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UNIQUENESS AND PARAMETER DEPENDENCE OF POSITIVE SOLUTIONS TO HIGHER ORDER BOUNDARY VALUE PROBLEMS WITH FRACTIONAL *Q*-DERIVATIVES

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Abstract The authors study a class of nonlinear higher order boundary value problem with fractional q-derivatives and dependence on a positive parameter λ. The existence, uniqueness, and dependence of positive solutions on λ are discussed. Two sequences are constructed so that they converge uniformly to the unique solution of the problems. Two examples are included in the paper. Numerical computations of the examples confirm their theoretical results.

Keywords Fractional q-calculus, boundary value problems, positive solutions, existence, uniqueness.

MSC(2000) 39A13, 34B18, 34A08.

1. Introduction and preliminaries on fractional q-calculus

In recent years, the subject of fractional calculus has gained considerable popularity and importance due mainly to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering. The monographs [29–32] are excellent sources for the theory and applications of fractional calculus. Among all the topics, the existence of positive solutions of boundary value problems (BVPs) for fractional differential equations is currently undergoing active investigation; see, for example, [2, 3, 9, 19, 25, 26, 28, 36] and the references therein.

Many efforts have also been made to develop the theory of discrete fractional calculus in various directions. For some recent work, we refer the reader to [6–8, 10–12, 20–22].

Early work on fractional q-calculus can be found in [1, 4]. Recently, there seems to be new interest in the study of this subject and many new developments have been made in the theory of fractional q-calculus ([5, 17, 18, 33]).

To the best of our knowledge, there are few results available in the literature to study the existence of *positive solutions* for BVPs with fractional q-derivatives; the only papers we know of are by El-Shahed and Al-Askar [14], El-Shahed and

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Hassan [15], Ferreira [17, 18], and Graef and Kong [23, 24]. Since finding positive solutions of BVPs is important in various fields of sciences, fractional q-calculus has tremendous potential for applications. In this paper, we will study the existence of positive solutions of a class of higher order BVPs with fractional q-derivatives.

To make this paper self-contained, below we recall some known facts on fractional q-calculus. The presentation here can be found in, for example, [1, 17, 18, 29, 33].

For $q \in (0,1)$, define

$$[a]_q = \frac{1-q^a}{1-q}, \quad a \in \mathbb{R}.$$

The q-analog of the Pochhammer symbol (the q-shifted factorial) is defined by

$$(a;q)_0 = 1, \quad (a;q)_k = \prod_{i=0}^{k-1}(1-aq^i), \quad k \in \mathbb{N} \cup \{\infty\}.$$

The q-analogue of the power function $(a-b)^k$ with $k \in \mathbb{N}_0 := \{0,1,2,\ldots\}$ is

$$(a-b)^{(0)} = 1, \quad (a-b)^{(k)} = \prod_{i=0}^{k-1}(a-bq^i), \quad k \in \mathbb{N},\ a,b \in \mathbb{R}.$$

The relationship between these two concepts is given by

$$(a-b)^{(k)} = a^k (b/a;q)_k, \quad a \neq 0.$$

Their natural expansions to the reals are

$$(a;q)_\gamma = \frac{(a;q)_\infty}{(aq^\gamma;q)_\infty}, \quad (a-b)^{(\gamma)} = a^\gamma \frac{(b/a;q)_\infty}{(q^\gamma b/a;q)_\infty}, \quad \gamma \in \mathbb{R}.$$

Clearly,

$$(a-b)^{(\gamma)} = a^\gamma (b/a;q)_\gamma, \quad a \neq 0,$$

and if $b=0$, then $a^{(\gamma)} = a^\gamma$. We also use the notation $0^{(\gamma)} = 0$ for $\gamma > 0$. The q-gamma function is defined by

$$\Gamma_q(x) = (q;q)_{x-1}(1-q)^{1-x}, \quad x \in \mathbb{R} \setminus \{0,-1,-2,\ldots\}.$$

Obviously, $\Gamma_q(x+1) = [x]_q \Gamma_q(x)$.

The q-derivative of a function h is defined by

$$(D_q h)(x) = \frac{h(x)-h(qx)}{(1-q)x} \quad \text{for } x \neq 0 \quad \text{and} \quad (D_q h)(0) = \lim_{x\to 0}(D_q h)(x),$$

and q-derivatives of higher order are given by

$$(D_q^0 h)(x) = h(x) \quad \text{and} \quad (D_q^k h)(x) = D_q(D_q^{k-1}h)(x),\ k \in \mathbb{N}.$$

The q-integral of a function h defined on the interval $[0,b]$ is given by

$$(I_q h)(x) = \int_0^x h(s)\,d_q s = x(1-q)\sum_{i=0}^{\infty} h(xq^i)q^i, \quad x \in [0,b].$$

