Journal of Applied Analysis and Computation
Volume 2, Number 4, November 2012

Website:http://jaac-online.com/
pp. 449–463

# EXISTENCE-UNIQUENESS PROBLEMS FOR INFINITE DIMENSIONAL STOCHASTIC DIFFERENTIAL EQUATIONS WITH DELAYS*

Daoyi Xu[1], Xiaohu Wang[1,†] and Zhiguo Yang[2]

**Abstract** The main aim of this paper is to develop the basic theory of a class of infinite dimensional stochastic differential equations with delays (IDSDEs) under local Lipschitz conditions. Firstly, we establish a global existence-uniqueness theorem for the IDSDEs under the global Lipschitz condition in $C$ without the linear growth condition. Secondly, the non-continuable solution for IDSDEs is given under the local Lipschitz condition in $C$. Then, the classical Itô's formula is improved and a global existence theorem for IDSDEs is obtained. Our new theorems give better results while conditions imposed are much weaker than some existing results. For example, we need only the local Lipschitz condition in $C$ but neither the linear growth condition nor the continuous condition on the time $t$. Finally, two examples are provided to show the effectiveness of the theoretical results.

**Keywords** Infinite dimensional stochastic differential equations, delay, existence, uniqueness.

**MSC(2000)** 34K30, 34K50, 60H15.

## 1. Introduction

As the most basic and interesting problem, existence-uniqueness for partial differential equations is very important. For example, the Clay Mathematics Institute Millennium Prize Problem on the incompressible Navier-Stokes equations asks for a proof of global existence of smooth solutions for all smooth data, or a proof of the converse [4]. This problem is open up to the present, which shows that existence-uniqueness problems for partial differential equations are also very complex. However, for the semilinear partial differential equations, the same results on existence-uniqueness as ordinary differential equations have been commendably established under the local Lipschitz conditions. It is come down to the following abstract Cauchy problem [9, 14, 16],

$$\begin{cases} u'(t) = Au(t) + f(t, u(t)), \quad t > 0, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

† the corresponding author. Email address:xiaohuwang111@163.com(X. Wang)
1 College of Mathematics, Sichuan University, Chengdu 610064, China
2 College of Mathematics and Software Science, Sichuan Normal University, Chengdu 610068, China
* The authors were supported by National Natural Science Foundation of China (No.11271270, 11101298, 11126229, 11201320), Research Fund for the Doctoral Program of Higher Education of China (No.20105134110001), Youth Foundation of Sichuan University (No.2011SCU11111) and Fundamental Research Funds for the Central Universities (No.2010SCU21006).

where $A$ is the infinitesimal operator of a $C_0$ semigroup $S(t) = e^{tA}, t \geq 0$.

Many actual systems have the property of delay effect, which is believed to occur in mechanics, physics, chemistry, biology, economics, etc [8, 12, 25]. Therefore, it is of significant importance to consider delay systems and many interesting results on the existence-uniqueness of partial differential equations with delays have been reported [11, 18, 20, 21].

On the other hand, in most dynamical systems which describe processes in engineering, physics and economics, stochastic components and random noise are included. The stochastic aspects of the models are used to capture the uncertainty about the environment in which the system is operating and the structure and parameters of the models of physical processes are being studied. Stochastic differential equations in infinite dimensional spaces are motivated by internal development of analysis and the theory of stochastic processes. Recently, studies on stochastic differential equations in infinite dimensional spaces have become a hot topic [1–3, 5, 10, 13, 15, 17, 19, 22, 26]. The important representative works are as follows.

Prato and Zabczyk [17] considered the following semilinear stochastic equation in a Hilbert space $H$

$$\begin{cases} \mathrm{d}u(t) = [Au(t) + f(t, u(t))]\mathrm{d}t + g(t, u(t))\mathrm{d}W(t), \quad t \in [0, T], \\ u(0) = u_0, \end{cases} \tag{1.2}$$

where $A$ is the infinitesimal operator of a $C_0$ semigroup $S(t) = e^{tA}, t \geq 0$. They proved that system (1.2) has a unique global solution if $f$ and $g$ satisfy global Lipschitz condition:

$$|f(t, x) - f(t, y)|_H + |g(t, x) - g(t, y)|_{L_2^0} \leq c|x - y|_H,$$

and linear growth condition:

$$|f(t, x)|_H^2 + |g(t, x)|_{L_2^0}^2 \leq c|x|_H^2,$$

where $c > 0$ is a constant, $|\cdot|_H$ and $|\cdot|_{L_2^0}$ are the norms in Hilbert spaces $H$ and $L_2^0$, respectively (see the definition below).

Following [3], Fu et al. [5] studied the following stochastic partial differential equation in a bounded domain $D \subset \mathbb{R}^n$

$$\begin{cases} \frac{\partial}{\partial t}u(t) = (\kappa\Delta - \alpha)u(t) + f(u(t)) + g(u(t))\frac{\partial}{\partial t}W(t), \quad t > 0, \\ u|_{\partial D} = 0, \quad u(0) = \phi(x), \end{cases} \tag{1.3}$$

where $\kappa$ and $\alpha$ are positive constants. They employed the local Lipschitz condition:

$$\|f(u) - f(v)\|_{L^2(D)}^2 \vee \|g(u) - g(v)\|_{L^2(D)}^2 \leq r_n\|u - v\|_\gamma^2,$$

for all $u, v \in H^\gamma$ ($H^\gamma$ denotes the domain of $A^\gamma$ in $H = L^2(D)$) with $\|u\|_\gamma \vee \|v\|_\gamma \leq n$ ($n = 1, 2, \dots$), where $\|\cdot\|_{L^2(D)}$ and $\|\cdot\|_\gamma$ represent the norm of $L^2(D)$ and $H^\gamma$ respectively, and the priori estimate

$$\mathbb{E}\|u(t)\|_\gamma^2 \leq K(t), \qquad t \geq 0,$$

where $K(t)$ is defined and finite for all $t > 0$. They gave a global existence-uniqueness theorem for (1.3) on the condition that $\gamma \in (0, \frac{1}{2})$ (see [5, Theorem 3.2]).

Taniguchi et al. [19] investigated the following IDSDE:

$$\begin{cases} \mathrm{d}u(t) = [Au(t) + f(t, u_t)]\mathrm{d}t + g(t, u_t)\mathrm{d}W(t), & t \geq t_0, \\ u_{t_0} = \phi, \end{cases} \tag{1.4}$$

where $A$ is a closed, densely defined linear operator and the generator of a certain analytic semigroup. By using analytic semigroups approach and fractional power operator arguments, the existence, uniqueness, and asymptotic behavior of mild solutions of (1.4) are given under the global Lipschitz conditions.

In [23] and [24], Xu et al. developed basic theories of existence-uniqueness for stochastic functional differential equations under the local Lipschitz conditions in $L^2(\Omega, C)$ and $C$, respectively. Motivated by the above discussions, our first aim is to establish a global existence-uniqueness theorem for the IDSDEs under the global Lipschitz condition in $C$ without the linear growth condition. Secondly, the non-continuable solution for IDSDEs is given under the local Lipschitz condition in $C$. Then, the classical Itô's formula is improved and a global existence theorem for IDSDEs is obtained. Our new theorems give better results while conditions imposed are much weaker than some existing results. For example, we need only the local Lipschitz condition in $C$ but neither the linear growth condition nor the continuous condition on the time $t$. Finally, two examples are provided to show the effectiveness of the theoretical results.

# 2. Preliminaries

In this section, we introduce some notations and recall some basic definitions.

Let $\mathbb{R}^n$ denote the $n$-dimensional Euclidean space and $\mathbb{R}^n_+$ the $n$-dimensional nonnegative Euclidean space.

