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EXISTENCE OF MULTIPLE LIMIT CYCLES IN CHEN SYSTEM*

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Abstract In this paper, the existence of multiple limit cycles for Chen system are investigated. By using the method of computing the singular point quantities, the simple and explicit parametric conditions can be determined to the number and stability of multiple limit cycles from Hopf bifurcation. Especially, at least 4 limit cycles can be obtained for the Chen system as a three-dimensional perturbed system.

Keywords Chen system, Hopf bifurcation, singular point quantities, center manifold.

MSC(2000) 34C23, 34C28, 34C40, 37G15.

1. Introduction

In this paper, we consider Chen system which is taken the following form

$$\begin{cases} \dot{x}_1 = a(x_2 - x_1), \\ \dot{x}_2 = (c-a)x_1 + cx_2 - x_1x_3, \\ \dot{x}_3 = x_1x_2 - bx_3, \end{cases} \tag{1.1}$$

where $a, b, c \in \mathrm{R}$. Since Lorenz found the first classical chaotic attractor in 1963, chaos has become one of the most interesting topics of research. In the past decades, extensive investigation on this topic has been carried out [1, 2, 5, 9]. Particularly, as the dual system of Lorenz model, since constructed by the authors of [3], Chen system have been given many investigations including the analysis of Hopf bifurcation (see [1, 3, 5, 8]). Nevertheless, some dynamics properties of Chen system have not been completely understood by mathematicians as well as Lorenz system's, for example, the multiple critical bifurcation. Here we investigate the stability and existence of multiple limit cycles by Hopf bifurcation, which is also helpful to make known the complete topological structure of the chaotic Chen's system.

In our process, not only under the condition: $a > 0, b > 0, c > 0$ in (1.1), but also for the case of general real parameter variables, the stability and critical bifurcation of Chen system are studied. Mainly the new method in [11] is applied to compute the singular point quantities of the fixed points for Chen system, which has been proved to be algebraic equivalent to the corresponding focal values. Thus when

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Hopf bifurcation being investigated, it is unnecessary to substitute an approximation form of the center manifold in order to obtain a two-dimensional system with center-focus type. The algorithm is linear and readily done with using computer symbol operation system such as Mathematica. In contrast to the more usual ones such as Liapunov functions-Poincaré normal form and integral averaging method (see [4]), it is convenient to compute the higher order focal values and solve the integrability at fixed point. And more our results are identical with and complementary to previous work on Hopf bifurcation in Chen system.

The rest of this paper is organized as follows. In Section 2, the corresponding singular point quantities are computed. In Section 3, the multiple Hopf bifurcations at the two symmetrical equilibria for the Chen system are investigated and an example of 4 limit cycles is given for the Chen system.

2. Singular point quantities of the equilibrium point

In this part, in order to present Chen system (1.1) with at least 4 limit cycles by Hopf bifurcation, we investigate the singular point quantities of the corresponding equilibrium point.

Evidently, Chen system always has the equilibrium $O(0,0,0)$. Suppose that $b\,(2c-a) > 0$ holds, for system (1.1) there exist another two fixed points $E_1 = (\sqrt{b(2c-a)}, \sqrt{b(2c-a)}, 2c-a)$ and $E_2 = (-\sqrt{b(2c-a)}, -\sqrt{b(2c-a)}, 2c-a)$. The equations in (1.1) are invariant under the transformation:

$$(x_1, x_2, x_3) \mapsto (-x_1, -x_2, x_3), \tag{2.1}$$

which means that if $(x_1(t), x_2(t), x_3(t))$ is a solution, then $(-x_1(t), -x_2(t), x_3(t))$ is a solution too. Therefore it is enough to analyze the stability and bifurcation of E_1.

On the other hand, we will consider the case of fixed points with the property: there is a pair of purely imaginary eigenvalues, that is, the equilibrium can undergo a generic Hopf bifurcation. However, one can figure out easily the Jacobian matrix of system (1.1) at the fixed point O as follows

$$Ao = \begin{pmatrix} -a & a & 0 \\ c-a & c & 0 \\ 0 & 0 & -b \end{pmatrix},$$

which never has a pair of purely imaginary eigenvalues. Therefore, it is unnecessary to discuss Hopf bifurcation for the fixed point O.

