begin{aligned}\langle Au,u\rangle \;&\geq \frac{1}{p_+}\frac{1}{2^{p_+-1}}\int_\Omega |\nabla u|^{p(x)}dx + C_0\int_\Omega |u|^{p(x)} \\ &\geq \min\{\frac{1}{p_+}\frac{1}{2^{p_+-1}}, C_0\}(\int_\Omega |\nabla u|^{p(x)}dx + \int_\Omega |u|^{p(x)}) \\ &\geq \min\{\frac{1}{p_+}\frac{1}{2^{p_+-1}}, C_0\}\rho_{1,p(x)}(u).\end{aligned} \tag{3.7}$$

Then $\dfrac{\langle Au,u\rangle}{\|u\|_{1,p(x)}} \to +\infty$ as $\|u\|_{1,p(x)} \to +\infty$.

Consequently we deduce that the operator A is coercive.

Besides, the operator A_n is bounded and hemi-continuous, thus for $F_n = (T_n(f), T_n(g)) \in E' \subset X_0'$, we can deduce the existence of a function $u_n \in X_0$ such that

$$\langle A_n u_n, v\rangle = \langle F_n, v\rangle \text{ for all } v \in X_0$$

i.e.

$$\begin{aligned}&\int_\Omega \Phi(\nabla u_n - \Theta(u_n))\nabla v\, dx + \int_\Omega |u_n|^{p(x)-2}u_n v\, dx + \int_\Omega T_n(\alpha(u_n))v\, dx \\ &+ \int_{\partial\Omega} T_n(\gamma(u_n))v\, d\sigma \\ =\; &\int_\Omega T_n(f)v\, dx + \int_{\partial\Omega} T_n(g)v\, d\sigma.\end{aligned} \tag{3.8}$$

Step2. A priori Estimates

Assertion 1. $(\nabla T_k(u_n))_{n\in\mathbb{N}}$ is bounded in $(L^{p_-}(\Omega))^N$.

Let $f_n = T_n(f)$ and $g_n = T_n(g)$ for all $n \in \mathbb{N}$, then $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ are sequences of bounded functions which converges strongly to $f \in L^1(\Omega)$ and to $g \in L^1(\partial\Omega)$ respectively. Moreover

$$\|f_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)} \text{ and } \|g_n\|_{L^1(\partial\Omega)} \leq \|g\|_{L^1(\partial\Omega)} \text{ for all } n \in \mathbb{N}.$$

Proof. We take $v = T_k(u_n)$ as test function in (3.8) to get

$$\begin{aligned} &\int_\Omega \Phi(\nabla u_n - \Theta(u_n))\nabla T_k(u_n)\,dx + \int_\Omega |u_n|^{p(x)-2}u_n T_k(u_n)dx \\ &+ \int_\Omega T_n(\alpha(u_n))T_k(u_n)\,dx + \int_{\partial\Omega} T_n(\gamma(u_n))T_k(u_n)\,d\sigma \\ = &\int_\Omega T_n(f)T_k(u_n)dx + \int_{\partial\Omega} T_n(g)T_k(u_n)d\sigma. \end{aligned} \tag{3.9}$$

The third and fourth terms in the left-hand side of equality above are nonnegative then

$$\begin{aligned} &\int_\Omega \Phi(\nabla u_n - \Theta(u_n))\nabla T_k(u_n)\,dx + \int_\Omega |u_n|^{p(x)-2}u_n T_k(u_n)dx \\ \leq\ & k(\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}). \end{aligned} \tag{3.10}$$

As

$$\begin{aligned} \int_\Omega |u_n|^{p(x)-2}u_n T_k(u_n)\,dx &\geq \int_{\{|u_n|\leq k\}} |T_k(u_n)|^{p(x)}\,dx + \int_{\{|u_n|>k\}} k^{p(x)}\,dx \\ &\geq \int_\Omega |T_k(u_n)|^{p(x)}\,dx \end{aligned}$$

then we deduce from (3.10) that

$$\begin{aligned} &\int_\Omega \Phi(\nabla T_k(u_n) - \Theta(u_n))\nabla T_k(u_n)\,dx + \int_\Omega |T_n(u_n)|^{p(x)}dx \\ \leq\ & k(\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}). \end{aligned} \tag{3.11}$$