If $a \in [0, b]$ and h is defined in the interval $[0, b]$, then its integral from a to b is defined by

$$\int_a^b h(s)d_qs = \int_0^b h(s)d_qs - \int_0^a h(s)d_qs.$$

Similar to derivatives, an operator I_q^k is given by

$$(I_q^0 h)(x) = h(x) \quad \text{and} \quad (I_q^k h)(x) = I_q(I_q^{k-1}h)(x), \ k \in \mathbb{N}.$$

The fundamental theorem of calculus applies to these operators D_q and I_q, i.e.,

$$(D_q I_q h)(x) = h(x),$$

and if h is continuous at $x = 0$, then

$$(I_q D_q h)(x) = h(x) - h(0).$$

Definition 1.1. Let $\nu \geq 0$ and h be a function defined on $[0, 1]$. The fractional q-integral of Riemann-Liouville type is given by $(I_q^0 h)(x) = h(x)$ and

$$(I_q^\nu h)(x) = \frac{1}{\Gamma_q(\nu)} \int_0^x (x - qs)^{(\nu-1)} h(s)d_qs, \quad \nu > 0, \ t \in [0, 1].$$

Definition 1.2. The fractional q-derivative of Riemann-Liouville type of order $\nu \geq 0$ is defined by $(D_q^0 h)(x) = h(x)$ and

$$(D_q^\nu h)(x) = (D_q^l I_q^{l-\nu} h)(x), \quad \nu > 0,$$

where l is the smallest integer greater than or equal to ν.

The rest of the paper is organized as follows. In Section 2, we introduce our problem and present our main results and two illustrative examples. All the proofs of the main results are given in Section 3.

2. Fractional boundary value problems

In this section, we are concerned with positive solutions of the higher order BVP with fractional q-derivatives consisting of the equation

$$-(D_q^\nu u)(t) = \lambda[f(t, u, u) + r(t, u)], \quad t \in (0, 1), \tag{2.1}$$

and the boundary condition (BC)

$$(D_q^i u)(0) = 0, \ i = 0, \ldots, n-2, \quad (D_q u)(1) = \sum_{j=1}^m a_j (D_q u)(t_j), \tag{2.2}$$

where $q \in (0, 1)$, $m \geq 1$ and $n \geq 2$ are integers, $n-1 < \nu \leq n$, $\lambda > 0$ is a parameter, $f : [0, 1] \times [0, \infty) \times [0, \infty) \to [0, \infty)$ and $r : [0, 1] \times [0, \infty) \to [0, \infty)$ are continuous and satisfy certain conditions given later, $a_j \geq 0$ and $t_j \in (0, 1)$ for $j = 1, \ldots, m$. By a *positive solution* of BVP (2.1), (2.2), we mean a function $u \in C[0, 1]$ such that

$u(t)$ satisfies (2.1) and (2.2), and $u(t) > 0$ on $(0, 1]$. Note that one special case of the equation (2.1) is given by

$$-(D_q^\nu u)(t) = \lambda f(t, u, u), \quad t \in (0, 1). \tag{2.3}$$

When $n = 3$, $\lambda = 1$, $f(t, u, u) = f(t, u)$, and $a_j = 0$ for $j = 1, \ldots, m$, BVP (2.3), (2.2) has been studied by Ferreira [17]. The well known Krasnosel'skii fixed point theorem was applied there to obtain an existence criterion for positive solutions. Very recently, Graef and Kong [23, 24] discussed the uniqueness, existence, and nonexistence of positive solutions of the general BVP (2.3), (2.2). In particular, in [23], the nonlinear term is allowed to be singular in the phase variable.

In this paper, by applying some recent results from mixed monotone operator theory (see Lemma 3.3 below), we obtain some new existence criteria for BVPs (2.1), (2.2) and (2.3), (2.2). In our theorems, we not only investigate the existence and uniqueness of positive solutions of our problems, but we also discuss the dependence of positive solutions on the parameter λ. Moreover, two sequences are constructed in such a way so that they converge uniformly to the unique positive solution of the problem. Two examples are provided to illustrate our theorems. Some numerical computations are performed to confirm our theoretic results. Our results extend and complement recent results on this subject in the literature, especially those in [17, 23, 24].