Let $U$ and $H$ be separable Hilbert spaces and let $\mathcal{L}(U, H)$ be the space of all bounded linear operators from $U$ to $H$. We denote the norms of elements in $U$, $H$ and $\mathcal{L}(U, H)$ by symbols $|\cdot|_U, |\cdot|_H$ and $|\cdot|_{\mathcal{L}(U,H)}$ respectively.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ satisfying the usual conditions. We are given a $Q$-Wiener process in the complete probability space$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$ and having values in $U$, i.e. $W(t)$ is defined as (see [17] for more details)

$$W(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\omega_n(t)e_n, \quad t \geq t_0,$$

where $\omega_n(t)(n = 1, 2, 3, \cdots)$ is a sequence of real valued one-dimensional standard Brownian motions mutually independent on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$; $\lambda_n$ $(n = 1, 2, 3, \cdots)$ are nonnegative real numbers such that $\sum\limits_{n\geq 1} \lambda_n < \infty$; $\{e_n\}_{n\geq 1}$ is a complete orthonormal basis in $U$, and $Q \in \mathcal{L}(U, U)$ is the incremental covariance operator of the process $W(t)$, which is a symmetric nonnegative trace class operator defined by

$$Qe_n = \lambda_n e_n, \quad n = 1, 2, 3, \cdots .$$

Denote $U_0 = Q^{\frac{1}{2}}U$, and let $L_2^0$ be the space of all Hilbert-Schmidt operators $L_2^0 = L_2(U_0, H)$ from $U_0$ to $H$. The space $L_2^0$ is a separable Hilbert space equipped with the following norm

$$|\Psi|^2_{L_2^0} = \mathrm{Tr}\,(\Psi\, Q\, \Psi^*).$$

$$\begin{aligned}&\mathcal{M}^2([a,b];H)\\ =\;&\{f : f \,\text{is}\, H\text{-valued-measurable}\, \mathcal{F}_t\text{-adapted process and } \mathbb{E}\int_a^b |f(t)|_H^2 \mathrm{d}t < \infty\}.\end{aligned}$$

Especially, we let $\mathcal{N}^2([a,b];H) = \{f : f$ is $H$-valued-measurable and $\int_a^b |f(t)|_H^2 \mathrm{d}t < \infty\}$. Similarly, we may define $\mathcal{M}^2([a,b];L_2^0)$, $\mathcal{N}^2([a,b];L_2^0)$ and $\mathcal{N}^1([a,b];\mathbb{R}^n)$.

Let $C(J;H)$ denote the space of all continuous functions from the interval $J$ into $H$ equipped with supremum norm. Let us fix a $\tau > 0$ and consider $c > 0$. If we have a function $u \in C([-\tau,c];H)$, for each $t \in [0,c]$ we denote by $u_t \in C([-\tau,0];H)$ the function defined by $u_t(s) = u(t+s), -\tau \le s \le 0$. Especially, let $C \triangleq C([-\tau,0];H)$ with the norm $\|\varphi\| = \sup\limits_{-\tau \le s \le 0} |\varphi(s)|_H$.

In this paper, we will study the following IDSDE

$$\begin{cases} \mathrm{d}u(t) = [A(t)u(t) + f(t,u_t)]\mathrm{d}t + g(t,u_t)\mathrm{d}W(t), \quad t \in [t_0,T), \\ u_{t_0} = \xi, \end{cases} \tag{2.1}$$

where, $T > t_0$ is a constant or $\infty$, $\xi = \{\xi(\theta) : -\tau \le \theta \le 0\}$ is an $\mathcal{F}_{t_0}$-measurable C-valued random variable such that $\mathbb{E}\|\xi\|^2 < \infty$. $A(t)$ is a family of linear operators in $H$, $f : [t_0,T) \times C \to H$ and $g : [t_0,T) \times C \to L_2^0$ are two nonlinear measurable mappings. We shall assume that $A(\cdot)$ generates an evolution operator $U(t,s)$ for $t_0 \le s \le t < T$. That is, $U(t,s)$ satisfies the following conditions (see for instance [16])

(i) $U(s,s) = I$ (the identity mapping in $H$), $U(t,r)U(r,s) = U(t,s)$ for $t_0 \le s \le r \le t < T$,

(ii) $(t,s) \to U(t,s)$ is strongly continuous for $t_0 \le s \le t < T$,

(iii) $\frac{\partial}{\partial t}U(t,s)u = A(t)U(t,s)u$ for all $u \in D(A(t)) \subset H$.

**Definition 2.1.** The mappings $f$ and $g$ in (2.1) are said to satisfy the local Lipschitz condition on $[t_0,b] \subset [t_0,T)$ if for any $n > 0$ there is a constant $K_n = K_n(b) > 0$ such that

$$|f(t,\phi) - f(t,\psi)|_H \le K_n\|\phi - \psi\|, \quad |g(t,\phi) - g(t,\psi)|_{L_2^0} \le K_n\|\phi - \psi\|, \tag{2.2}$$

for all $t \in [t_0,b]$ and $\varphi,\psi \in C$ with $\|\varphi\| \vee \|\psi\| \le n$. The mappings $f$ and $g$ are said to satisfy the local Lipschitz condition on $[t_0,T)$ if (2.2) holds for any $b \in [t_0,T)$.

**Definition 2.2.** A predictable $H$-valued process $u(t)$, $t \in [t_0,T)$, is said to be a mild solution of (2.1) if

$$\mathbb{P}\Big(\int_{t_0}^t |u(s)|_H^2 \mathrm{d}s < +\infty\Big) = 1,$$

and for arbitrary $t \in [t_0,T)$, we have

$$u(t) = U(t,t_0)\xi(0) + \int_{t_0}^t U(t,s)f(s,u_s)\mathrm{d}s + \int_{t_0}^t U(t,s)g(s,u_s)\mathrm{d}W(s).$$

**Definition 2.3.** A predictable $H$-valued process $u(t)$, $t \in [t_0,T)$, is said to be a strong solution of (2.1) if $u(t)$ takes values in $D(A(t))$ almost everywhere,

$$\mathbb{P}\Big(\int_{t_0}^t |A(s)u(s)|_H \,\mathrm{d}s < +\infty\Big) = 1, \quad \text{a.s.}$$

And for arbitrary $t \in [t_0, T)$, we have

$$u(t) = \xi + \int_{t_0}^{t} A(s)u(s)\mathrm{d}s + \int_{t_0}^{t} f(s, u_s)\mathrm{d}s + \int_{t_0}^{t} g(s, u_s)\mathrm{d}W(s).$$

**Definition 2.4.** The mild solution $u(t)$ of (2.1) is said to explode at $\bar{t} > t_0$ if

$$\mathbb{P}(\sup_{s\in(\bar{t}-\varepsilon,\ \bar{t}\ ],\ \varepsilon\to 0} |u(s)|_H = \infty) > 0.$$

## 3. Basic results on existence and uniqueness

**Lemma 3.1.** *For any $b \in [t_0, T)$, assume that*

$$f(t, 0) \in \mathcal{N}^2([t_0, b]; H), \quad g(t, 0) \in \mathcal{N}^2([t_0, b]; L_2^0), \tag{3.1}$$

*in addition, $f$ and $g$ in (2.1) satisfy the global Lipschitz condition in $[t_0, b] \times C$, that is,*

$$|f(t, \phi) - f(t, \psi)|_H \le L\|\phi - \psi\|, \quad |g(t, \phi) - g(t, \psi)|_{L_2^0} \le L\|\phi - \psi\|, \tag{3.2}$$