Next, we consider the Jacobian matrix of system (1.1) at the fixed point E_1 as follows

$$A = \begin{pmatrix} -a & a & 0 \\ -c & c & -\sqrt{b(2c-a)} \\ \sqrt{b(2c-a)} & \sqrt{b(2c-a)} & -b \end{pmatrix}.$$

To satisfy the necessary eigenvalue condition, namely A has a pair of purely imaginary eigenvalues $\pm\mathbf{i}\,\omega(\omega > 0)$, we need $b = (c^2 + 3ac - 2a^2)/c$, then

$$\omega = \sqrt{c^2 + 3ac - 2a^2}. \tag{2.2}$$

And more by transforming the equilibrium E_1 to the origin, setting $\mathbf{x} = (x_1, x_2, x_3) = (\tilde{x}_1 + \sqrt{b(2c-a)}, \tilde{x}_2 + \sqrt{b(2c-a)}, \tilde{x}_3 + (2c-a))$, then system (1.1) takes the form

$$\dot{\mathbf{x}} = A\begin{pmatrix} x_1 + \sqrt{b(2c-a)} \\ x_2 + \sqrt{b(2c-a)} \\ x_3 + (2c-a) \end{pmatrix} + \begin{pmatrix} 0 \\ -(x_1 + \sqrt{b(2c-a)})(x_3 + (2c-a)) \\ (x_1 + \sqrt{b(2c-a)})(x_2 + \sqrt{b(2c-a)}) \end{pmatrix}. \quad (2.3)$$

Here we used x_i instead of $\tilde{x}_i$ for $i = 1, 2, 3$. Thus one can construct a matrix P which transforms A to be a block-diagonal one, i.e. using the nondegenerate transformation $\mathbf{x} = P\,\mathbf{y}$, such that

$$P^{-1}AP = \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & -2ad_1/c \end{pmatrix},$$

where $\mathbf{y} = (y_1, y_2, y_3)$ and

$$P = \begin{pmatrix} \frac{a(a-c)}{d_0\sqrt{cd_1}} & -\frac{a\omega}{d_0\sqrt{cd_1}} & -\frac{c}{2\omega}\sqrt{\frac{c}{d_1}} \\ \frac{c^2+2ac-a^2}{d_0\sqrt{cd_1}} & \frac{\omega}{d_0}\sqrt{\frac{c}{d_1}} & \frac{(3c-2a)\omega}{2}\sqrt{\frac{c}{d_1}} \\ 0 & 1 & 1 \end{pmatrix}$$

and $d_0 = a + c$, $d_1 = 2c - a$.

From the above conditions, one can also get comprehensively that: $b(2c-a) > 0$, $bc > 0$ and $2a^2 - 3ac - c^2 < 0$, i.e.,

$$c > 0, b > 0, \frac{(3-\sqrt{17})\,c}{4} < a < \frac{(3+\sqrt{17})\,c}{4} \quad \text{or} \quad c < 0, b < 0, \frac{(3+\sqrt{17})\,c}{4} < a < \frac{(3-\sqrt{17})\,c}{4} \quad (2.4)$$

must be all satisfied for Hopf bifurcation of E_i. It is useful to the following discussion.

Moreover, after a time scaling: $t \to t/\omega$, we can get a new system from the system (2.3):

$$\begin{aligned} \dot{\mathbf{y}} &= \tfrac{1}{\omega}[P^{-1} * \mathrm{diag}(P\mathbf{y} + E)AP\mathbf{y}] \\ &= \tfrac{1}{\omega}[P^{-1}AP\ \mathbf{y} + P^{-1} * \mathrm{diag}(P\ \mathbf{y})AP\ \mathbf{y}]. \end{aligned} \quad (2.5)$$