Recall that the characteristic function χ on an interval $I \subseteq \mathbb{R}$ is given by

$$\chi_I(t) = \begin{cases} 1, & t \in I, \\ 0, & t \notin I. \end{cases}$$

Define a function $G : [0, 1] \times [0, 1] \to \mathbb{R}$ by

$$\begin{aligned} G(t, s) = \; & \tfrac{t^{\nu-1}}{\left(1-\sum_{j=1}^m a_j t_j^{\nu-2}\right)\Gamma_q(\nu)} \left((1-s)^{(\nu-2)} - \textstyle\sum_{j=1}^m a_j (t_j - s)^{(\nu-2)} \chi_{[0,t_j]}(s) \right) \\ & - \tfrac{1}{\Gamma_q(\nu)} (t-s)^{(\nu-1)} \chi_{[0,t]}(s). \end{aligned} \tag{2.4}$$

Notice that, for the special case where $a_j = 0$ for $j = 1, \ldots, m$, $G(t, s)$ can be written as

$$G(t, s) = \frac{1}{\Gamma_q(\nu)} \begin{cases} (1-s)^{(\nu-2)} t^{\nu-1} - (t-s)^{(\nu-1)}, & 0 \le s \le t \le 1, \\ (1-s)^{(\nu-2)} t^{\nu-1}, & 0 \le t \le s \le 1. \end{cases}$$

We need to make use of the following assumptions.

(H1) $0 \le \sum_{j=1}^m a_j t_j^{\nu-2} < 1$.

(H2) For any $(t, x, y) \in [0, 1] \times [0, \infty) \times [0, \infty)$, $f(t, x, y)$ is increasing in x for any fixed t and y, and decreasing in y for any fixed t and x.

(H3) There exists $\alpha \in (0, 1)$ such that

$$f(t, \kappa x, \kappa^{-1} y) \ge \kappa^\alpha f(t, x, y)$$

for $t \in [0, 1]$, $\kappa \in (0, 1)$, $x \in [0, \infty)$, and $y \in [0, \infty)$.

(H4) For any $(t, x) \in [0, 1] \times [0, \infty)$, $r(t, x)$ is increasing in x for any fixed t and

$$\int_0^1 G(1, qs) r(s, 0) d_q s > 0.$$

(H5) There exists $\eta > 0$ such that

$$f(t,x,y) \geq \eta r(t,x) \quad \text{for } t \in [0,1],\ x \in [0,\infty),\ \text{and } y \in [0,\infty).$$

(H6) $r(t,\kappa x) \geq \kappa r(t,x)$ for $t \in [0,1]$, $\kappa \in (0,1)$, and $x \in [0,\infty)$.

(H7) For α given in (H3) with $\alpha \in (0,1/2)$, we have

$$r(t,\kappa x) \geq \kappa^{\alpha} r(t,x) \quad \text{for } t \in [0,1],\ \kappa \in (0,1),\ \text{and } x \in [0,\infty).$$

It is obvious that (H7) is stronger than (H6), i.e, (H7) implies (H6).

Remark 2.1. We would like to make a few comments on the format of the nonlinear term f in (2.1). As we mentioned earlier, the analysis of this paper mainly relies on the mixed monotone operator theory. To apply such theory, one alternative way in the literature is to write the nonlinearity as $f(t,x)$ and assume, among others, that $f(t,x)$ can be decomposed into $f(t,x) = g(t,x) + h(t,x)$, where $g : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous and nondecreasing in the second argument, and $h : [0,1] \times (0,\infty) \to [0,\infty)$ is continuous and nonincreasing in the second argument, and that there exists $\alpha \in (0,1)$ such that

$$g(t,\kappa x) \geq \kappa^{\alpha} g(t,x) \tag{2.5}$$

and

$$h(t,\kappa^{-1} x) \geq \kappa^{\alpha} h(t,x) \tag{2.6}$$

for $t \in [0,1]$, $\kappa \in (0,1)$, and $x > 0$. The reader may refer to [13] for a related discussion.

Here, in (2.1), the nonlinear term is written as a function of three arguments. Then, to apply the mixed monotone operator theory, we need to assume that the above conditions (H2) and (H3) are satisfied. By writing f this way, a larger class of functions can be covered. For instance, if $f(t,x,y) = x^{1/3}(y+1)^{-1/2}$, then, $f(t,x,x)$ cannot be decomposed into a sum of two functions g and h satisfying (2.5) and (2.6), but $f(t,x,y)$ does satisfy (H2) and (H3) with $\alpha = 5/6$.

Throughout this paper, let $C[0,1]$ be the Banach space of continuous functions equipped with the norm $||u|| = \max_{t\in[0,1]} |u(t)|$. Define the cone $P \subset C[0,1]$ by

$$P = \{u \in C[0,1] \ :\ u(t) \geq 0 \text{ for } t \in [0,1]\}. \tag{2.7}$$

Let $w(t) = t^{\nu-1}$. We also define a smaller cone $P_w \subset P$ by

$$P_w = \{u \in P \ :\ \text{there exist } c, d > 0 \text{ such that } cw(t) \leq u(t) \leq dw(t) \text{ on } [0,1]\}. \tag{2.8}$$

Our first theorem provides some results for BVP (2.1), (2.2).

Theorem 2.1. *Assume that (H1)–(H6) hold. Then:*

(1) BVP (2.1), (2.2) has a unique positive solution $u_\lambda(t)$ in P.