*for all $t \in [t_0, b]$ and $\varphi, \psi \in C$, where $L$ is a positive constant. Then (2.1) has a unique mild solution $u(t)$ for $t \in [t_0 - \tau, b]$.*

**Proof.** To prove this theorem by using the classical contraction mapping principle, we denote by $\mathcal{H}_T$ the Banach space of all the $H$-valued predictable process $Y(t)$ for $t \in [t_0 - \tau, T]$ such that

$$\|Y\|_{\mathcal{H}_T} = \big(\mathbb{E} \sup_{t\in[t_0-\tau,T]} |Y(t)|_H^2\big)^{\frac{1}{2}} < +\infty.$$

Choose $t^* \in [t_0, b]$ such that

$$2L^2(t^* - t_0)[N^2(b)(t^* - t_0) + 4] < 1, \tag{3.3}$$

where $N(b) = \sup_{t_0\le s\le t\le b} \|U(t, s)\|$. Let $\Gamma(u)(t) =$

$$\begin{cases} U(t, t_0)\xi(0) + \int_{t_0}^{t} U(t, s)f(s, u_s)\mathrm{d}s + \int_{t_0}^{t} U(t, s)g(s, u_s)\mathrm{d}W(s), \ t \in [t_0, t^*], \\ \xi(t - t_0), \ t \in [t_0 - \tau, t_0]. \end{cases} \tag{3.4}$$

Firstly, we will show that $\Gamma$ maps $\mathcal{H}_{t^*}$ into $\mathcal{H}_{t^*}$. To this end, let $u \in \mathcal{H}_{t^*}$, by Jensen's inequality, then we have for $t \in [t_0, t^*]$

$$\begin{aligned} & \mathbb{E} \sup_{t_0-\tau\le t\le t^*} |\Gamma(u)(t)|_H^2 \\ = \ & \mathbb{E} \sup_{t_0\le t\le t^*} |U(t, t_0)\xi(0) + \textstyle\int_{t_0}^{t} U(t, s)f(s, u_s)\mathrm{d}s + \int_{t_0}^{t} U(t, s)g(s, u_s)\mathrm{d}W(s)|_H^2 \\ \le \ & 3\mathbb{E} \sup_{t_0\le t\le t^*} \Big[|U(t, t_0)\xi(0)|_H^2 + |\textstyle\int_{t_0}^{t} U(t, s)f(s, u_s)\mathrm{d}s|_H^2 \\ & +|\textstyle\int_{t_0}^{t} U(t, s)g(s, u_s)\mathrm{d}W(s)|_H^2\Big] \\ \le \ & 3N^2(b)\mathbb{E}\|\xi\|^2 + 3\mathbb{E} \sup_{t_0\le t\le t^*} \big[\textstyle\int_{t_0}^{t} |U(t, s)f(s, u_s)|_H\mathrm{d}s\big]^2 \\ & +3\mathbb{E} \sup_{t_0\le t\le t^*} |\textstyle\int_{t_0}^{t} U(t, s)g(s, u_s)\mathrm{d}W(s)|_H^2. \end{aligned} \tag{3.5}$$

It follows from (3.1) that there is a positive constant $M$ such that

$$\int_{t_0}^{t} |f(s,0)|_H^2 \mathrm{d}s \vee \int_{t_0}^{t} |g(s,0)|_{L_2^0}^2 \mathrm{d}s \le M, \quad \forall\, t \in [t_0, t^*]. \tag{3.6}$$

By Hölder's inequality and (3.6), we can get

$$\begin{aligned} & \mathbb{E} \sup_{t_0 \le t \le t^*} \Big[ \int_{t_0}^{t} |U(t,s) f(s,u_s)|_H \mathrm{d}s \Big]^2 \\ \le\ & N^2(b) \mathbb{E} \sup_{t_0 \le t \le t^*} \Big[ \int_{t_0}^{t} |f(s,u_s)|_H \mathrm{d}s \Big]^2 \\ \le\ & N^2(b)(t^*-t_0) \mathbb{E} \int_{t_0}^{t^*} |f(s,u_s)|_H^2 \mathrm{d}s \\ \le\ & 2N^2(b)(t^*-t_0) \mathbb{E} \int_{t_0}^{t^*} \Big[ |f(s,u_s)-f(s,0)|_H^2 + |f(s,0)|_H^2 \Big] \mathrm{d}s \\ \le\ & 2N^2(b)(t^*-t_0) \Big[ L^2(t^*-t_0) \mathbb{E} \sup_{t \in [t_0-\tau, t^*]} |u(t)|_H^2 + M \Big]. \end{aligned} \tag{3.7}$$

By Lemma 7.2 in [17, p.182] and (3.6), we find that

$$\begin{aligned} & \mathbb{E} \sup_{t_0 \le t \le t^*} | \int_{t_0}^{t} U(t,s) g(s,u_s) \mathrm{d}W(s)|_H^2 \\ \le\ & 4N^2(b) \mathbb{E} \Big[ \int_{t_0}^{t^*} |g(s,u_s)|_{L_2^0}^2 ds \Big] \\ \le\ & 8N^2(b) \mathbb{E} \int_{t_0}^{t^*} \Big[ |g(s,u_s)-g(s,0)|_{L_2^0}^2 + |g(s,0)|_{L_2^0}^2 \Big] \mathrm{d}s \\ \le\ & 8N^2(b) \Big[ L^2(t^*-t_0) \mathbb{E} \sup_{t \in [t_0-\tau, t^*]} |u(t)|_H^2 + M \Big]. \end{aligned} \tag{3.8}$$

Then (3.5)-(3.8) together imply that $\|\Gamma(u)\|_{\mathcal{H}_{t^*}} < +\infty$ if $u \in \mathcal{H}_{t^*}$. Thus $\Gamma$ maps $\mathcal{H}_{t^*}$ into $\mathcal{H}_{t^*}$.

Now we will show that $\Gamma$ has a unique fixed point. For any $u, v \in \mathcal{H}_{t^*}$, we have

$$\begin{aligned} & \mathbb{E} \sup_{t_0-\tau \le t \le t^*} \Big| \Gamma(u)(t) - \Gamma(v)(t) \Big|_H^2 \\ =\ & \mathbb{E} \sup_{t_0 \le t \le t^*} \Big| \int_{t_0}^{t} U(t,s)[f(s,u_s)-f(s,v_s)] \mathrm{d}s \\ & + \int_{t_0}^{t} U(t,s)[g(s,u_s)-g(s,v_s)] \mathrm{d}W(s) \Big|_H^2 \\ \le\ & 2\mathbb{E} \sup_{t_0 \le t \le t^*} \Big[ | \int_{t_0}^{t} U(t,s)[f(s,u_s)-f(s,v_s)] \mathrm{d}s|_H^2 \\ & + | \int_{t_0}^{t} U(t,s)[g(s,u_s)-g(s,v_s)] \mathrm{d}W(s)|_H^2 \Big] \\ \le\ & 2(t^*-t_0) N^2(b) \int_{t_0}^{t^*} \mathbb{E}|f(s,u_s)-f(s,v_s)|_H^2 \mathrm{d}s \\ & + 8\mathbb{E}[ \int_{t_0}^{t^*} |g(s,u_s)-g(s,v_s)|_{L_2^0}^2 \mathrm{d}s] \\ \le\ & 2L^2[N^2(b)(t^*-t_0)+4] \int_{t_0}^{t^*} \mathbb{E}\|u_s - v_s\|^2 \mathrm{d}s \\ \le\ & 2L^2(t^*-t_0)[N^2(b)(t^*-t_0)+4] \|u_t - v_t\|_{\mathcal{H}_{t^*}}^2, \end{aligned} \tag{3.9}$$

which shows that $\Gamma$ is contractive by using (3.3). Consequently then $\Gamma$ has a unique fixed point $u$ in $\mathcal{H}_{t^*}$, which is the unique mild solution of (2.1). Moreover, by induction, we can get that (2.1) has a mild solution on intervals $[t_0-\tau, t^*]$, $[t^*-\tau, 2t^*-t_0], \cdots$. Therefore, we have that (2.1) has a unique mild solution $u(t)$ for $t\in[t_0-\tau, b]$. □