By the same way as in the proof of the coerciveness of A_n, we get

$$\begin{aligned} \rho_{1,p(x)}(T_k(u_n)) &\leq kC_1(\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}) \\ &\leq kC_2, \end{aligned} \tag{3.12}$$

where $C_2 = \text{const}(f, g, p_-, p_+)$. Therefore,

$$\|T_k(u_n)\|_{1,p(x)} \leq 1 + (kC_2)^{\frac{1}{p_-}}\ . \tag{3.13}$$

We deduce that for any $k > 0$, the sequence $(T_k(u_n))_{n\in\mathbb{N}}$ is uniformly bounded in $W^{1,p(x)}(\Omega)$ and so in $W^{1,p_-}(\Omega)$. Then, up to a subsequence we can assume that for any $k > 0$,

$$T_k(u_n) \rightharpoonup v_k \qquad \text{in} \quad W^{1,p_-}(\Omega)$$

and by the compact imbedding, we have

$$T_k(u_n) \to v_k \qquad \text{in} \quad L^{p_-}(\Omega) \qquad \text{and a.e. in } \Omega.$$

□

Assertion 2. $(u_n)_{n\in\mathbb{N}}$ converges in measure to some function u.

To prove this, we show that u_n is a Cauchy sequence in measure.

Let $k > 0$ be large enough. Using $T_k(u_n)$ as a test function in (3.8), we get

$$\rho_{1,p(.)}(T_k(u_n)) \leq k(\|f\|_{L^1(\Omega)}) + \|g\|_{L^1(\partial\Omega)})$$

which yields,

$$\int_{\{|u_n|>k\}} k^{p(x)} dx \leq k(\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}).$$

It follows that

$$\text{meas}\{|u_n| > k\} \leq k^{1-p_-}(\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}).$$

Therefore

$$\text{meas}\{|u_n| > k\} \to 0 \text{ as } k \to +\infty \text{ since } 1 - p_- < 0. \tag{3.14}$$

Moreover, for every fixed $t > 0$ and every positive $k > 0$, we know that

$$\{|u_n - u_m| > t\} \subset \{|u_n| > k\} \cup \{|u_m| > k\} \cup \{|T_k(u_n) - T_k(u_m)| > t\}$$

and hence

$$\begin{aligned} \text{meas } (\{|u_n - u_m| > t\}) \leq \quad & \text{meas } (\{|u_n| > k\}) + \text{meas } (\{|u_m| > k\}) \\ & + \text{meas } (\{|T_k(u_n) - T_k(u_m)| > t\}). \end{aligned} \tag{3.15}$$

Let $\epsilon > 0$. Using (3.14), we choose $k = k(\epsilon)$ such that

$$\text{meas}(\{|u_n| > k\}) \leq \frac{\epsilon}{3} \qquad \text{and} \qquad \text{meas}(\{|u_m| > k\}) \leq \frac{\epsilon}{3}. \tag{3.16}$$

Since $T_k(u_n)$ converges strongly in $L^{p_-}(\Omega)$, then it is a Cauchy sequence in $L^{p_-}(\Omega)$. Thus

$$\text{meas } (\{|T_k(u_n) - T_k(u_m)| > t\}) \leq \frac{1}{t^{p_-}} \int_\Omega |T_k(u_n) - T_k(u_m)|^{p_-} dx \leq \frac{\epsilon}{3}, \tag{3.17}$$

for all $n, m \geq n_0(t, \epsilon)$.

Finally, from (3.15), (3.16) and (3.17) we obtain

$$\text{meas}(\{|u_n - u_m| > t\}) \leq \epsilon \text{ for all } n, m \geq n_0(t, \epsilon) \tag{3.18}$$

which proves that the sequence $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in measure and then converges almost everywhere to some measurable function u. Therefore,

$$\begin{aligned} & T_k(u_n) \rightharpoonup T_k(u) \quad \text{in } W^{1,p_-}(\Omega), \\ & T_k(u_n) \to T_k(u) \quad \text{in} \quad L^{p_-}(\Omega) \text{ and a.e. in } \Omega. \end{aligned} \tag{3.19}$$

Assertion 3. $(\nabla u_n)_{n\in\mathbb{N}}$ converges in measure to the weak gradient of u.

