Journal of Applied Analysis and Computation
Volume 3, Number 1, February 2013
Website:http://jaac-online.com/
pp. 95–104

A NUMERICAL SOLUTION OF NONLINEAR VOLTERRA-FREDHOLM INTEGRAL EQUATIONS

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Abstract
In this paper, a numerical procedure for solving a class of nonlinear Volterra-Fredholm integral equations is presented. The method is based upon the globally defined sinc basis functions. Properties of the sinc procedure are utilized to reduce the computation of the nonlinear integral equations to some algebraic equations. Illustrative examples are included to demonstrate the validity and applicability of the method.

Keywords Nonlinear, Volterra-Fredholm, integral equations, sinc function.

MSC(2000) 41A30, 45G10.

1. Introduction

Consider the nonlinear Volterra-Fredholm integral equations of the form

$$y(x) = g(x) + (KQy)(x), \qquad x \in \Gamma = [a, b], \tag{1.1}$$

where

$$(KQy)(x) = \lambda_1 \int_a^x F_1(x,t) q_1(t, u(t)) dt + \lambda_2 \int_a^b F_2(x,t) q_2(t, u(t)) dt.$$

In this equations the functions g, q_1, q_2, and the kernels F_1, F_2 given and u is the unknown function to be determined. The existence and the uniqueness are discussed and given in Refs. [2] and [9].

The nonlinear Volterra-Fredholm integral equation (1.1) arises from various physical and biological models. The essential features of these models are of wide applicable [1], [3], [4] and [11].

Several numerical methods for approximating the solution of nonlinear Volterra-Fredholm integral equations are known. The numerical solutions of the nonlinear Volterra-Fredholm integral equations by using homotopy perturbation method was introduced in [5]. Minggen et al. [7], used the representation of the exact solution for the nonlinear Volterra-Fredholm integral equations in the reproducing kernel space. The exact solution is given by the form of series. Its approximate solution is obtained by truncating the series and a new numerical approximate method. Ordokhani [8], applied the rationalized Haar functions to approximate of the nonlinear Volterra-Fredholm-Hammerstein integral equations. Also, in [12], Yalçinbaş developed the Taylor polynomial solutions for the nonlinear Volterra-Fredholm integral equations.

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The organization of the paper is as follows. In Section 2, we describe the basic formulation of sinc approximation required for our subsequent development. Section 3 is devoted to the solution of Eq. (1.1) by using sinc method. Finally numerical examples are given in section 4 to illustrate the efficiency of the presented method.

2. Prroperties of sinc function

The sinc function properties and the sinc method are discussed thoroughly in [6], [10]. For any $h > 0$, the sinc basis functions are given by

$$S(j,h)(z) = \text{sinc}(\frac{z-jh}{h}), \quad j = 0, \pm 1, \pm 2, \cdots , \tag{2.1}$$

where

$$\text{sinc}(z) = \begin{cases} \frac{\sin(\pi z)}{\pi z}, & z \neq 0; \\ 1, & z = 0. \end{cases} \tag{2.2}$$

The sinc function for the interpolating points $z_k = kh$ is given by

$$S(j,h)(kh) = \delta_{jk}^{(0)} = \begin{cases} 1, & \text{k=j}; \\ 0, & k \neq j. \end{cases} \tag{2.3}$$

Let

$$\delta_{kj}^{(-1)} = \frac{1}{2} + \int_0^{k-j} \frac{\sin(\pi t)}{\pi t} dt, \tag{2.4}$$

then define a matrix $I^{(-1)} = [\delta_{kj}^{(-1)}]$ whose $(k,j)th$ entry is given by $\delta_{kj}^{(-1)}$. They are based in the infinite strip D_d in the complex plane

$$D_d = \{w = u + iv : |v| < d \leq \frac{\pi}{2}\}. \tag{2.5}$$

To construct approximation on the interval (a,b), we consider the conformal map

$$\phi(z) = \ln\left(\frac{z-a}{b-z}\right). \tag{2.6}$$

The map ϕ carries the eye-shaped region

$$D = \left\{ z = x + iy : \left| arg(\frac{z-a}{b-z}) \right| < d \leq \frac{\pi}{2} \right\}. \tag{2.7}$$