(2) For any $u_0, v_0 \in P_w$, consider the sequences $\{u_n\}$ and $\{v_n\}$ defined by

$$u_{n+1}(t) = \lambda \int_0^1 G(t,qs)[f(s,u_n(s),v_n(s)) + r(s,u_n(s))]d_q s \tag{2.9}$$

and

$$v_{n+1}(t) = \lambda \int_0^1 G(t,qs)[f(s,v_n(s),u_n(s)) + r(s,v_n(s))]d_qs \tag{2.10}$$

for $n = 0,1,2,\ldots$. *Then,* $||u_n - u_\lambda|| \to 0$ *and* $||v_n - u_\lambda|| \to 0$ *as* $n \to \infty$.

(3) *If, in addition, (H7) holds, then the unique solution* $u_\lambda(t)$ *satisfies the following properties:*

(a) $u_\lambda(t)$ *is strictly increasing in* λ, *i.e.,* $\lambda_1 > \lambda_2 > 0$ *implies* $u_{\lambda_1}(t) > u_{\lambda_2}(t)$;

(b) $\lim_{\lambda\to 0^+} ||u_\lambda|| = 0$ *and* $\lim_{\lambda\to\infty} ||u_\lambda|| = \infty$;

(c) $u_\lambda(t)$ *is continuous in* λ, *i.e.,* $\lambda \to \lambda_0 > 0$ *implies* $||u_\lambda - u_{\lambda_0}|| \to 0$.

The next result is for BVP (2.3), (2.2).

Theorem 2.2. *Assume that (H1), (H2), and (H3) hold. Then,*

(1) *BVP* (2.3), (2.2) *has a unique positive solution* $u_\lambda(t)$ *in* P.

(2) *For any* $u_0, v_0 \in P_w$, *consider the sequences* $\{u_n\}$ *and* $\{v_n\}$ *defined by*

$$u_{n+1}(t) = \lambda \int_0^1 G(t,qs)f(s,u_n(s),v_n(s))d_qs \tag{2.11}$$

and

$$v_{n+1}(t) = \lambda \int_0^1 G(t,qs)f(s,v_n(s),u_n(s))d_qs \tag{2.12}$$

for $n = 0,1,2,\ldots$. *Then,* $||u_n - u_\lambda|| \to 0$ *and* $||v_n - u_\lambda|| \to 0$ *as* $n \to \infty$.

(3) *If, in addition,* $\alpha \in (0,1/2)$, *then the unique solution* $u_\lambda(t)$ *satisfies the three properties specified in the conclusion (3) of Theorem 2.1.*

We end this section with the following two examples.

Example 2.1. In BVP (2.1), (2.2) let

$$\nu = 5/2,\ q = 1/2,\ m = 1,\ n = 3,\ a_1 = 0,\ t_1 = 1/4,$$
$$f(t,x,y) = x^{1/3} + y^{-1/4},\ \text{and } r(t,x) = x^{1/3}.$$

Then, it is easy to see that conditions (H1)–(H7) hold. In fact, in (H3) and (H7), we can take $\alpha = 1/3$, and in (H5), we can choose $\eta = 1$. Therefore, the conclusions of Theorem 2.1 hold. In fact, with the help of MATLAB, we performed the following computations.

(1) For $\lambda = 100$, the first 15 iterations of the two sequences $\{u_n(t)\}$ and $\{v_n(t)\}$ given in (2.9) and (2.10) are computed numerically with $u_0(t) = t^{3/2}$ and $v_0(t) = 50t^{3/2}$. By Theorem 2.1, both $\{u_n(t)\}$ and $\{v_n(t)\}$ converge uniformly to the unique solution $u_\lambda(t)$ of BVP (2.1), (2.2). For this choice of the initial functions, the differences between the u_n's (between the v_n's) are small for $n \geq 6$ (see Figures 1 and 2).

(2) For several different values of λ, the unique solutions $u(t)$ of BVP (2.1), (2.2) are computed numerically. The computations are consistent with properties (a) and (b) of conclusion (3) (see Figure 3).