Indeed, let ϵ, t, k, ν are positive real numbers (it is assumed that $\nu < 1$) and let $n \in \mathbb{N}$. We have

$$\begin{aligned} \{|\nabla u_n - \nabla u| > t\} \subset \quad & \Big\{\{|u_n| > k\} \cup \{|u| > k\} \cup \{|\nabla T_k(u_n)| > k\} \\ & \cup\{|\nabla T_k(u)| > k\} \cup \{|u_n - u| > \nu\} \cup G\Big\}, \end{aligned}$$

where

$$G = \{|\nabla u_n - \nabla u| > t, |u_n| \le k, |u| \le k, |\nabla T_k(u_n)| \le k, |\nabla T_k(u)| \le k, |u_n - u| \le \nu\}.$$

The same method used for proof of Assertion 2 allows us to obtain for k sufficiently large,

$$\mathrm{meas}(\{|u_n| > k\} \cup \{|u| > k\} \cup \{|\nabla T_k(u_n)| > k\} \cup \{|\nabla T_k(u)| > k\}) \le \frac{\epsilon}{3}. \quad (3.20)$$

On the other hand, the application

$$\mathcal{A} : (s, \xi_1, \xi_2) \mapsto (\Phi(\xi_1 - \Theta(s)) - \Phi(\xi_2 - \Theta(s)))(\xi_1 - \xi_2)$$

is continuous and the set

$$\mathcal{K} := \{(s, \xi_1, \xi_2) \in I\!R \times I\!R^N \times I\!R^N, |s| \le k, |\xi_1| \le k, |\xi_2| \le k, |\xi_1 - \xi_2| > t\}$$

is compact and

$$(\Phi(\xi_1 - \Theta(s)) - \Phi(\xi_2 - \Theta(s)))(\xi_1 - \xi_2) > 0, \quad \forall \xi_1 \ne \xi_2.$$

Then, the application $\mathcal{A}$ has its minimum on $\mathcal{K}$, we denote it by β. Therefore, we have $\beta > 0$ and

$$\begin{aligned} \int_G \beta dx &\le \int_G [\Phi(\nabla u_n - \Theta(u_n)) - \Phi(\nabla u - \Theta(u_n))][\nabla u_n - \nabla u] dx \\ &\le \int_\Omega [\Phi(\nabla u_n - \Theta(u_n)) - \Phi(\nabla T_k(u) - \Theta(T_{k+\nu}(u_n)))] \\ &\quad \nabla T_\nu(T_{k+\nu}(u_n) - T_k(u)) dx. \end{aligned}$$

We take $v = T_\nu(T_{k+\nu}(u_n) - T_k(u))$ in (3.8) to obtain

$$\begin{aligned} &\int_\Omega \Phi(\nabla u_n - \Theta(u_n)) \nabla T_\nu(T_{k+\nu}(u_n) - T_k(u)) dx \\ &+ \int_\Omega |u_n|^{p(x)-2} u_n T_\nu(T_{k+\nu}(u_n) - T_k(u)) dx \\ = \; &- \int_\Omega T_n(\alpha(u_n)) T_\nu(T_{k+\nu}(u_n) - T_k(u)) dx \\ &- \int_{\partial\Omega} T_n(\gamma(u_n)) T_\nu(T_{k+\nu}(u_n) - T_k(u)) d\sigma \\ &+ \int_\Omega T_n(f) T_\nu(T_{k+\nu}(u_n) - T_k(u)) dx + \int_{\partial\Omega} T_n(g) T_\nu(T_{k+\nu}(u_n) - T_k(u)) d\sigma \\ \le \; &\nu(\|T_n(\alpha(u_n))\|_{L^1(\Omega)} + \|T_n(\gamma(u_n))\|_{L^1(\partial\Omega)} + \|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}). \end{aligned}$$

Then,

$$\begin{aligned} &\int_\Omega \Phi(\nabla u_n - \Theta(u_n)) \nabla T_\nu(T_{k+\nu}(u_n) - T_k(u)) dx \\ \le \; &\nu(\|T_n(\alpha(u_n))\|_{L^1(\Omega)} + \|T_n(\gamma(u_n))\|_{L^1(\partial\Omega)} + \|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}) \\ &- \int_\Omega |u_n|^{p(x)-2} u_n T_\nu(T_{k+\nu}(u_n) - T_k(u)). \end{aligned}$$