The function

$$z = \phi^{-1}(w) = \frac{a + be^w}{1 + e^w} \tag{2.8}$$

is an inverse mapping of $w = \phi(z)$. We define the range of ϕ^{-1} on the real line as

$$\Gamma = \{\psi(u) = \phi^{-1}(u) \in D : -\infty < u < \infty\}. \tag{2.9}$$

The sinc grid points $z_k \in (a,b)$ in D will be denoted by x_k because they are real. For the evenly spaced nodes $\{kh\}_{k=-\infty}^{\infty}$ on the real line, the image which corresponds to these nodes is denoted by

$$x_k = \phi^{-1}(kh) = \frac{a + be^{kh}}{1 + e^{kh}} , \quad k = \pm 1, \pm 2, \ldots . \tag{2.10}$$

Definition 2.1. A function $y(z)$ is in the space $L_\alpha(D)$ if and only if y(z) is analytic in D and there exists a constant, $C > 0$, such that

$$|y(z)| \leq C \frac{|\rho(z)|^\alpha}{[1 + |\rho(z)|]^{2\alpha}}, \quad z \in D, \quad 0 < \alpha \leq 1. \tag{2.11}$$

Theorem 2.1. *Let $y(z) \in L_\alpha(D)$, $0 < \alpha \leq 1$ and $d > 0$, let N be a positive integer, and let h be selected by the formula*

$$h = (\frac{\pi d}{\alpha N})^{\frac{1}{2}}, \tag{2.12}$$

then there exists positive constant c_1, independent of N, such that

$$\sup_{z\in\Gamma} \left| y(z) - \sum_{j=-N}^{N} y(z_j) S(j,h) \circ \phi(z) \right| \leq c_1 e^{-(\pi d \alpha N)^{\frac{1}{2}}}. \tag{2.13}$$

Theorem 2.2. *Let $\frac{y}{\phi'} \in L_\alpha(D)$, let $\delta_{kj}^{(-1)}$ be defined as in* (2.4), *and let $h = (\frac{\pi d}{\alpha N})^{\frac{1}{2}}$. Then there exists a constant c_2, which is independent of N, such that*

$$\left| \int_a^{z_k} y(t)dt - h \sum_{j=-N}^{N} \delta_{kj}^{(-1)} \frac{y(z_j)}{\phi'(z_j)} \right| \leq c_2 e^{-(\pi d \alpha N)^{\frac{1}{2}}}. \tag{2.14}$$

Theorem 2.3. *Let $\frac{y}{\phi'} \in L_\alpha(D)$, let N be a positive integer and let h be selected by* (2.12), *then there exist positive constant c_3, independent of N, such that*

$$\left| \int_\Gamma y(z)dz - h \sum_{j=-N}^{N} \frac{y(z_j)}{\phi'(z_j)} \right| \leq c_3 e^{-(\pi d \alpha N)^{\frac{1}{2}}}. \tag{2.15}$$

3. The approximate solution of nonlinear Volterra-Fredholm integral equations

Let $y(x)$ be the exact solution of the integral equation (1.1) and let $y(x) \in L_\alpha(D)$. We approximate the solution of (1.1) by the following linear combinations of the sinc functions and auxiliary functions:

$$y(x) = \sum_{j=-N}^{N} y(x_j)\alpha_j(x), \quad x \in [a,b], \tag{3.1}$$

where

$$\alpha_j(x) = \begin{cases} \beta_a(x), & j = -N, \\ S(j,h) \circ \phi(x), & j = -N+1, \ldots, N-1, \\ \beta_b(x), & j = N. \end{cases} \tag{3.2}$$

In the above relation, auxiliary basis functions $\beta_a(x)$ and $\beta_b(x)$ are defined by

$$\beta_a(x) = \frac{1}{1+\rho(x)}, \qquad \beta_b(x) = \frac{\rho(x)}{1+\rho(x)}, \tag{3.3}$$

and satisfied the following conditions:

$$\lim_{x\to a}\beta_a(x)=1,\quad \lim_{x\to b}\beta_a(x)=0,$$
$$\lim_{x\to a}\beta_b(x)=0,\quad \lim_{x\to b}\beta_b(x)=1, \tag{3.4}$$

where $\rho(x)=e^{\phi(x)}$.