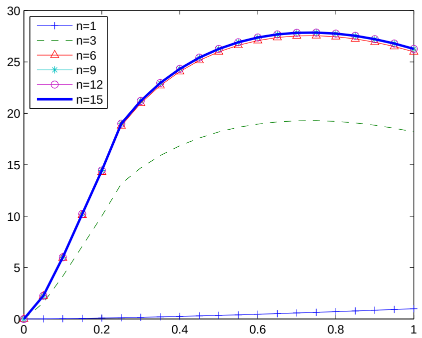

Figure 1. $u_n \to u_\lambda$

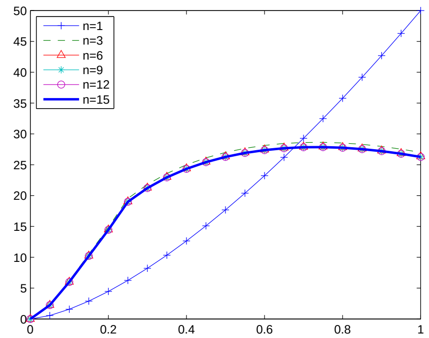

Figure 2. $v_n \to u_\lambda$

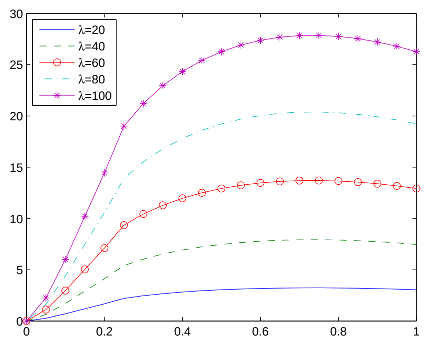

Figure 3. Dependence on λ

Example 2.2. In BVP (2.3), (2.2) let

$$v = 5/2,\ q = 1/2,\ m = 1,\ a_1 = 1/2,\ t_1 = 1/4,\ \text{and}\ f(t,x,y) = x^{1/3}(y+1)^{-1/2}.$$

Then, it is easy to see that conditions (H1)–(H3) hold. In fact, in (H3), we can take $\alpha = 5/6$. Therefore, the conclusions of Theorem 2.2 hold. In fact, with the help of MATLAB, we performed the following computations.

(1) For $\lambda = 100$, the first 25 iterations of the two sequences $\{u_n(t)\}$ and $\{v_n(t)\}$ given in (2.11) and (2.12) are computed numerically with $u_0(t) = t^{3/2}$ and $v_0(t) = 50t^{3/2}$. By Theorem 2.2, both $\{u_n(t)\}$ and $\{v_n(t)\}$ converge uniformly to the unique solution $u_\lambda(t)$ of BVP (2.3), (2.2). For this choice of the initial functions, the differences between the u_n's (between the v_n's) are small for $n \geq 19$ (see Figures 4 and 5).

(2) For several different values of λ, the unique solution $u(t)$ of BVP (2.3), (2.2) are computed numerically. The computations are consistent with properties (a) and (b) of conclusion (3) (see Figure 6).

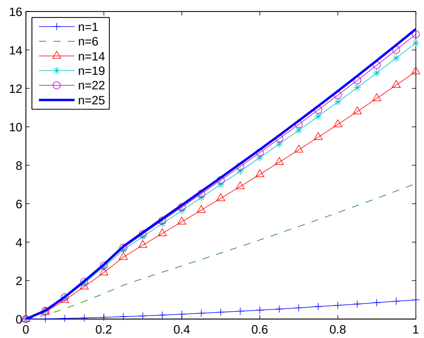

Figure 4. $u_n \to u_\lambda$

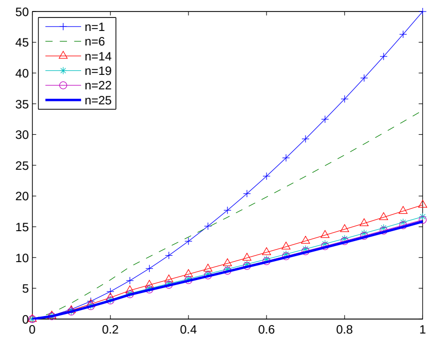

Figure 5. $v_n \to u_\lambda$

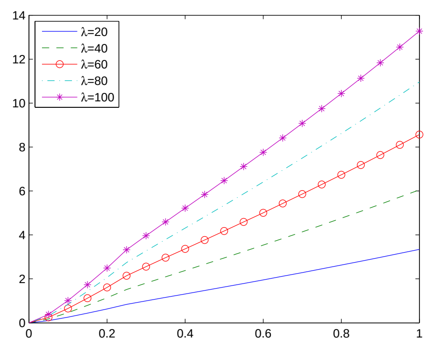

Figure 6. Dependence on λ

3. Proofs of the main results

Lemma 3.1 below gives some properties of the function $G(t, s)$ defined by (2.4).

Lemma 3.1. *Assume (H1) holds. The function $G(t, s)$ has the following properties:*

(a) $G(t, qs) \geq 0$ *for* $t, s \in [0, 1]$*;*

(b) $G(t, qs) \geq t^{\nu-1} G(1, qs)$ *for* $t, s \in [0, 1]$*;*

(c) $G(t, qs) \leq t^{\nu-1} k(qs)$ *for* $t, s \in [0, 1]$, *where*

$$k(s) = \frac{1}{\left(1 - \sum_{j=1}^{m} a_j t_j^{\nu-2}\right) \Gamma_q(\nu)} \left((1-s)^{(\nu-2)} - \sum_{j=1}^{m} a_j (t_j - s)^{(\nu-2)} \chi_{[0,t_j]}(s) \right).$$

Proof. Parts (a) and (b) were proved in [24, Lemma 2.1]. Part (c) follows directly from the definition of $G(t, s)$. □

The following result follows directly from [24, Lemma 2.2] and shows that $G(t, s)$ can be used to obtain the equivalent integral forms for some given BVPs.