Taking now $v = \frac{1}{k}T_k(u_n)$ in (3.8), we get

$$\int_\Omega T_n(\alpha(u_n))\frac{1}{k}T_k(u_n)dx + \int_{\partial\Omega} T_n(\gamma(u_n))\frac{1}{k}T_k(u_n)d\sigma \leq \|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}. \tag{3.21}$$

Since

$$\lim_{k\to 0}\frac{1}{k}T_k(u_n) = \mathrm{sign}_0(u_n)$$

and

$$\mathrm{sign}_0(u_n) = \mathrm{sign}_0(T_n(\alpha(u_n))) = \mathrm{sign}_0(T_n(\gamma(u_n))),$$

hence, by passing to the limit as $k \to 0$, we have

$$\|T_n(\alpha(u_n))\|_{L^1(\Omega)} + \|T_n(\gamma(u_n))\|_{L^1(\partial\Omega)} \leq \|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}.$$

Therefore,

$$\begin{aligned} & \left|\int_\Omega(\Phi(\nabla u_n - \Theta(u_n))\nabla T_\nu(T_{k+\nu}(u_n) - T_k(u))dx\right| \\ \leq\ & \nu C_3 + \int_\Omega |u_n|^{p(x)-1}\left|T_\nu(T_{k+\nu}(u_n) - T_k(u))\right| dx. \end{aligned} \tag{3.22}$$

So, since

$$|u_n|^{p(x)-1} \to |u|^{p(x)-1} \text{ in } L^{p'(x)}(\Omega)$$

and

$$T_{k+\nu}(u_n) \to T_{k+\nu}(u) \text{ in } L^{p(x)}(\Omega),$$

hence, by using (H_3), we obtain

$$\Theta(T_{k+\nu}(u_n)) \to \Theta(T_{k+\nu}(u)) \text{ in } L^{p(x)}(\Omega). \tag{3.23}$$

Thus,

$$\Phi(\nabla T_k(u) - \Theta(T_{k+\nu}(u_n))) \to \Phi(\nabla T_k(u) - \Theta(T_{k+\nu}(u)) \text{ in } L^{p'(x)}(\Omega) \tag{3.24}$$

and since

$$\nabla T_\nu(T_{k+\nu}(u_n) - T_k(u)) \rightharpoonup \nabla T_\nu(T_{k+\nu}(u) - T_k(u)) \text{ in } L^{p(x)}(\Omega), \tag{3.25}$$

we deduce that

$$\begin{aligned} & \lim_{n\to\infty}\int_\Omega \Phi(\nabla T_k(u) - \Theta(T_{k+\nu}(u_n)))\nabla T_\nu(T_{k+\nu}(u_n) - T_k(u))dx \\ =\ & \int_\Omega \Phi(\nabla T_k(u) - \Theta(T_{k+\nu}(u)))\nabla T_\nu(T_{k+\nu}(u) - T_k(u))dx. \end{aligned} \tag{3.26}$$

Now, since

$$\lim_{\nu\to 0}\nabla T_\nu(T_{k+\nu}(u) - T_k(u)) = 0,$$

and as $\nu < 1$, thanks to (H_3), we get

$$\begin{aligned} & \Phi(\nabla T_k(u) - \Theta(T_{k+\nu}(u)))\nabla T_\nu(T_{k+\nu}(u) - T_k(u)) \\ \leq\ & C_4(|T_{k+1}(u)|^{p(x)-1} + |\nabla T_k(u)|^{p(x)-1})|\nabla T_1(T_{k+1}(u) - T_k(u))|. \end{aligned}$$

Although,

$$(|T_{k+1}(u)|^{p(x)-1} + |\nabla T_k(u)|^{p(x)-1})|\nabla T_1(T_{k+1}(u) - T_k(u))| \in L^1(\Omega),$$

thanks to the Lebesgue dominated convergence theorem, we obtain

$$\lim_{\nu\to 0}\int_\Omega \Phi(\nabla T_k(u) - \Theta(T_{k+\nu}(u)))\nabla T_\nu(T_{k+\nu}(u) - T_k(u))dx = 0.$$