**Theorem 3.1.** *Assume the condition (3.1) holds. If $f$ and $g$ satisfy the the local Lipschitz condition on $[t_0, T)$, then there must be $\bar{t}\in(t_0, T)$ such that (2.1) has a unique mild solution $u(t)$ for $t\in[t_0-\tau, \bar{t})$. Moreover, $u(t)$ explodes at $\bar{t}$ if $\bar{t}<T$. Otherwise, the solution $u(t)$ exists globally in $[t_0-\tau, T)$.*

**Proof.** Since $f$ and $g$ satisfy the local Lipschitz condition on the interval $[t_0, b]$, for any $n>0$, there exist positive constant $K_n$ such that

$$|f(t,\phi)-f(t,\psi)|_H\le K_n\|\phi-\psi\|, \quad |g(t,\phi)-g(t,\psi)|_{L_2^0}\le K_n\|\phi-\psi\|, \tag{3.10}$$

for all $t\in[t_0, b]$ and $\varphi, \psi\in C$ with $\|\varphi\|\vee\|\psi\|\le n$. For the above $n$, define functions $f_n$ and $g_n$ as follows:

$$f_n(t,u_t)=f(t,\frac{n\wedge\|u_t\|}{\|u_t\|}u_t), \;\; g_n(t,u_t)=g(t,\frac{n\wedge\|u_t\|}{\|u_t\|}u_t), \tag{3.11}$$

where we set $\frac{\|u_t\|}{\|u_t\|}=1$ when $u_t\equiv 0$. Then it is obvious that $f_n$ and $g_n$ satisfy the global Lipschitz condition (3.2) on $[t_0, b]\times C$. By Lemma 3.1, the following equation,

$$\begin{aligned} u_n(t)= \;\; & U(t,t_0)u_n(t_0)+\textstyle\int_{t_0}^t U(t,s)f_n(s,(u_n)_s)\mathrm{d}s \\ & +\textstyle\int_{t_0}^t U(t,s)g_n(s,(u_n)_s)\mathrm{d}W(s), \end{aligned} \tag{3.12}$$

$$(u_n)_{t_0}=u_n(t_0+s)=\left\{\begin{array}{ll} \xi, & \text{if } \|\xi\|\le n, \\ 0, & \text{if } \|\xi\|>n, \end{array}\right. \quad s\in[-\tau, 0], \tag{3.13}$$

has a unique mild solution $u_n(t)$ on $[t_0-\tau, b]$.

Define a sequence of stopping time $\delta_n$ by

$$\delta_n=b\wedge\inf\{t\in(t_0, b]: |u_n(t)|_H\ge n\}.$$

From (3.11) and (3.13), for $t\in[t_0,\delta_n]$, we have known that

$$\begin{aligned} f_{n+1}(s,(u_n)_s)=f_n(s,(u_n)_s)=f(s,(u_n)_s), \\ g_{n+1}(s,(u_n)_s)=g_n(s,(u_n)_s)=g(s,(u_n)_s). \end{aligned} \tag{3.14}$$

That is, (3.12) and the following equation for $t\in[t_0, b]$,

$$\begin{aligned} u_{n+1}(t)= \;\; & U(t,t_0)u_{n+1}(t_0)+\textstyle\int_{t_0}^t U(t,s)f_{n+1}(s,(u_{n+1})_s)\mathrm{d}s \\ & +\textstyle\int_{t_0}^t U(t,s)g_{n+1}(s,(u_{n+1})_s)\mathrm{d}W(s) \end{aligned}$$

have the same coefficients for $t\in[t_0,\delta_n]$ and their initial data overlap in $D=\{u_t\in C: \|u_t\|\le n\}$. Thus, by the similar proof of Theorem 5.2.1 [6], we can get that

$$u_{n+1}(t)=u_n(t), \; t\in[t_0-\tau,\delta_n], \;\; a.s..$$

This further implies that $\delta_n$ is increasing in $n$. So we can define $\delta=\lim_{n\to\infty}\delta_n$.

Now we suppose that

$$\Omega(t)=\{\omega\in\Omega: \delta\in[t_0, t]\subseteq[t_0, b]\}, \tag{3.15}$$

$$\bar{t}=\sup_{t_0\le t\le b}\{s\in[t_0, t]: \mathbb{P}(\Omega(s))=0\}. \tag{3.16}$$

From the definition of $\bar{t}$, there must be a sequence $\{t_k : t_k \in [\bar{t}, b]\}$ with $\lim_{k\to\infty} t_k = \bar{t}$ such that

$$\mathbb{P}(\Omega(t_k)) > 0. \tag{3.17}$$

For the above given $t_k$, we can choose an integer $N_{t_k}$ satisfying

$$N_{t_k} > N \triangleq \mathbb{E}\|\xi\|^2, \quad \text{and} \quad \mathbb{P}(\Omega(t_k))N_{t_k}^2 > N + 1.$$

Let

$$\Omega_n(t) = \{\sup_{t_0-\tau\leq s\leq t} |u_n(s)|_H > n\}.$$

Then we have $\lim_{n\to\infty} \Omega_n(t) = \Omega(t)$ and $\mathbb{P}(\Omega(t_k)) \leq \mathbb{P}(\Omega_{N_{t_k}}(t_k))$. So, we can get

$$\mathbb{E}(|u_{N_{t_k}}(\delta_{N_{t_k}})|_H^2) \geq \mathbb{E}(I_{\Omega_{N_{t_k}}} |u_{N_{t_k}}(\delta_{N_{t_k}})|_H^2) \geq \mathbb{P}(\Omega(t_k))N_{t_k}^2 > N + 1, \tag{3.18}$$

where $I_{(\cdot)}$ denote the indicator function of $(\cdot)$. If $\bar{t} = t_0$, noting that $t_0 \leq \delta_{N_{t_k}} \leq t_k$, we have

$$\delta_{N_{t_k}} \to t_0, \quad \text{when} \quad t_k \to t_0.$$

This together with (3.18) implies that

$$\mathbb{E}(|\xi(0)|_H^2) > N + 1 = \mathbb{E}\|\xi\|^2 + 1,$$

which contradicts the initial condition $\mathbb{E}\|\xi\|^2 = N$. Therefore, we get $\bar{t} > t_0$. For $\hat{t} \in [t_0, \bar{t})$, we have

$$\lim_{n\to\infty} \mathbb{P}(\Omega_n(t)) = 0 \ \text{ for } \ t \in [t_0, \hat{t}]. \tag{3.19}$$

For this case, we can prove $u(t)$ defined by

$$u(t,\omega) = \lim_{n\to\infty} u_n(t,\omega), \ \text{ if } \ \omega \notin \lim_{n\to\infty} \Omega_n(\hat{t}), \ \ \forall\, t \in [t_0, \hat{t}] \tag{3.20}$$

is the solution of (2.1). In fact, $u(t,\omega) = u_n(t,\omega)$ if $\omega \notin \Omega_n(\hat{t})$ for $t \in [t_0, \hat{t}]$, and we get

$$\|u_t\| \leq n \ \ a.s. \ \text{ for } \ \omega \notin \Omega_n(\hat{t}), \ t \in [t_0, \hat{t}]. \tag{3.21}$$