In order to investigate Hopf bifurcation of system (2.5), firstly by means of transformation: $y_1 = (z+w)/2$, $y_2 = (z-w)\mathbf{i}/2$, $y_3 = u$, $t = -T\,\mathbf{i}$, the system (2.5) can also be transformed into the following form

$$\begin{cases} \frac{\mathrm{d}z}{\mathrm{d}T} = z + a_{101}uz + a_{011}uw + a_{110}zw + a_{200}z^2 + a_{020}w^2 + a_{002}u^2 = Z, \\ \frac{\mathrm{d}w}{\mathrm{d}T} = -(w + b_{011}uz + b_{101}uw + b_{110}zw + b_{020}z^2 + b_{200}w^2 + b_{002}u^2) = -W, \\ \frac{\mathrm{d}u}{\mathrm{d}T} = d_{001}\,u + d_{101}uz + d_{011}uw + d_{110}zw + d_{200}z^2 + d_{020}w^2 + d_{002}u^2 = U, \end{cases} \quad (2.6)$$

where $u \in \mathrm{R}$, $z, w, T \in \mathrm{C}$, and

$$
\begin{aligned}
a_{200} &= -\tfrac{a(2a^3+a^2c-11ac^2-2c^3)}{2d_0d_2} + \tfrac{a(2a^4-8a^3c+9a^2c^2+ac^3-2c^4)}{2d_0d_2\omega}\mathbf{i},\\
a_{020} &= \tfrac{a(2a^5+7a^4c-38a^3c^2+27a^2c^3+10ac^4+4c^5)}{2d_0^2d_1d_2} - \tfrac{a^2(2a^5+10a^4c-59a^3c^2+64a^2c^3-ac^4-4c^5)}{2d_0^2d_1d_2\omega}\mathbf{i},\\
a_{002} &= -\tfrac{c^2(8a^4-24a^3c+6a^2c^2+17ac^3+7c^4)}{8d_1d_2\omega^2} + \tfrac{c^2(2a^2-4ac+c^2)(4a^3-8a^2c-ac^2+3c^3)}{8d_1d_2\omega^3}\mathbf{i},\\
a_{101} &= \tfrac{4a^5-12a^4c+4a^3c^2+9a^2c^3-2ac^4-c^5}{2d_2\omega^2} + \tfrac{a(4a-5c)c^2}{2d_2\omega}\mathbf{i},\\
a_{011} &= -\tfrac{(a^2-ac-c^2)(12a^5-44a^4c+40a^3c^2+a^2c^3-c^5)}{2d_0d_1d_2\omega^2} - \tfrac{(a^2-ac-c^2)(8a^4-24a^3c+16a^2c^2+ac^3+c^4)}{2d_0d_1d_2\omega}\mathbf{i},\\
a_{110} &= -\tfrac{a(6a^3-9a^2c-4ac^2-c^3)}{d_0d_2} + \tfrac{a^2(6a^3-14a^2c+5ac^2+c^3)}{d_0d_2\omega}\mathbf{i},\\
b_{kjl} &= \bar{a}_{kjl} \ \ (kjl = 200, 020, 002, 101, 011, 110),\\
d_{200} &= \tfrac{2a^2(2a^4-10a^3c+9a^2c^2+4ac^3-c^4)}{d_0^2d_2c} - \tfrac{2a^2(a^3-6a^2c+4ac^2+3c^3)\omega}{d_0^2d_2c}\mathbf{i},\\
d_{020} &= -\tfrac{2a^2(2a^4-10a^3c+9a^2c^2+4ac^3-c^4)}{d_0^2d_2c} - \tfrac{2a^2(a^3-6a^2c+4ac^2+3c^3)\omega}{d_0^2d_2c}\mathbf{i},\\
d_{002} &= -\tfrac{a(4a-5c)c^2}{2d_2\omega}\mathbf{i},\\
d_{101} &= -\tfrac{a(4a^3-10a^2c+7ac^2+c^3)}{d_0d_2} - \tfrac{2a(a-c)(4a^3-7a^2c+c^3)}{d_0d_2\omega}\mathbf{i},\\
d_{011} &= \tfrac{a(4a^3-10a^2c+7ac^2+c^3)}{d_0d_2} - \tfrac{2a(a-c)(4a^3-7a^2c+c^3)}{d_0d_2\omega}\mathbf{i},\\
d_{110} &= \tfrac{4a^2(a^2-ac-c^2)\omega}{cd_0d_2}\mathbf{i},\\
d_{001} &= \tfrac{2ad_1}{c\omega}\mathbf{i},
\end{aligned}
$$

where $d_2 = 4a^4 - 16a^3c + 14a^2c^2 + 3ac^3 + c^4$.