Next, we take $\delta > 0$ such that $\nu = \frac{\delta}{4C}$. Then, there exists $n_1 > 0$ such that

$$\int_\Omega \Phi(\nabla T_k(u) - \Theta(T_{k+\nu}(u_n)))\nabla T_\nu(T_{k+\nu}(u_n) - T_k(u))dx \le \frac{\delta}{2}, \qquad \forall n > n_1 \quad (3.27)$$

and since

$$\int_\Omega |u_n|^{p(x)-1}\,|T_\nu(T_{k+\nu}(u_n) - T_k(u))|\,dx \to 0 \text{ as } \nu \to 0,$$

then

$$\nu C_3 + \int_\Omega |u_n|^{p(x)-1}\,|T_\nu(T_{k+\nu}(u_n) - T_k(u))|\,dx \le \frac{\delta}{2}. \qquad (3.28)$$

Consequently, the two inequalities (3.22) and (3.28) implies that

$$\int_G \beta \le \delta.$$

From lemma 2.3, it follows that

$$\text{meas}(G) \le \frac{\epsilon}{3}. \qquad (3.29)$$

Finally, the assertion 2 gives the existence of $n_2 \in I\!N$, such that

$$\text{meas}(\{|u_n - u| > \nu\}) \le \frac{\epsilon}{3}, \forall n \ge n_2. \qquad (3.30)$$

Consequently, the previous results provide us the existence of $n_0 = \max(n_1, n_2)$, such that

$$\text{meas}(\{|\nabla u_n - \nabla u| > t\}) \le \epsilon, \forall n \ge n_0.$$

Hence, ∇u_n converges in measure to ∇u.

Assertion 4. $(u_n)_{n\in I\!N}$ converges a.e. on $\partial\Omega$ to some function v.

We know that the trace operator is compact from $W^{1,1}(\Omega)$ into $L^1(\partial\Omega)$, then there exists a constant $C_5 > 0$ such that

$$\|T_k(u_n) - T_k(u)\|_{L^1(\partial\Omega)} \le C_5\|T_k(u_n) - T_k(u)\|_{W^{1,1}(\Omega)}.$$

Therefore,

$$T_k(u_n) \to T_k(u) \text{ in } L^1(\partial\Omega) \text{ and a.e. in } \partial\Omega.$$

Therefore, there exists $A \subset \partial\Omega$ such that $T_k(u_n)$ converges to $T_k(u)$ on $\partial\Omega\backslash A$ with $\mu(A) = 0$, where μ is the area measure on $\partial\Omega$.

For every $k > 0$, let $A_k = \{x \in \partial\Omega : |T_k(u(x))| < k\}$ and $B = \partial\Omega \setminus \bigcup_{k>0} A_k$.

We have

$$\mu(B) = \frac{1}{k}\int_B |T_k(u)|d\sigma \quad \begin{aligned} &\leq \frac{C_4}{k}\|T_k(u)\|_{W^{1,1}(\Omega)} \\ &\leq \frac{C_6}{k}\|T_k(u)\|_{1,p(x)}. \end{aligned} \tag{3.31}$$

We know that $\rho_{1,p(.)}(T_k(u_n)) \leq kM$ where M is a positive constant which doesn't depend on n. Then,

$$\int_\Omega |T_k(u_n)|^{p(x)}\,dx + \int_\Omega |\nabla T_k(u_n)|^{p(x)}\,dx \leq kM. \tag{3.32}$$

We now use the Fatou's lemma in (3.32) to get

$$\int_\Omega |T_k(u)|^{p(x)}\,dx + \int_\Omega |\nabla T_k(u)|^{p(x)}\,dx \leq kM,$$

which is equivalent to

$$\rho_{1,p(.)}(T_k(u)) \leq kM. \tag{3.33}$$

According to (3.33), we deduce that

$$\|T_k(u)\|_{W^{1,p(x)}(\Omega)} \leq C_7\left(k^{\frac{1}{p_-}} + k^{\frac{1}{p_+}}\right).$$

Therefore, we get by letting $k \to +\infty$ in (3.31) that $\mu(B) = 0$.