Lemma 3.1. *Let $y(x)\in L_\alpha(D)$, let N be a positive integer, and let h be selected by the formula*

$$h=(\frac{\pi d}{\alpha N})^{\frac{1}{2}}, \tag{3.5}$$

then there exists positive constant c_4, independent of N, such that

$$\sup_{x\in\Gamma}\left|y(x)-\sum_{j=-N}^{N}y(x_j)\alpha_j(x)\right|\le c_4e^{-(\pi d\alpha N)^{\frac{1}{2}}}, \tag{3.6}$$

where $\alpha_j(x)$ is defined in (3.2).

Proof. By using Theorem 2.1 and Eq. (3.1) it follows that

$$\sup_{x\in\Gamma}\left|y(x)-\sum_{j=-N}^{N}y(x_j)\alpha_j(x)\right|\le S_0+S_1+S_2, \tag{3.7}$$

where

$$S_0=\sup_{x\in\Gamma}\left|y(x)-\textstyle\sum_{j=-N+1}^{N-1}y(x_j)S(j,h)\circ\phi(x)\right|,$$
$$S_1=\sup_{x\in\Gamma}\left|y(x_{-N})\beta_a(x)\right|,\quad S_2=\sup_{x\in\Gamma}\left|y(x_N)\beta_b(x)\right|.$$

A bound for S_0 is obtained by assumption $y(x)\in L_\alpha(D)$ and Theorem 2.1 as follows

$$\begin{aligned} S_0 &\le \sup_{x\in\Gamma}\left|y(x)-\textstyle\sum_{j=-N}^{N}y(x_j)S(j,h)\circ\phi(x)\right| \\ &\quad+\sup_{x\in\Gamma}\left|y(x_{-N})S(-N,h)\circ\phi(x)\right|+\sup_{x\in\Gamma}\left|y(x_N)S(N,h)\circ\phi(x)\right| \\ &\le C_4e^{-(\pi d\alpha N)^{1/2}}+C_5e^{-\alpha Nh}+C_6e^{-\alpha Nh}. \end{aligned} \tag{3.8}$$

Similarly, by considering the Eqs. (2.11) and (3.1)-(3.4), S_1 and S_2 are also bounded

$$S_1\le C_7e^{-\alpha Nh},\quad S_2\le C_8e^{-\alpha Nh}. \tag{3.9}$$

Finally, the result (3.6) follows by using the relations (3.8), (3.9) and taking h as in (3.5). □

Lemma 3.2. *Let $y(x)$ is defined as* (3.1)*, let $\frac{y}{\phi'}\in L_\alpha(D)$, and let h be selected by* (3.5) *then*

$$\begin{aligned} \int_a^{x_k}y(t)dt = \quad & hy(t_{-N})\textstyle\sum_{j=-N}^{N}\delta_{kj}^{(-1)}\frac{w_a(t_j)}{\phi'(t_j)}+h\sum_{j=-N+1}^{N-1}\delta_{kj}^{(-1)}\frac{y(t_j)}{\phi'(t_j)} \\ & +hy(t_N)\textstyle\sum_{j=-N}^{N}\delta_{kj}^{(-1)}\frac{w_b(t_j)}{\phi'(t_j)}+\mathcal{O}\big(exp-(\pi d\alpha N)^{1/2}\big). \end{aligned} \tag{3.10}$$

Proof. By applying Theorem 2.3 and assumption $\frac{y}{\phi'} \in L_\alpha(D)$, we have

$$\int_a^{x_k} y(t)dt = h\sum_{l=-N}^{N} \delta_{kl}^{(-1)} \frac{y(t_l)}{\phi'(t_l)} + \mathcal{O}\big(exp{-}(\pi d\alpha N)^{1/2}\big). \tag{3.11}$$

Using the explicitly form of $y(t)$ from (3.1), the collocation result is written as

$$\begin{aligned}\int_a^{x_k} y(t)dt = \quad & hy(t_{-N})\textstyle\sum_{l=-N}^{N}\delta_{kl}^{(-1)}\frac{w_a(t_l)}{\phi'(t_l)}\\ &+h\textstyle\sum_{j=-N+1}^{N-1} y(t_j)\{\sum_{l=-N}^{N}\delta_{kl}^{(-1)}\frac{S(j,h)\circ\phi(t_l)}{\phi'(t_l)}\}\\ &+hy(t_N)\textstyle\sum_{l=-N}^{N}\delta_{kl}^{(-1)}\frac{w_b(t_l)}{\phi'(t_l)} + \mathcal{O}\big(exp{-}(\pi d\alpha N)^{1/2}\big).\end{aligned} \tag{3.12}$$