According to Theorem 3.1 in [11], we have

Theorem 2.1. *For the system (2.6), setting* $c_{110} = 1, c_{101} = c_{011} = c_{200} = c_{020} = 0, c_{kk0} = 0, k = 2, 3, \cdots$, *we can derive successively and uniquely the terms of the following formal series:*

$$F(z, w, u) = zw + \sum_{\alpha+\beta+\gamma=3}^{\infty} c_{\alpha\beta\gamma} z^{\alpha} w^{\beta} u^{\gamma} \tag{2.7}$$

such that

$$\frac{\mathrm{d}F}{\mathrm{d}T} = \frac{\partial F}{\partial z}Z - \frac{\partial F}{\partial y}W + \frac{\partial F}{\partial u}U = \sum_{m=1}^{\infty} \mu_m (zw)^{m+1} \tag{2.8}$$

and if $\alpha \neq \beta$ *or* $\alpha = \beta, \gamma \neq 0$, $c_{\alpha\beta\gamma}$ *is determined by following recursive formula:*

$$
\begin{aligned}
c_{\alpha\beta\gamma} = \ & \tfrac{1}{\beta-\alpha-\gamma d_{001}}[d_{200}(1+\gamma)c_{\alpha-2,\beta,\gamma+1} - b_{020}(1+\beta)c_{\alpha-2,\beta+1,\gamma}\\
&+ d_{110}(1+\gamma)c_{\alpha-1,\beta-1,\gamma+1} - (a_{200} + b_{110}\beta - a_{200}\alpha - d_{101}\gamma)c_{\alpha-1,\beta,\gamma}\\
&- b_{011}(1+\beta)c_{\alpha-1,\beta+1,\gamma-1} + d_{020}(\gamma+1)c_{\alpha,\beta-2,\gamma+1}\\
&+ (b_{200} - b_{200}\beta + a_{110}\alpha + d_{011}\gamma)c_{\alpha,\beta-1,\gamma}\\
&- (d_{002} + b_{101}\beta - a_{101}\alpha - d_{002}\gamma)c_{\alpha,\beta,\gamma-1}\\
&- b_{002}(1+\beta)c_{\alpha,\beta+1,\gamma-2} + a_{020}(1+\alpha)c_{\alpha+1,\beta-2,\gamma}\\
&+ a_{011}(1+\alpha)c_{\alpha+1,\beta-1,\gamma-1} + a_{002}(1+\alpha)c_{\alpha+1,\beta,\gamma-2}]
\end{aligned}
$$

and for any positive integer m, μ_m *is determined by the following recursive formula:*

$$
\begin{aligned}
\mu_m = \ & d_{200}c_{m-2,m,1} - b_{020}(1+m)c_{m-2,m+1,0} + d_{110}c_{m-1,m-1,1}\\
&- (a_{200} - a_{200}m + b_{110}m)c_{m-1,m,0} + d_{020}c_{m,m-2,1}\\
&+ (b_{200} + a_{110}m - b_{200}m)c_{m,m-1,0} + a_{020}(1+m)c_{m+1,m-2,0},
\end{aligned}
$$

where $c_{110} = 1, c_{101} = c_{011} = c_{200} = c_{020} = 0, c_{kk0} = 0$, $k = 2, 3, \cdots$.

Definition 2.1 ([11])**.** For the system (2.6), μ_m in (2.8) is called the m-th singular point quantity at the origin, $m = 1, 2, \cdots$.

Lemma 2.1 ([11])**.** *For system (2.6), the singular point quantity* μ_m *is algebraic equivalent to the m-th focal value* v_{2m+1} *at the origin of system (2.5), i.e., for any positive integer* $m = 2, 3, \cdots$, *if* $v_3 = v_5 = \cdots = v_{2m-1} = 0$ *and* $\mu_1 = \mu_2 = \cdots = \mu_{m-1} = 0$ *hold, the* $v_{2m+1} = \mathbf{i}\pi\mu_m$.