Combining with (3.21), we have for $\omega \notin \Omega_n(\hat{t})$,

$$f_n(t, (u_n)_t) = f(t, u_t), \ \ g_n(t, (u_n)_t) = g(t, u_t), \ a.s.\ \forall\, t \in [t_0, \hat{t}]. \tag{3.22}$$

From (3.12) and (3.22), we have for $\omega \notin \Omega_n(\hat{t})$ and $t \in [t_0, \hat{t}]$,

$$\begin{aligned} u_n(t) = \ & U(t, t_0)u_n(t_0) + \int_{t_0}^t U(t,s)f(s,(u_n)_s)\mathrm{d}s \\ & + \int_{t_0}^t U(t,s)g(s,(u_n)_s)\mathrm{d}W(s), \ \ a.s.. \end{aligned} \tag{3.23}$$

This implies $u(t) = u_n(t)$ for $\omega \notin \Omega_n(\hat{t})$, which is also the mild solution of (2.1) for $\omega \notin \Omega_n(\hat{t})$. Combining (3.19), we get $u(t)$ defined by (3.20) is the mild solution of (2.1) for all almost $\omega \in \Omega$ and $t \in [t_0, \hat{t}]$. From the arbitrariness of taking $\hat{t}$, the solution $u(t)$ exists in $[t_0 - \tau, \bar{t})$.

From the procedure of the above proof, it is obvious that the mild solution $u(t)$ explodes at $\bar{t}$ defined by (3.16) if there is a $t > 0$ such that $\bar{t} < t$ or $\bar{t} = t$ and $\mathbb{P}(\Omega(t)) > 0$. Otherwise, from the arbitrariness of choosing $t$, the mild solution $u(t)$ exists in $[t_0 - \tau, T)$. Then, the proof is completed. □

**Remark 3.1.** In Theorem 3.1, the condition (3.1) is necessary, for example, consider the following equation:

$$du(t) = [u_{xx} + f(t, u)]dt, \quad u(0) = u_0, \tag{3.24}$$

where $f(t, u) = u(t) + \alpha(t)$, $\alpha(t) = \frac{1}{t}$ for $t > 0$ and $\alpha(0) = 0$. It is obvious that $f$ in (3.24) satisfies the Lipschitz condition, but the condition (3.1) is not satisfied. However, (3.24) does not have a solution.

# 4. Global existence

Since the stochastic convolution in Definition 2.2 is no longer a martingale, we can not employ Itô's formula for mild solutions directly on most occasions of our arguments. We encounter a difficulty that we need strong solution in order to use Itô's formula. We can handle this problem, however, by introducing approximating systems with strong solutions and using a limiting argument. This idea has appeared in [10, 13] for semilinear stochastic evolution equations under the global Lipschitz condition. Motived by the works mentioned above, we shall establish corresponding results for IDSDEs under the local Lipschitz condition. By the approximation method, we can establish a useful result, which can be applied as the Itô's formula for the mild solution of (2.1). Finally we will obtain a global existence theorem for IDSDEs.

To this end, we introduce an approximating system of (2.1) as follows:

$$\begin{cases} \mathrm{d}u(t) = [A(t)u(t) + R(\lambda)f(t, u_t)]\mathrm{d}t + R(\lambda)g(t, u_t)\mathrm{d}W(t), & t \in [t_0, T), \\ u_{t_0}(s) = R(\lambda)\xi(s) \in D(A(t)), \quad s \in [-\tau, 0], \end{cases} \tag{4.1}$$

where $\lambda \in \rho(A(t))$, the resolvent set of $A(t)$ and $R(\lambda) = \lambda R(\lambda, A(t))$, $R(\lambda, A(t))$ is the resolvent operator of $A(t)$.

**Lemma 4.1.** *Under the hypothesis of Theorem 3.1, for any fixed $\lambda$, system (4.1) has a unique strong solution $u_t^\lambda \in D(A(t))$ in $t \in [t_0, \bar{t})$.*

**Proof.** By Theorem 3.1, system (4.1) has a unique mild solution $u_t^\lambda$ in $[t_0, \bar{t})$. Thus, it suffices to prove that the mild solution $u_t^\lambda$ is also a strong solution of system (4.1). By the closed graph theorem, $R(\lambda, A(t))$ is bounded. Then $A(t)R(\lambda) = \lambda^2 R(\lambda, A(t)) - \lambda I$ is a bounded operator. Thus, we have almost surely

$$\int_{t_0}^{\bar{t}} \int_{t_0}^{t} |A(t)U(t, r)R(\lambda)f(r, u_r^\lambda)|_H \mathrm{d}r\mathrm{d}t < \infty,$$

and

$$\int_{t_0}^{\bar{t}} \int_{t_0}^{t} \mathrm{Tr}((A(t)U(t, r)R(\lambda)g(r, u_r^\lambda))Q(A(t)U(t, r)R(\lambda)g(r, u_r^\lambda))^*)\mathrm{d}r\mathrm{d}t < \infty.$$

Thus, by the classic Fubini theorem, we have for $t \in [t_0, \bar{t})$

$$\begin{aligned} & \textstyle\int_{t_0}^{t} \int_{t_0}^{s} A(s)U(s, r)R(\lambda)f(r, u_r^\lambda)\mathrm{d}r\mathrm{d}s \\ = \; & \textstyle\int_{t_0}^{t} \int_{r}^{t} A(s)U(s, r)R(\lambda)f(r, u_r^\lambda)\mathrm{d}s\mathrm{d}r \\ = \; & \textstyle\int_{t_0}^{t} U(t, r)R(\lambda)f(r, u_r^\lambda)\mathrm{d}r - \int_{t_0}^{t} R(\lambda)f(r, u_r^\lambda)\mathrm{d}r. \end{aligned} \tag{4.2}$$

On the other hand, in view of stochastic Fubini theorem [17, Theorem 4.18], we also have

$$\begin{aligned} & \int_{t_0}^t \int_{t_0}^s A(s)U(s,r)R(\lambda)g(r,u_r^\lambda)\mathrm{d}W(r)\mathrm{d}s \\ = & \int_{t_0}^t \int_r^t A(s)U(s,r)R(\lambda)g(r,u_r^\lambda)\mathrm{d}s\,\mathrm{d}W(r) \\ = & \int_{t_0}^t U(t,r)R(\lambda)g(r,u_r^\lambda)\mathrm{d}W(r) - \int_{t_0}^t g(r,u_r^\lambda)\mathrm{d}W(r). \end{aligned} \tag{4.3}$$

Hence, $A(t)u^\lambda(t)$ is integrable almost surely for $t \in [t_0, \bar{t})$,

$$\begin{aligned} & \int_{t_0}^t A(s)u^\lambda(s)\mathrm{d}s \\ = & U(t,t_0)\xi(0) - \xi(0) + \int_{t_0}^t U(t,r)R(\lambda)f(r,u_r^\lambda)\mathrm{d}r - \int_{t_0}^t R(\lambda)f(r,u_r^\lambda)\mathrm{d}r \\ & + \int_{t_0}^t U(t,r)R(\lambda)g(r,u_r^\lambda)\mathrm{d}W(r) - \int_{t_0}^t R(\lambda)g(r,u_r^\lambda)\mathrm{d}W(r) \\ = & u^\lambda(t) - \xi(0) - \int_{t_0}^t R(\lambda)f(r,u_r^\lambda)\mathrm{d}r - \int_{t_0}^t R(\lambda)g(r,u_r^\lambda)\mathrm{d}W(r). \end{aligned}$$

So, the mild solution $u_t^\lambda \in D(A(t))$ is also a strong solution of (4.1) in $[t_0, \bar{t})$. □

Let $C^{1,2}([t_0,T) \times H; \mathbb{R}_+)$ denote the space of all real-valued functions $\Psi(t,x) : [t_0,T) \times H \to \mathbb{R}_+$ with properties:

(i) $\Psi(t,x)$ is differentiable in $t \in [t_0,T)$ and $\Psi_t(t,x)$ is continuous on $[t_0,T) \times H$,

(ii) $\Psi(t,x)$ is twice Frechet differentiable in $x$, $\Psi_x(t,x) \in H$ and $\Psi_{xx}(t,x) \in \mathcal{L}(H,H)$ are continuous on $[t_0,T) \times H$.