Lemma 3.2. *Assume that (H1) holds and $w \in C[0,1]$. Then, $u(t)$ is a solution of the BVP consisting of the equation*

$$-(D_q^\nu u)(t) = w(t), \quad t \in (0,1),$$

and BC (2.2) *if and only if*

$$u(t) = \int_0^1 G(t, qs) w(s) d_q s.$$

To prove our results, we also need to recall some knowledge from the mixed monotone theory. Let $(X, ||\cdot||)$ be a real Banach space. By θ we denote the zero element of X. Recall that a nonempty closed convex subset $P \subset X$ is called a *cone* if it satisfies: (i) $u \in P$ and $k > 0$ implies $ku \in P$; (ii) $u \in P$ and $-u \in P$ implies $u = \theta$. A cone P is said to be *normal* if there exists a constant $C > 0$ such that, for all $u, v \in X$, $\theta \le u \le v$ implies $||u|| \le C||v||$. The constant C is called the *normality constant* of P.

Below, we assume that X is partially ordered by a normal cone $P \subset X$, i.e., $u \le v$ if and only if $v - u \in P$. If $u \le v$ and $u \ne v$, then we write $u < v$ or $v > u$.

For any $u, v \in X$, we use the notation $u \sim v$ to mean that there exist $c > 0$ and $d > 0$ such that $cv \le u \le dv$. Clearly, $\sim$ is an equivalent relation.

In the following, we let $w \in X$ be such that $w > \theta$ (i.e., $w \ge \theta$ and $w \ne \theta$) and define $P_w = \{u \in X \ : \ u \sim w\}$. It is easy to see that $P_w \subset P$. We now recall several definitions.

Definition 3.1. An operator $A : P_w \times P_w \to X$ is called mixed monotone if $A(x, y)$ is nondecreasing in x and nonincreasing in y, i.e., for $x_1,\ x_2,\ y_1,\ y_2 \in P_w$, we have

$$x_1 \le x_2 \text{ and } y_1 \ge y_2 \text{ implies } A(x_1, y_1) \le A(x_2, y_2).$$

Moreover, an element $u \in P_w$ is said to be a fixed point of A if $A(u, u) = u$.

Definition 3.2. An operator $B : P_w \to X$ is called sub-homogeneous if it satisfies

$$B(\kappa u) \ge \kappa B u \quad \text{for all } u \in P_w \text{ and } \kappa \in (0,1).$$

Definition 3.3. Let $\alpha \in [0,1)$. An operator $B : P_w \to X$ is called α-concave if it satisfies

$$B(\kappa u) \ge \kappa^\alpha B u \quad \text{for all } u \in P_w \text{ and } \kappa \in (0,1).$$

Remark 3.1. From the definitions, we see that if B is α-concave, then it is also sub-homogeneous.

Let $A : P_w \times P_w \to X$ and $B : P_w \to X$ be two operators, and $\lambda > 0$. Consider the two operator equations on P_w

$$\lambda(A(u,u) + Bu) = u \tag{3.1}$$

and

$$\lambda A(u,u) = u. \tag{3.2}$$

The following lemma is crucial in the proofs of our results. For its proof, see [27, Lemma 2.1], [34, Corollary 2.3], and [35, Theorem 2.1 and Theorem 2.3].

Lemma 3.3. *Let $\alpha \in (0,1)$ and $A : P_w \times P_w \to X$ be a mixed monotone operator satisfying*

$$A(\kappa u, \kappa^{-1} v) \geq \kappa^{\alpha} A(u,v) \quad \text{for all } u, v \in P_w \text{ and } \kappa \in (0,1). \tag{3.3}$$

(A) Assume that $B : P_w \to X$ is an increasing sub-homogeneous operator and the following conditions hold:

(i) $A(w,w) \in P_w$ and $Bw \in P_w$;

(ii) there exists a constant $\eta > 0$ such that $A(u,v) \geq \eta Bu$ for all $u, v \in P_w$.