The (3.10) is given straightforwardly by using sinc function properties and setting $l = j$. □

Lemma 3.3. *Let* $y(x)$ *is defined as in* (3.1), *let* $\frac{y}{\phi'} \in L_\alpha(D)$, *and let* h *be selected by* (3.5) *then*

$$\begin{aligned}\int_\Gamma y(t)dt = \quad & hy(t_{-N})\textstyle\sum_{j=-N}^{N}\frac{w_a(t_j)}{\phi'(t_j)} + h\sum_{j=-N+1}^{N-1}\frac{y(t_j)}{\phi'(t_j)}\\ &+hy(t_N)\textstyle\sum_{j=-N}^{N}\frac{w_b(t_j)}{\phi'(t_j)} + \mathcal{O}\big(exp{-}(\pi d\alpha N)^{1/2}\big).\end{aligned} \tag{3.13}$$

Proof. The proof is similar to Lemma 3.2. □

Now, let $y(x)$ be the exact solution of (1.1) that is approximated by the following expansion

$$y_N(x) = \sum_{j=-N}^{N} y_j\alpha_j(x), \quad x \in [a,b], \tag{3.14}$$

where $\alpha_j(x)$ is defined as (3.2). By replacing approximate solution (3.14) in the Eq. (1.1), it follows that

$$y_N(x) = g(x) + \lambda_1\int_a^x F_1(x,t)q_1(t,y_N(t))dt + \lambda_2\int_a^b F_2(x,t)q_2(t,y_N(t))dt, \tag{3.15}$$

for convenience, we consider

$$q_1(t,y_N(t)) = Q_N^1(t), \quad q_2(t,y_N(t)) = Q_N^2(t). \tag{3.16}$$

Thus the term is written as

$$\sum_{j=-N}^{N} y_j\alpha_j(x) = g(x) + \lambda_1\int_a^x F_1(x,t)Q_N^1(t)dt + \lambda_2\int_a^b F_2(x,t)Q_N^2(t)dt. \tag{3.17}$$

Let $\frac{F_i}{\phi'}Q_N^i \in L_\alpha(D)$, $i = 1,2$. Having substituted sinc-collocation points $x = x_k$ for $k = -N,\ldots,N$ and having applied the Lemmas 3.1, 3.2 and 3.3, we obtain

$$\begin{aligned}& y_{-N}\beta_a(x_k) + \textstyle\sum_{j=-N+1}^{N-1} y_j S(j,h)\circ\phi(x_k) + y_N\beta_b(x_k)\\ =\quad & g(x_k) + \lambda_1 h\textstyle\sum_{l=-N}^{N}\delta_{kl}^{(-1)}\frac{F_1(x_k,t_l)}{\phi'(t_l)}Q_N^1(t_l)\\ & +\lambda_2 h\textstyle\sum_{l=-N}^{N}\frac{F_2(x_k,t_l)}{\phi'(t_l)}Q_N^2(t_l), \quad k=-N,\ldots,N,\end{aligned} \tag{3.18}$$

where

$$Q_N^i(t_l) = q_i\Big(t_l, y_{-N}\beta_a(x_k) + \sum_{j=-N+1}^{N-1} y_j S(j,h)\circ\phi(x_k) + y_N\beta_b(x_k)\Big), \quad i=1,2.$$

We then rewrite these equations in the matrix form which are the nonlinear system

$$AY - \lambda_1 h\big(I^{(-1)}\circ\tilde{F}_1\big)\tilde{Q}^1 - \lambda_2 h\tilde{F}_2\tilde{Q}^2 = \tilde{G}. \tag{3.19}$$