From Lemma 4.1, we can give a useful theorem, which can be applied as the Itô's formula for the mild solution of (2.1).

**Lemma 4.2.** *Under the hypothesis of Theorem 3.1, let $\Psi \in C^{1,2}([t_0,T) \times H; \mathbb{R}_+)$, $u(t)$ be the mild solution of (2.1) for $t \in [t_0, \bar{t})$, $\tau_n$ the random variable equal to the time at which the process $u(t)$ first leaves $U_n = \{|u|_H < n\}$, $\tau_n(t) = \tau_n \wedge t$. Then, for any $t_0 \le s \le t < \bar{t}$, it holds that*

$$\mathbb{E}\Big(\Psi(\tau_n(t), u(\tau_n(t)))\Big) = \mathbb{E}\Psi(s,u(s)) + \mathbb{E}\int_s^{\tau_n(t)} \mathcal{L}\Psi(r,u(r))\mathrm{d}r, \tag{4.4}$$

*where the operator $\mathcal{L}$ is defined by*

$$\begin{aligned} \mathcal{L}\Psi(t,u(t)) = \; & \Psi_t(t,u(t)) + \langle \Psi_u(t,u(t)), A(t)u(t) + f(t,u_t)\rangle_H \\ & + \tfrac{1}{2}\mathrm{Tr}\Big[\Psi_{uu}(t,u(t))g(t,u_t)Q\big(g(t,u_t)\big)^*\Big], \end{aligned}$$

*where $\langle \cdot, \cdot \rangle_H$ denotes the inner product of Hilbert space $H$.*

**Proof.** From Proposition 1.3.6 [13, P.26], there exists a subsequence $u^k(t)$ of the strong solution $u^\lambda(t)$ of (4.1) such that for any $t \in [t_0, \bar{t})$, $u^k(t) \to u(t)$ almost surely as $k \to \infty$, uniformly with respect to $[t_0, t]$, where $k \in \rho(A(t))$.

Let $\tau_n^k$ the random variable equal to the time at which the process $u^k(t)$ first leaves $U_n = \{|u|_H < n\}$ and $\tau_n^k(t) = \tau_n^k \wedge t$. It is well known that the process $y(t) = u^k(\tau_n^k(t))$, obtained by the stopping time has an Itô differential for $t \in [t_0, \bar{t})$

$$\mathrm{d}y(t) = I_{[t_0,\tau_n^k]}[A(t)y(t) + R(k)f(t,y_t)]\mathrm{d}t + I_{[t_0,\tau_n^k]}R(k)g(t,y_t)\mathrm{d}W(t). \tag{4.5}$$

Therefore, by using Itô formula to $\Psi$ and taking expectation, we can deduce that for any $t_0 \le s \le t < \bar{t}$,

$$\mathbb{E}\Big(\Psi(\tau_n^k(t), u^k(\tau_n^k(t)))\Big) = \mathbb{E}\Big(\Psi(s, u^k(s))\Big) + \mathbb{E}\int_s^{\tau_n^k(t)} \mathcal{L}_k\Psi(r, u^k(r))\mathrm{d}r,$$

where

$$\begin{aligned}\mathcal{L}_k\Psi(t,u) = \quad & \Psi_t(t,u) + \langle \Psi_u(t,u), A(t)u + R(k)f(t,u_t)\rangle_H \\ & + \tfrac{1}{2}\mathrm{Tr}\Big[\Psi_{uu}(t,u)R(k)g(t,u_t)Q\big(R(k)g(t,u_t)\big)^*\Big].\end{aligned}$$

Consequently, letting $k \to \infty$, we have $\tau_n^k(t) \to \tau_n(t)$ and (4.4) holds. □

By Lemma 4.2, we shall give some sufficient conditions on global existence without the continuity of $\mathbb{E}u(t)$.

**Theorem 4.1.** *Let the conditions of Theorem 3.1 hold. Suppose that there are functions $a(\cdot) \in \mathcal{N}^1([t_0,t];\mathbb{R}_+)$ and $b(\cdot) \in C([t_0,t];\mathbb{R}_+)$ for any $t \in [t_0,T)$, and $V \in C^{1,2}([t_0,T) \times H; \mathbb{R}_+)$ such that*

$$\lim_{|u|_H \to \infty} [\inf_{t_0 \le t < T} V(t,u(t))] = \infty, \tag{4.6}$$

$$\mathcal{L}V(t,u(t)) \le a(t) + b(t)V(t,u(t)), \quad \forall \ t \in [t_0,T), \tag{4.7}$$

*whenever $V(t+s, u(t+s)) \le V(t,u(t))$ for any $s \in [-\tau, 0]$. Then there exists a unique global mild solution $u(t)$ of (2.1). Moreover, there exists a function $L_0(t)$ with $\sup_{s\in[t_0,t]} L_0(s) < \infty$ for any $t \in [t_0,T)$ such that*

$$\mathbb{E}V(t,u(t)) \le e^{\int_{t_0}^t b(s)\mathrm{d}s} L_0(t). \tag{4.8}$$

**Proof.** Let

$$w(t,u(t)) = V(t,u(t))e^{-\int_{t_0}^t b(s)\mathrm{d}s}, \tag{4.9}$$

then from Condition (4.7) and the continuity of $b(t)$,

$$\mathcal{L}w(t,u(t)) \le e^{-\int_{t_0}^t b(s)\mathrm{d}s}[a(t) + b(t)V - b(t)V)] = a(t)e^{-\int_{t_0}^t b(s)\mathrm{d}s} \triangleq h(t), \tag{4.10}$$

whenever $V(t+s,u(t+s)) \le V(t,u(t))$ for any $s \in [-\tau,0]$.

Let $\bar{w}(t,u(t)) = \sup_{t_0-\tau \le s \le t} w(s,u(s))$ for $t \in [t_0,T)$, then there is a $s_0 = s_0(\omega) \in [t_0-\tau, t]$ such that $\bar{w}(t,u(t)) = w(s_0,u(s_0))$ and either $s_0 = t$ or $s_0 < t$ and $w(s,u(s)) < w(s_0,u(s_0))$ for $s_0 < s \le t$.

If $s_0 < t$, then for $h > 0$ sufficiently small $\bar{w}(t+h,u(t+h)) = \bar{w}(t,u(t))$ and $\mathcal{L}\bar{w}(t,u(t)) = 0$. If $s_0 = t$, then $\bar{w}(t,u(t)) = w(t,u(t))$, that is, $w(t,u(t)) \ge w(t+s,u(t+s))$ for any $s \in [-\tau,0]$, which implies that $V(t+s,u(t+s)) \le V(t,u(t))$ for any $s \in [-\tau,0]$. From (4.10), we get $\mathcal{L}\bar{w}(t,u(t)) = \mathcal{L}w(t,u(t)) \le h(t)$. Therefore, we have

$$\mathcal{L}\bar{w}(t,u(t)) \le h(t), \tag{4.11}$$

where $h(\cdot) \in \mathcal{N}^1([t_0,t];\mathbb{R}_+)$ for any $t \in [t_0,T)$, which can be implied by $a(\cdot) \in \mathcal{N}^1([t_0,t];\mathbb{R}_+)$ and $b(\cdot) \in C([t_0,t];\mathbb{R}_+)$ for any $t \in [t_0,T)$.