Then:

(1) for any $\lambda > 0$, Eq. (3.1) has a unique solution u_λ in P_w;

(2) for any initial values $u_0, v_0 \in P_w$, consider the sequences $\{u_n\}$ and $\{v_n\}$ defined by

$$\begin{aligned} u_n &= \lambda(A(u_{n-1}, v_{n-1}) + Bu_{n-1}) \\ v_n &= \lambda(A(v_{n-1}, u_{n-1}) + Bv_{n-1}), \end{aligned} \quad n = 1, 2, \ldots.$$

Then, $||u_n - u_\lambda|| \to 0$ and $||v_n - u_\lambda|| \to 0$ as $n \to \infty$;

(3) if we further assume that $\alpha \in (0, 1/2)$ and B is α-concave, then the unique solution u_λ satisfies the properties:

(a) u_λ is strictly increasing in λ, that is, if $\lambda_1 > \lambda_2 > 0$, then $u_{\lambda_1} > u_{\lambda_2}$;

(b) $\lim_{\lambda \to 0^+} ||u_\lambda|| = 0$ and $\lim_{\lambda \to \infty} ||u_\lambda|| = \infty$;

(c) u_λ is continuous in λ, that is, if $\lambda \to \lambda_0 > 0$, then $||u_\lambda - u_{\lambda_0}|| \to 0$.

(B) Assume that $A(w,w) \in P_w$. Then:

(1) for any $\lambda > 0$, Eq. (3.2) has a unique solution u_λ in P_w;

(2) for any initial values $u_0, v_0 \in P_w$, consider the sequences $\{u_n\}$ and $\{v_n\}$ defined by

$$u_n = \lambda A(u_{n-1}, v_{n-1}), \quad v_n = \lambda A(v_{n-1}, u_{n-1}), \; n = 1, 2, \ldots.$$

Then, $||u_n - u_\lambda|| \to 0$ and $||v_n - u_\lambda|| \to 0$ as $n \to \infty$;

(3) if we further assume that $\alpha \in (0, 1/2)$, then the unique solution u_λ satisfies the three properties (a), (b), and (c) specified in (3) of part (A).

Now, we prove Theorem 2.1.

Proof of Theorem 2.1. Clearly, the cone P defined by (2.7) is normal. Let $w(t) = t^{\nu-1}$ and P_w be defined by (2.8). Define two operators $A_\lambda : P_w \times P_w \to X$ and $B_\lambda : P_w \to X$ by

$$A_\lambda(u,v)(t) = \lambda \int_0^1 G(t, qs) f(s, u(s), v(s)) d_q s$$

and

$$B_\lambda(u)(t) = \lambda \int_0^1 G(t, qs) r(s, u(s)) d_q s.$$

Then, by Lemma 3.2, we see that $u(t)$ is a solution of BVP (2.1), (2.2) if and only if $u = A(u,u) + Bu$. Moreover, from the monotonicity of f and r assumed in (H2) and (H4), A_λ is mixed monotone and B is increasing. For $u, v \in P_w$ and $\kappa \in (0,1)$, from (H3), we have

$$\begin{aligned} A_\lambda(\kappa u, \kappa^{-1} v)(t) &= \lambda \int_0^1 G(t,qs) f(s, \kappa u(s), \kappa^{-1} v(s)) d_q s \\ &\geq \kappa^\alpha \lambda \int_0^1 G(t,qs) f(s, u(s), v(s)) d_q s \\ &= \kappa^\alpha A_\lambda(u,v)(t), \end{aligned}$$

i.e., (3.3) of Lemma 3.3 holds. Similarly, by (H6),

$$\begin{aligned} B_\lambda(\kappa u)(t) &= \lambda \int_0^1 G(t,qs) r(s, \kappa u(s)) d_q s \\ &\geq \kappa \lambda \int_0^1 G(t,qs) r(s, u(s)) d_q s \\ &= \kappa B_\lambda(u)(t), \end{aligned}$$

i.e., B_λ is sub-homogeneous.

Note that $0 \leq w(t) = t^{\nu-1} \leq 1$ on $[0,1]$. Then, from Lemma 3.1 and (H2),

$$\begin{aligned} A_\lambda(w,w)(t) &= \lambda \int_0^1 G(t,qs) f(s, w(s), w(s)) d_q s \\ &\geq \lambda t^{\nu-1} \int_0^1 G(1,qs) f(s,0,1) d_q s \\ &= c_1 w(t), \end{aligned}$$

and

$$\begin{aligned} A_\lambda(w,w)(t) &= \lambda \int_0^1 G(t,qs) f(s, w(s), w(s)) d_q s \\ &\leq \lambda t^{\nu-1} \int_0^1 k(qs) f(s,1,0) d_q s \\ &= d_1 w(t) \end{aligned}$$

where

$$c_1 = \lambda \int_0^1 G(1,qs) f(s,0,1) d_q s \tag{3.4}$$

and

$$d_1 = \lambda \int_0^1 k(qs) f(s,1,0) d_q s. \tag{3.5}$$

By (H2), (H4), and (H5), we have

$$f(s,1,0) \geq f(s,0,1) \geq \eta r(s,0) \geq 0. \tag{3.6}$$

Then, from (H4) and (3.4)–(3.6), we have $d_1 \geq c_1 > 0$. Thus, $A_\lambda(w,w) \in P_w$. Similarly, from Lemma 3.1 and (H4),