Let us now define in $\partial\Omega$ the function v by

$$v(x) = T_k(u(x)) \text{ if } x \in A_k.$$

We take $x \in \partial\Omega\setminus(A \cup B)$; then there exists $k > 0$ such that $x \in A_k$ and we have

$$u_n(x) - v(x) = (u_n(x) - T_k(u_n(x))) + (T_k(u_n(x)) - T_k(u(x))).$$

Since $x \in A_k$, we have $|T_k(u(x))| < k$ and so $|T_k(u_n(x))| < k$, from which we deduce that $|u_n(x)| < k$.

Therefore,

$$u_n(x) - v(x) = (T_k(u_n(x)) - T_k(u(x))) \to 0, \text{ as } n \to +\infty.$$

This means that u_n converges to v a.e. on $\partial\Omega$.

Assertion 5. u is an entropy solution of the problem (1.1).

Since the sequence $(\nabla T_k(u_n))_{n\in\mathbb{N}}$ converges in measure to $\nabla T_k(u)$, then by (3.13) and Lemma 2.2, we get

$$\nabla T_k(u_n) \to \nabla T_k(u) \quad \text{in } \left(L^1(\Omega)\right)^N. \tag{3.34}$$

Consequently, assertions 2, 4 and (3.34) give $u \in \mathcal{T}_{tr}^{1,p(x)}(\Omega)$.

Let $\varphi \in W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$, we take $v = T_k(u_n - \varphi)$ as test function in (3.8) to get

$$\begin{aligned} &\int_\Omega \Phi(\nabla u_n - \Theta(u_n))\nabla T_k(u_n - \varphi)\,dx + \int_\Omega |u_n|^{p(x)-2}u_n T_k(u_n - \varphi)dx \\ &+ \int_\Omega T_n(\alpha(u_n))T_k(u_n - \varphi)\,dx + \int_{\partial\Omega} T_n(\gamma(u_n))T_k(u_n - \varphi)\,d\sigma \\ = &\int_\Omega T_n(f)T_k(u_n - \varphi)\,dx + \int_{\partial\Omega} T_n(g)T_k(u_n - \varphi)\,d\sigma. \end{aligned} \tag{3.35}$$

Let $\overline{k} = k + \|\varphi\|_\infty$, we have

$$\begin{aligned}
&\int_\Omega \Phi(\nabla u_n - \Theta(u_n))\nabla T_k(u_n - \varphi)\,dx \\
= \quad &\int_\Omega \Phi(\nabla T_{\overline{k}}(u_n) - \Theta(T_{\overline{k}}(u_n)))\nabla T_k(T_{\overline{k}}(u_n) - \varphi)\,dx \\
= \quad &\int_\Omega \Phi(\nabla T_{\overline{k}}(u_n) - \Theta(u_n))\nabla T_{\overline{k}}(u_n)\chi_{\Omega(n,\overline{k})}\,dx \\
&- \int_\Omega \Phi(\nabla T_{\overline{k}}(u_n) - \Theta(u_n))\nabla\varphi\chi_{\Omega(n,\overline{k})}\,dx,
\end{aligned}$$

where $\Omega(n,\overline{k}) = \{|T_{\overline{k}}(u_n) - \varphi| \le k\}$ and χ_B the characteristic function of a measurable set $B \in I\!R^d$.

The inequality (3.35) can be written as

$$\begin{aligned}
&\int_\Omega \left(\Phi(\nabla T_{\overline{k}}(u_n) - \Theta(T_{\overline{k}}(u_n)))\nabla T_{\overline{k}}(u_n) + \frac{1}{p(x)}|\Theta(T_{\overline{k}}(u_n))|^{p(x)}\right)\chi_{\Omega(n,\overline{k})}\,dx \\
&- \int_\Omega \Phi(\nabla T_{\overline{k}}(u_n) - \Theta(T_{\overline{k}}(u_n)))\nabla\varphi\chi_{\Omega(n,\overline{k})}\,dx + \int_\Omega |u_n|^{p(x)-2}u_n T_k(u_n-\varphi)dx \\
&+ \int_\Omega T_n(\alpha(u_n))T_k(u_n-\varphi)\,dx + \int_{\partial\Omega} T_n(\gamma(u_n))T_k(u_n-\varphi)\,d\sigma \\
= \quad &\int_\Omega T_n(f)T_k(u_n-\varphi)\,dx + \int_{\partial\Omega} T_n(g)T_k(u_n-\varphi)\,d\sigma \\
&+ \int_\Omega \frac{1}{p(x)}|\Theta(T_{\overline{k}}(u_n))|^{p(x)}\chi_{\Omega(n,\overline{k})}\,dx.
\end{aligned} \tag{3.36}$$