Therefore, by Lemma 4.2, we can get from (4.11) that

$$\begin{aligned} & \mathbb{E}\bar{w}(\tau_n(t), u((\tau_n(t)))) = \mathbb{E}\bar{w}(t_0,u(t_0)) + \mathbb{E}\textstyle\int_{t_0}^{\tau_n(t)} \mathcal{L}\bar{w}(s,u(s))\mathrm{d}s \\ \le \quad & e^{\int_{t_0-\tau}^{t_0} b(s)\mathrm{d}s}\mathbb{E}\sup_{-\tau\le s\le 0} V(t_0+s,u(t_0+s)) + \textstyle\int_{t_0}^t h(s)\mathrm{d}s \triangleq L_0(t). \end{aligned} \tag{4.12}$$

By the Chebyshev's inequality, we have for any $t \in (t_0, T)$

$$\mathbb{P}(\tau_n \le t) = \mathbb{P}(|u(\tau_n(t))|_H \ge n) \le \frac{L_0(t)}{\inf_{t\in[t_0,T),|u(t)|_H\ge n} V(t,u(t))e^{-\int_{t_0}^t b(s)\mathrm{d}s}}. \quad (4.13)$$

We claim that $\bar{t} = T$. Otherwise, $\bar{t} < T$ and the mild solution $u(t)$ explodes at $\bar{t}$ by Theorem 3.1. Letting $n \to \infty$, using $\sup_{s\in[t_0,t]} L_0(s) < \infty$ for any given $t \in [\bar{t}, T)$ and (4.13), we can get $\mathbb{P}(\delta \le t) = 0$, which is a contradiction. So $u(t)$ exists globally on $[t_0 - \tau, T)$. □

## 5. Examples

In this section, we shall present two examples in order to illustrate our results.

**Example 5.1.** Consider the stochastic delay reaction-diffusion equation

$$\begin{cases} \mathrm{d}u(t,x) = & [\frac{\partial^2}{\partial x^2}u(t,x) + \alpha(t)u(t-\tau,x) - \beta(t)u^3(t,x)]\mathrm{d}t \\ & +\gamma(t)\frac{u^2(t-\tau)}{1+|u(t-\tau)|}\mathrm{d}\omega(t), \qquad (t,x) \in \mathbb{R}_+ \times [0,\pi], \\ u(t,0) = & u(t,\pi) = 0, \qquad t \ge 0, \\ u(s,x) = & \xi(s,x), \qquad (t,x) \in [-\tau, 0] \times [0,\pi], \end{cases} \quad (5.1)$$

where $\omega(t)$ is a real standard Wiener process, $\tau \ge 0$ and $\alpha(t), \beta(t), \gamma(t)$ are positive continuous function.

Denote by $A = \frac{\partial^2}{\partial x^2}$ and $H = L^2(0,\pi)$, then $H$ is a Hilbert space with the inner product

$$\langle u, v\rangle_H = \int_0^\pi u(t,x)v(t,x)\mathrm{d}x,$$

and the norm

$$|u|_H = \{\langle u,u\rangle_H\}^{\frac{1}{2}} = \Big(\int_0^\pi |u(t,x)|^2\mathrm{d}x\Big)^{\frac{1}{2}}.$$

The operator $A$ has the domain

$$D(A) = \{u \in H : \frac{\partial u(x)}{\partial x}, \frac{\partial^2 u(x)}{\partial x^2} \in H, u(0) = u(\pi) = 0\}.$$

Then it is known that

$$\langle Au, u\rangle_H \le -|u|_H^2, \quad u \in D(A).$$

One can compute immediately that $A$ generates a $C_0$ semigroup $S(t)$ satisfying $\|S(t)\| \le e^{-t}$ for all $t \ge 0$. We can observe that the $C_0$ semigroup $S(t)$ leads to an evolution system $\{U(t,s)\}_{0\le s\le t}$ by means of the relation

$$U(t,s) = S(t-s), \quad 0 \le s \le t.$$

Let $f(t,u_t) = \alpha(t)u(t-\tau) - \beta(t)u^3(t)$ and $g(t,u_t) = \gamma(t)\frac{u^2(t-\tau)}{1+|u(t-\tau)|}$. Then system (5.1) can be written in the following abstract form

$$\begin{cases} \mathrm{d}u(t) = [Au(t) + f(t,u_t)]dt + g(t,u_t)\mathrm{d}W(t), \quad (t,x) \in [0,\infty) \times [0,\pi], \\ u(s,x) = \xi(s,x), \quad (t,x) \in [-\tau, 0] \times [0,\pi]. \end{cases} \quad (5.2)$$

Notice that $f$ and $g$ satisfy local and global Lipschitz conditions for the second arguments, respectively.

On the other hand, let $V(t,u(t)) = |u(t)|_H^2$. Then the operator $\mathcal{L}$ defined in Lemma 4.2 has the form

$$\begin{aligned} & \mathcal{L}V(t,u(t),u(t-\tau)) \\ \leq\ & -2|u(t)|_H^2 + 2\langle u(t), \alpha(t)u(t-\tau)\rangle_H - 2\langle u(t), \beta(t)u^3(t)\rangle_H + \gamma^2(t)|u(t-\tau)|_H^2 \\ \leq\ & (\tfrac{1}{2}\alpha^2(t) + \gamma^2(t))|u(t-\tau)|_H^2 \\ \leq\ & (\tfrac{1}{2}\alpha^2(t) + \gamma^2(t))V(t,u(t)), \end{aligned}$$

whenever $|u(t+s)|_H^2 \leq |u(t)|_H^2, \forall\, s \in [-\tau, 0]$. Then it follows from Theorem 4.1 that system (5.2) has a unique global mild solution.

**Example 5.2.** We consider a non-autonomous stochastic Lotka-Volterra competitive system with diffusion

$$\begin{aligned} & \mathrm{d}u_i(t,x) - \Delta u_i(t,x)\mathrm{d}t \\ =\ & u_i(t,x)[(b_i - \sum_{j=1}^{n} a_{ij}u_j(t,x))\mathrm{d}t + \sigma_i \mathrm{d}\omega(t)], \quad i = 1,\dots,n,\ x \in \mathcal{O}, t \geq 0, \end{aligned} \tag{5.3}$$

with boundary condition $\frac{\partial u_i}{\partial n} = 0$ on $\partial\mathcal{O}$ and initial value $u_i(0,x) = \psi_i(x)$, where $\mathcal{O}$ is a bounded open subset in $\mathbb{R}^n$, $\omega(t)$ is a one-dimensional standard Brownian motion, $b_i$, $a_{ij}$ and $\sigma_i$ are all nonnegative constants.

We can also write system (5.3) as follows

$$\mathrm{d}u - \Delta u \mathrm{d}t = f(u)\mathrm{d}t + g(u)\mathrm{d}\omega(t), \quad x \in \mathcal{O}, t \geq 0, \tag{5.4}$$

with boundary condition $\frac{\partial u}{\partial n} = 0$ on $\partial\mathcal{O}$ and initial value $u(0,x) = \psi(x)$, where $u = (u_1,\cdots,u_n)^T$, $\psi(x) = (\psi_1,\cdots,\psi_n)^T$, $b = (b_1,\cdots,b_n)^T$, $a = (a_{ij})_{n\times n}$, $\sigma = (\sigma_1,\cdots,\sigma_n)^T$, $f(u) = \mathrm{diag}(u_1,\cdots,u_n)(b - au)$, $g(u) = \mathrm{diag}(u_1,\cdots,u_n)\sigma$.