$$\begin{aligned} B_\lambda(w)(t) &= \lambda \int_0^1 G(t,qs) r(s, w(s)) d_q s \\ &\geq \lambda t^{\nu-1} \int_0^1 G(1,qs) r(s, w(s)) d_q s \\ &\geq c_2 w(t) \end{aligned}$$

and

$$\begin{aligned} B_\lambda(w)(t) &= \lambda \int_0^1 G(t,qs) r(s, w(s)) d_q s \\ &\leq \lambda t^{\nu-1} \int_0^1 k(qs) r(s, w(s)) d_q s \\ &= d_2 w(t), \end{aligned}$$

where

$$c_2 = \lambda \int_0^1 G(1, qs) r(s, w(s)) d_q s \quad \text{and} \quad d_2 = \lambda \int_0^1 k(qs) r(s, w(s)) d_q s.$$

Then, using the fact that $r(s, w(s)) \geq r(z, 0)$ on $[0, 1]$ and (H4), we see that $B_\lambda w \in P_w$. Hence, the condition (i) of part (A) of Lemma 3.3 is satisfied.

For $u, v \in P_w$, from (H5), it follows that

$$\begin{aligned} A_\lambda(u, v)(t) &= \lambda \textstyle\int_0^1 G(t, qs) f(s, u(s), v(s)) d_q s \\ &\geq \eta\lambda \textstyle\int_0^1 G(t, qs) r(s, u(s)) d_q s \\ &\geq \eta B_\lambda(u)(t). \end{aligned}$$

Then, the condition (ii) of part (A) of Lemma 3.3 holds. Therefore, by the conclusion (1) of Lemma 3.3 (A), we see that, for any $\lambda > 0$, BVP (2.1), (2.2) has a unique solution $u_\lambda(t)$ in P_w, which is obviously positive, and from the conclusion (2) of Lemma 3.3 (A), the conclusion (2) of Theorem 2.1 holds. We now prove the following claim.

Claim: If, for any $\lambda > 0$, we assume that $u_\lambda(t)$ is a positive solution of BVP (2.1), (2.2), then $u_\lambda \in P_w$.

In fact, if $u_\lambda(t)$ is a positive solution of BVP (2.1), (2.2), then, by Lemma 3.2, we have

$$u_\lambda(t) = \lambda \int_0^1 G(t, qs)[f(s, u_\lambda(s), u_\lambda(s)) + r(s, u_\lambda(s))] d_q s.$$

Then, from Lemma 3.1, (H2), and (H4), it follows that

$$\begin{aligned} u_\lambda(t) &\geq \lambda t^{\nu-1} \textstyle\int_0^1 G(1, qs)[f(s, 0, ||u_\lambda||) + r(s, 0)] d_q s \\ &= c_3 w(t) \end{aligned}$$

and

$$\begin{aligned} u_\lambda(t) &\leq \lambda t^{\nu-1} \textstyle\int_0^1 k(qs)[f(s, ||u||, 0) + r(s, ||u||)] d_q s \\ &= d_3 w(t), \end{aligned}$$

where

$$c_3 = \lambda \int_0^1 G(1, qs)[f(s, 0, ||u_\lambda||) + r(s, 0)] d_q s$$

and

$$d_3 = \lambda \int_0^1 k(qs)[f(s, ||u||, 0) + r(s, ||u||)] d_q s.$$

As in (3.6), we have

$$f(s, ||u||, 0) \geq f(s, 0, ||u_\lambda||) \geq \eta r(s, 0) \geq 0$$

and

$$r(s, ||u||) \geq r(s, 0) \geq 0.$$

Then, by (H4), we see that $d_3 \geq c_3 > 0$. This shows that $u_\lambda \in P_w$, i.e., the claim is true.

Now, by the claim, we see that BVP (2.1), (2.2) has a unique positive solution in P.

Finally, if (H7) holds, we have $\alpha \in (0, 1/2)$ and

$$\begin{aligned} B_\lambda(\kappa u)(t) &= \lambda \int_0^1 G(t, qs) r(s, \kappa u(s)) d_q s \\ &\geq \kappa^\alpha \lambda \int_0^1 G(t, qs) r(s, u(s)) d_q s \\ &= \kappa^\alpha B_\lambda(u)(t), \end{aligned}$$

i.e., B_λ is α-concave. Thus, the conclusion (3) of Theorem 2.1 follows from the last conclusion of Lemma 3.3 (A). This completes the proof of Theorem 2.1. □

Using part (B) of Lemma 3.3, by an argument similar (but much simpler) to the proof of Theorem 2.1, we can prove Theorem 2.2; the details are omitted.

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