Since

$$\nabla T_{\overline{k}}(u_n) \rightharpoonup \nabla T_{\overline{k}}(u) \quad \text{in} \quad L^{p(x)}(\Omega)$$

and

$$\Theta(T_{\overline{k}}(u_n)) \longrightarrow \Theta(T_{\overline{k}}(u)) \quad \text{in} \quad L^{p(x)}(\Omega), \tag{3.37}$$

then

$$\nabla T_{\overline{k}}(u_n) - \Theta(T_{\overline{k}}(u_n)) \rightharpoonup \nabla T_{\overline{k}}(u) - \Theta(T_{\overline{k}}(u)) \text{ in } L^{p(x)}(\Omega).$$

Thus,

$$\Phi(\nabla T_{\overline{k}}(u_n) - \Theta(T_{\overline{k}}(u_n))) \rightharpoonup \Phi(\nabla T_{\overline{k}}(u) - \Theta(T_{\overline{k}}(u))) \quad \text{in} \quad L^{p'(x)}(\Omega). \tag{3.38}$$

Furthermore,

$$\nabla\varphi\chi_{\Omega(n,\overline{k})} \longrightarrow \nabla\varphi\chi_{\Omega(\overline{k})} \quad \text{in} \quad L^{p(x)}(\Omega)$$

with $\Omega(\overline{k}) = \{|T_{\overline{k}}(u) - \varphi| \le k\}$. Then

$$\begin{aligned}
&\int_\Omega \Phi(\nabla T_{\overline{k}}(u_n) - \Theta(T_{\overline{k}}(u_n)))\nabla\varphi\chi_{\Omega(n,\overline{k})}\,dx \\
\longrightarrow \quad &\int_\Omega \Phi(\nabla T_{\overline{k}}(u) - \Theta(T_{\overline{k}}(u)))\nabla\varphi\chi_{\Omega(\overline{k})}\,dx.
\end{aligned} \tag{3.39}$$

According to (H_3) and the properties of the truncation function, we get

$$|\Theta(T_{\overline{k}}(u_n))|^{p(x)} \le (C\overline{k})^{p(x)}.$$

Using (3.37) and the Lebesgue dominated convergence theorem, we obtain

$$\int_\Omega \frac{1}{p(x)}|\Theta(T_{\overline{k}}(u_n))|^{p(x)}\chi_{\Omega(n,\overline{k})}\,dx \longrightarrow \int_\Omega \frac{1}{p(x)}|\Theta(T_{\overline{k}}(u))|^{p(x)}\chi_{\Omega(\overline{k})}\,dx.$$

Now, since

$$\left(\Phi(\nabla T_{\overline{k}}(u_n)-\Theta(T_{\overline{k}}(u_n)))\nabla T_{\overline{k}}(u_n)+\frac{1}{p(x)}|\Theta(T_{\overline{k}}(u_n))|^{p(x)}\right)\chi_{\Omega(n,\overline{k})}\geq 0 \text{ a.e. in } \Omega,$$

we obtain by using Fatou's lemma

$$\begin{aligned} &\int_\Omega \left(\Phi(\nabla T_{\overline{k}}(u)-\Theta(T_{\overline{k}}(u)))\nabla T_{\overline{k}}(u)+\frac{1}{p(x)}|\Theta(T_{\overline{k}}(u))|^{p(x)}\right)\chi_{\Omega(\overline{k})}\,dx \\ \leq\ & \liminf_{n\to\infty}\Bigg(\int_\Omega \Big(\Phi(\nabla T_{\overline{k}}(u_n)-\Theta(T_{\overline{k}}(u_n)))\nabla T_{\overline{k}}(u_n) \\ &+\frac{1}{p(x)}|\Theta(T_{\overline{k}}(u_n))|^{p(x)}\Big)\chi_{\Omega(n,\overline{k})}\,dx\Bigg). \end{aligned}$$

We then pass to the limit as $n \to \infty$ in the equality (3.36) to concludes that u satisfy relation (3.1).

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