Let $H = L^2(\mathcal{O})$ be a Hilbert space with the inner product

$$\langle u, v\rangle_H = \int_{\mathcal{O}} u(t,x)\cdot v(t,x)\mathrm{d}x,$$

and the norm

$$|u|_H = \{\langle u,u\rangle_H\}^{\frac{1}{2}} = \Big(\int_{\mathcal{O}} |u(t,x)|^2 \mathrm{d}x\Big)^{\frac{1}{2}}.$$

Let $A_i^0$ be the operator on $\mathrm{Dom}(A_i^0) \subset H$ defined by

$$A_i^0 y_i = \Delta y_i,$$

where $y_i \in \mathrm{D}(A_i^0) = \{y_i \in W^{2,2}(\overline{\mathcal{O}}) | \frac{\partial y_i}{\partial n} = 0 \text{ on } \partial\mathcal{O}\}$. Then $A_i$, the closure of $A_i^0$, is a generator of a $C_0$ semigroup $\{S_i(t)\}_{t\geq 0}$ on $H$. And $A = (A_1,\cdots,A_n)^T$ is a generator of a $C_0$ semigroup $S(t) = (S_1(t),\cdots,S_n(t))^T$ on $H$.

Obviously, $f$ satisfies the local Lipschitz condition, $g$ satisfies the global Lipschitz condition. Furthermore $f(0) = g(0) = 0$. If $\psi \equiv 0$ $\mathbb{P}$-a.s. then $u \equiv 0$ is the unique solution of (5.4). By the comparison theorems for stochastic differential equations in infinite dimensions (see [7, Theorem 2.1]), we can get that the mild solution of (5.4) is nonnegative.

We define $V : H \to \mathbb{R}_+$ by

$$V(u) = |u|_H^2.$$

It is obvious that $V(u) \geq 0$ and $V(u) \to \infty$ as $|u|_H \to \infty$. We have

$$\begin{aligned} & \mathcal{L}V(u(t,x)) \\ = \; & \langle Au, 2u\rangle_H + \langle f(u), 2u\rangle_H + |g(u)|_H^2 \\ \leq \; & -2\int_{\mathcal{O}} \sum_{i=1}^n |\nabla u_i|^2 \mathrm{d}x + \int_{\mathcal{O}} \sum_{i=1}^n \Big(b_i u_i - u_i \sum_{j=1}^n a_{ij} u_j\Big) u_i \mathrm{d}x + \int_{\mathcal{O}} \sum_{i=1}^n \sigma_i^2 u_i^2 \mathrm{d}x \\ \leq \; & \int_{\mathcal{O}} \sum_{i=1}^n (b_i + \sigma_i^2) u_i^2 \mathrm{d}x \\ \leq \; & \hat{b} V(u), \end{aligned}$$

where $\hat{b} = \max_{1\leq i\leq n}\{b_i + \sigma_i^2\}$. Then, from Theorem 4.1, we can find that the mild solution of (5.3) exists on $[0, \infty)$.

## Acknowledgements

The authors would like to thank Dr. Lingying Teng, Shujun Long and Bing Li for their helpful comments and suggestions.

## References

[1] P. Balasubramaniam and S.K. Ntouyas, *Global existence for semilinear stochastic delay evolution equations with nonlocal conditions*, Soochow J. Math., 27(2001), 331-342.

[2] P. Balasubramaniama, J.Y. Park and A.V.A. Kumar, *Existence of solutions for semilinear neutral stochastic functional differential equations with nonlocal conditions*, Nonlinear Anal., 71(2009), 1049-1058.

[3] P.L. Chow, *Stochastic partial dierential equations*, Chapman & Hall/CRC, New York, 2007.

[4] C. Fefferman, *Existence and smoothness of the Navier-Stokes equations*, The Millennium Problems, 2000.

[5] H.B. Fu, D.M. Cao and J.Q. Duan, *A sufficient condition for non-explosion for a class of stochastic partial differential equations*, Interdisciplinary Mathematical Sciences, 8(2010), 131-142.

[6] A. Friedman, *Stochastic differential equations and applications*, Academic Press, New York, 1975.

[7] C. Geiß and R. Manthey, *Comparison theorems for stochastic differential equations in finite and infinite dimensions*, Stochastic Process Appl., 53(1994), 23-35.

[8] J.K. Hale and S.M.V. Lunel, *Introduction to functional differential equations*, Springer-Verlag, New York, 1993.

[9] D. Henry, *Geometric theory of semilinear parabolic equations*, Springer-Verlag, Berlin, 1981.

[10] A. Ichikawa, *Stability of semilinear stochastic evolution equations*, J. Math. Anal. Appl., 90(1982), 12-44.

[11] S. Kato, *Existence, uniqueness, and continuous dependence of solutions of delay-differential equations with infinite delays in a Banach space*, J. Math. Anal. Appl., 195(1995), 82-91.

[12] X.X. Liao and X.R. Mao, *Exponential stability of stochastic delay interval systems*, System Control Lett., 40(2000), 171-181.

[13] K. Liu, *Stability of infinite dimensional stochastic differential equations with applications*, Pitman Monographs Ser. Pure Appl. Math., 135, Chapman & Hall/CRC, 2006.

[14] R.C. McOwen, *Partial differential equations*, Pearson Education, New Jersey, 2003.

[15] H.W. Ning and B. Liu, *Local existence-uniqueness and continuation of solutions for delay stochastic evolution equations*, Appl. Anal., 88(2009), 563-577.

[16] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, New York, 1983.

[17] G. Da Prato and J. Zabczyk, *Stochastic equations in infinite dimensions*, Cambridge University Press, Cambridge, UK, 1992.

[18] A.G. Rutkas and L.A. Vlasenko, *Existence, uniqueness and continuous dependence for implicit semilinear functional differential equations*, Nonlinear Anal., 55(2003), 125-139.

[19] T. Taniguchi, K. Liu and A. Truman, *Existence, uniqueness, and asymptotic behavior of mild solution to stochastic functional differential equations in Hilbert spaces*, J. Differential Equations, 181(2002), 72-91.

[20] C.C. Travis and G.F. Webb, *Existence and stability for partial functional differential equations*, Trans. Amer. Math. Soc., 200(1974), 395-418.

[21] C.C. Travis and G.F. Webb, *Existence, stability and compactness in the $\alpha$-norm for partial functional differential equations*, Trans. Amer. Math. Soc., 240(1978), 129-143.

[22] L.S. Wang, Z. Zhang and Y.F. Wang, *Stochastic exponential stability of the delayed reaction-diffusion recurrent neural networks with Markovian jumping parameters*, Phys. Lett. A, 372(2008), 3201-3209.

[23] D.Y. Xu, Z.G. Yang and Y.M. Huang, *Existence-uniqueness and continuation theorems for stochastic functional differential equations*, J. Differential Equations, 245(2008), 1681-1703.

[24] D.Y. Xu, X.H. Wang and Z. G. Yang, *Further results on existence-uniqueness for stochastic functional differential equations*, Sci. China Ser. A, to appear.

[25] H.Y. Zhao and N. Ding, *Dynamic analysis of stochastic Cohen-Grossberg neural networks with time delays*, Appl. Math. Comput., 183(2006), 464-470.

[26] S.F. Zhou, F.Q. Yin and Z.G. Ouyang, *Random attractor for damped nonlinear wave equations with white noise*, SIAM J. Appl. Dynam. Sys., 4(2005), 883-903.