The notation " $\circ$ " denotes the Hadamard matrix multiplication. $I^{(-1)} = \big[\delta_{kj}^{(-1)}\big]$, $\tilde{F}_i = \big[F_i(x_k,t_j)\big]$, $i=1,2$, where $I^{(-1)}$ and $\tilde{F}_i$ are square matrices of order $(2N+1)\times(2N+1)$,

$$\begin{aligned}
A &= \begin{pmatrix} \beta_a(x_{-N}) & 0 & \cdots & 0 & \beta_b(x_{-N}) \\ \beta_a(x_{-N+1}) & 1 & \cdots & 0 & \beta_b(x_{-N+1}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \beta_a(x_{N-1}) & 0 & \cdots & 1 & \beta_b(x_{N-1}) \\ \beta_a(x_N) & 0 & \cdots & 0 & \beta_b(x_N) \end{pmatrix}, \\
Y &= [y_{-N}, y_{-N+1}, \ldots, y_N]^T, \\
\tilde{Q}^i &= \left[\frac{Q_N^i(t_{-N})}{\phi'(t_{-N})}, \frac{Q_N^i(t_{-N+1})}{\phi'(t_{-N+1})}, \ldots, \frac{Q_N^i(t_{N-1})}{\phi'(t_{N-1})}, \frac{Q_N^i(t_N)}{\phi'(t_N)}\right]^T, \quad i=1,2, \\
\tilde{G} &= [g(x_{-N}), g(x_{-N+1}), \ldots, g(x_N)]^T.
\end{aligned} \tag{3.20}$$

The above nonlinear system consists of $2N+1$ equations with $2N+1$ unknowns $\{y_j\}_{j=-N}^N$. Solving this nonlinear system by *Newton's method*, we can obtain an approximation to the solution of (1.1):

$$y_N(x) = \sum_{j=-N}^{N} y_j\alpha_j(x), \quad x\in[a,b], \tag{3.21}$$

where $\alpha_j(x)$ is defined as (3.2). Each Newton iteration step involves evaluation of the vector $\mathbf{F}^{(l)}$, the Jacobian matrix $\mathbf{J}^{(l)}$ and $\Delta Y^{(l)}$. Whenever the distance between two iteration is less than a given tolerance, ϵ, then the algorithm is to stop.

$$\|Y^{(l+1)} - Y^{(l)}\| \le \epsilon.$$

Algorithm of the method

- initialize: $Y = Y^{(0)}$.
- for $l = 0,1,2,\ldots$.
- $\mathbf{F}^{(l)} = AY\big|^{(l)} - \lambda_1 h\big(I^{(-1)}\circ\tilde{F}_1\big)\tilde{Q}^1\big|^{(l)} - \lambda_2 h\tilde{F}_2\tilde{Q}^2\big|^{(l)} - \tilde{G}$.
- if $\|\mathbf{F}^{(l)}\|$ is small enough, stop.
- compute $\mathbf{J}^{(l)}$.
- solve $\mathbf{J}^{(l)}\Delta Y^{(l)} = -\mathbf{F}(Y^{(l)})$.
- $Y^{(l+1)} = Y^{(l)} + \Delta Y^{(l)}$.
- end.

4. Numerical examples

In order to illustrate the performance of the sinc method in solving of the nonlinear Volterra-Fredholm integral equations and justify the accuracy and efficiency of the method, we consider the following examples. The examples have been solved by presented method with different values of N and α, $0 < \alpha \leq 1$. In all examples, we take $\alpha = \frac{1}{2}$, $d = \frac{\pi}{2}$, which yields $h = \pi(\frac{1}{N})^{\frac{1}{2}}$. The errors are reported on the set of sinc grid points

$$S = \{x_{-N}, \ldots, x_0, \ldots, x_N\},$$

$$x_k = \frac{a + be^{kh}}{1 + e^{kh}}, \quad k = -N, \ldots, N. \tag{4.1}$$

The maximum error on the sinc grid points is

$$\|E_s(h)\|_\infty = \max_{-N \leq j \leq N} |u(x_j) - u_N(x_j)|. \tag{4.2}$$

We stopped the numbers of iteration in the Newton method when we achieved the accuracy $\epsilon = 10^{-4}$. The numerical results are tabulated in Tables 4.1, 4.2 and shown in Figures 4.1 and 4.2.

Example 4.1. Consider the following nonlinear Volterra-Fredholm integral equation with the exact solution $y(x) = 1 - x$,

$$y(x) = g(x) + \int_0^x F_1(x,t)q_1(t, y(t))dt + \int_0^1 F_2(x,t)q_2(t, y(t))dt, \quad x \in \Gamma = [0,1], \tag{4.3}$$

where

$F_1(x,t) = \sin(x - t), \qquad F_2(x,t) = x - t,$

$q_1(t, y(t)) = \cos(y(t)), \qquad q_2(t, y(t)) = 1 + y^2(x),$

$g(x) = \frac{1}{12}(19 - 28x + 6\sin 1x \cos x - 6\cos 1x \sin x + 6\sin 1 \sin x).$

The Example 4.1 is solved for different values of N. The maximum of absolute errors on the sinc grid S are tabulated in Table 4.1. This table indicates that as N increases the errors are decrease more rapidly. The exact and approximate solutions of Example 4.1 are shown in Fig. 4.1 for $N = 1$ and $N = 5$.

N	h	$\|E_s(h)\|_\infty$
5	1.40496	2.42903×10^{-3}
10	0.99346	1.97337×10^{-4}
20	0.70248	5.97438×10^{-6}
30	0.57357	3.93851×10^{-7}
40	0.49673	3.92648×10^{-8}
50	0.44429	5.10671×10^{-9}
60	0.40558	8.03961×10^{-10}

Table 4.1: Results for Example 1.

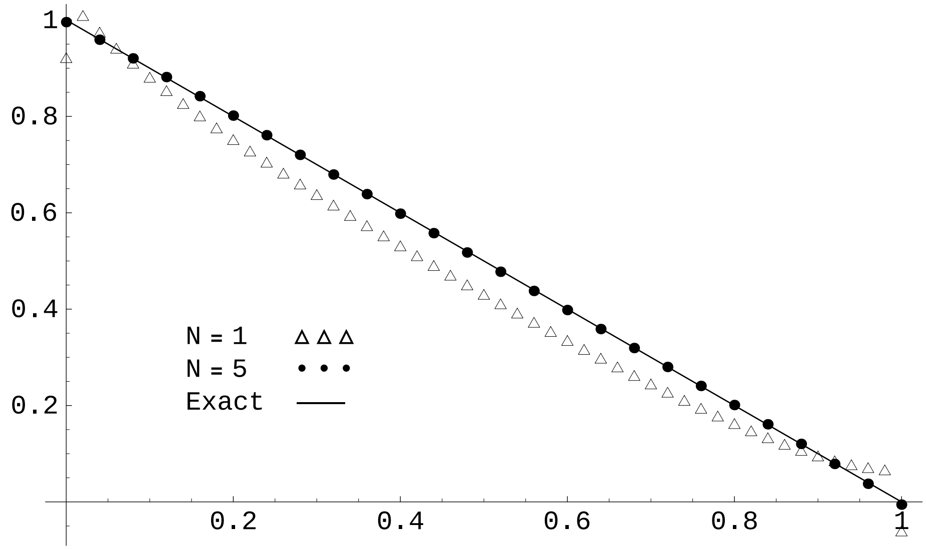

Figure 4.1. Exact and approximate solutions for Example 4.1,($N = 1, 5$)

Example 4.2. Consider the integral equation

$$y(x) = 2x - \frac{1}{12}x^4 - \frac{5}{3} + \frac{1}{4}\int_0^x (x-t)[y(t)]^2 dt + \int_0^1 (1+t)y(t)dt, \quad x \in \Gamma = [0,1], \quad (4.4)$$

with exact solution $y(x) = 2x$.

The approximate solution is calculated for different values of N, $\alpha = \frac{1}{2}$, $d = \frac{\pi}{2}$ and $h = \pi(\frac{1}{N})^{\frac{1}{2}}$. The maximum absolute errors on the sinc grid S are tabulated in Table 4.2. This table indicates that as N increases the errors are decrease rapidly. The exact and approximate solutions of Example 4.2 are shown in Fig. 4.2 for $N = 1$ and $N = 5$.

N	h	$\|E_s(h)\|_\infty$
5	1.40496	3.64982×10^{-3}
10	0.99346	2.92645×10^{-4}
20	0.70248	8.34053×10^{-6}
30	0.57357	5.23380×10^{-7}
40	0.49673	4.97018×10^{-8}
50	0.44429	6.15092×10^{-9}
60	0.40558	9.18873×10^{-10}

Table 4.2: Results for Example 4.2.

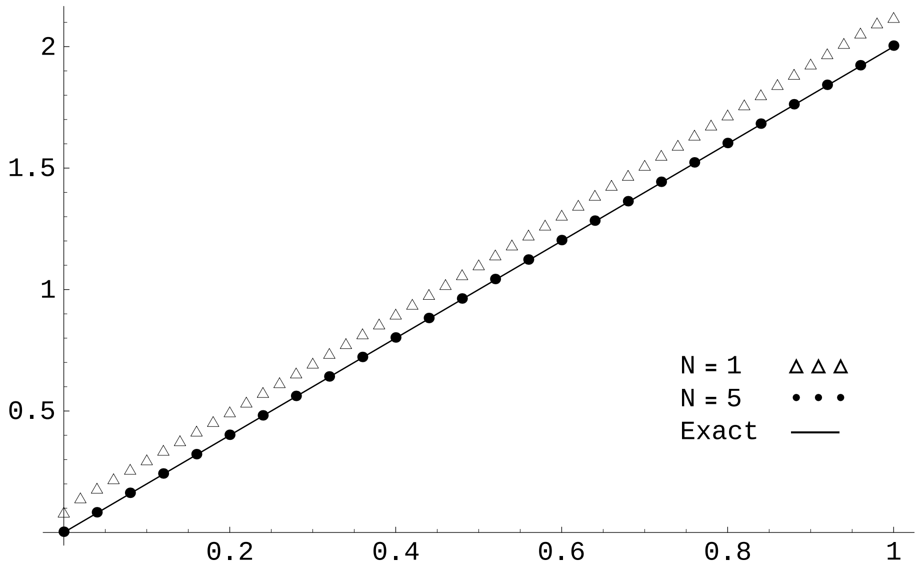

Figure 4.2. Exact and approximate solutions for Example 4.2,($N = 1, 5$)

In general, the above Figures 4.1 and 4.2 show that for larger values of $N(N \geq 5)$, the approximate solutions are indistinguishable(for the given scale) from the exact solution.

Conclusion

The sinc functions are used to solve the nonlinear Volterra-Fredholm integral equations. The numerical examples show that the accuracy improve with increasing the number of sinc grid points N.

References

[1] H.O. Bakodah and M.A. Darwish, *On discrete Adomian decomposition method with Chebyshev abscissa for nonlinear integral equations of Hammerstein Type*, Advances in Pure Mathematics, 2 (2012), 310-313.

[2] H. Brunner, *On the numerical solution of nonlinear Volterra-Fredholm integral equation by collocation methods*, SIAM J. Numer. Anal., 27 (1990), 987-1000.

[3] M.A. Darwish, *Fredholm-Volterra integral equation with singular kernel*, Korean J. Comput. Appl. Math., 6 (1999), 163-174.

[4] M.A. Darwish, *Note on stability theorems for nonlinear mixed integral equations*, Korean J. Comput. Appl. Math., 6 (1999), 633-637.

[5] M. Ghasemi, M. Tavassoli Kajani and E. Babolian, *Numerical solutions of the nonlinear Volterra-Fredholm integral equations by using homotopy perturbation method*, Appl. Math. Comput., 188 (2007), 446-449.

[6] J. Lund and K. Bowers, *Sinc methods for quadrature and differential equations*, SIAM, Philadelphia, PA, 1992.

[7] C. Minggen and D. Hong, *Representation of exact solution for the nonlinear Volterra-Fredholm integral equations*, Appl. Math. Comput., 182 (2006), 1795-1802.

[8] Y. Ordokhani, *Solution of nonlinear Volterra-Fredholm-Hammerstein integral equations via rationalized Haar functions*, Appl. Math. Comput., 180 (2006), 436-443.

[9] B. G. Pachpatte, *On a nonlinear Volterra-Fredholm integral equation*, Sarajevo J. Math., 16 (2008), 61-71.

[10] F. Stenger, *Numerical methods based on Sinc and analytic functions*, Springer-Verlag, New York, 1993.

[11] A. M. Wazwaz, *A reliable treatment for mixed Volterra-Fredholm integral equations*, Appl. Math. Comput., 127 (2002), 405-414.

[12] S. Yalçinbaş, *Taylor polynomial solutions of nonlinear Volterra-Fredholm integral equations*, Appl. Math. Comput., 127 (2002), 195-206.