Now applying the recursive formulas in Theorem 2.1 in the Mathematica symbolic computation system, we can obtain the first two singular point quantities easily:

$$\begin{aligned} \mu_1 &= \mathbf{i}\, a^2(a-c)\omega f_1/(d_0 d_1 d_2 d_3), \\ \mu_2 &= \mathbf{i}\, a^3(a-c) f_2/(3\omega d_0^3 d_1^3 d_2^3 d_3^2 d_4), \end{aligned} \tag{2.9}$$

where $d_3 = a^4 - 4a^3c + 2a^2c^2 + 3ac^3 + c^4$, $d_4 = 4a^3 - 20a^2c + 18ac^2 + 9c^3$ and

$$\begin{aligned} f_1 = \ & (a-4c)(2a^3 - 2a^2c - 2ac^2 - c^3), \\ f_2 = \ & 1568a^{22} - 36128a^{21}c + 356480a^{20}c^2 - 1941440a^{19}c^3 + 6178504a^{18}c^4 \\ & -10564816a^{17}c^5 + 4177594a^{16}c^6 + 18011318a^{15}c^7 - 28847216a^{14}c^8 \\ & -1501417a^{13}c^9 + 34159380a^{12}c^{10} - 12029641a^{11}c^{11} - 21438405a^{10}c^{12} \\ & +6477957a^9c^{13} + 11655208a^8c^{14} - 207621a^7c^{15} - 3893740a^6c^{16} \\ & -1373755a^5c^{17} + 252636a^4c^{18} + 333003a^3c^{19} + 112625a^2c^{20} \\ & +18978ac^{21} + 1404c^{22}. \end{aligned}$$

In fact, if letting $\mu_1 = 0$, then one can simplify μ_2 as follows

$$\mu_2 = 9\,\mathbf{i}\, a^3(a-c)\, g_2/(4\omega d_0^3 d_1^3 d_2^3 d_3^2 d_4), \tag{2.10}$$

where $g_2 = c^{19}(29410863522a^3 - 29410865128a^2c - 29410861563ac^2 - 14705430308c^3)$.

Similarly if letting $\mu_1 = \mu_2 = 0$ continuously, then by computing, we can get easily

$$\mu_3 = \mu_4 = \cdots = \mu_{20} = 0. \tag{2.11}$$

Thus from above singular point quantities $\mu_i, i = 1, 2$, and Lemma 2.1, then we have

Theorem 2.2. *For the flow on center manifold of the system (2.5), the first 2 focal values of the origin as follows*

$$v_3 = \mathbf{i}\pi\mu_1, \qquad v_5 = \mathbf{i}\pi\mu_2,$$

in the above expression of v_5, *we have let* $v_3 = 0$.

3. Hopf bifurcation of the two symmetrical equilibriums

In this section, we present there exist at least 4 limit cycles by Hopf bifurcation for Chen system. That is, if we change the parameters of system (1.1) slightly, the two symmetrical equilibriums will undergo a generic Hopf bifurcation, i.e. E_1 and E_2 are all surrounded by two smaller limit cycles respectively.

Firstly whether or not the first two focal values vanish simultaneously is investigated. The equations $\mu_1 = 0$ and $\mu_2 = 0$ are coupled, then we obtain only one case: $a - c = 0$ because of guaranteeing the equilibrium property, $a \neq 0$ can not hold. Meanwhile we consider $\mu_1 = 0$ and $\mu_2 \neq 0$, i.e., $f_1 = 0, g_2 \neq 0$, and from the conditions in (2.4), then $a = 4c$ should be excluded. Therefore, we have

Lemma 3.1. *The origin of system (2.5) or the equilibrium $E_i\,(i = 1, 2)$ of system (1.1) is a weak focus of order 2 if and only if*

$$2a^3 - 2a^2c - 2ac^2 - c^3 = 0, \text{ i.e. } a = \kappa_0 c, \tag{3.1}$$

where $\kappa_0 = \frac{1}{6}[2 + (98 + 18\sqrt{17})^{1/3} + 16(98 + 18\sqrt{17})^{-1/3}] \approx 1.73991.$

Now we can apply the method of constructing limit cycles to investigate the bifurcation of limit cycles from the origin for perturbed system (2.5), which has been introduced in [6, 12] (also see [7, 10]). Starting from the critical values (c^*, a^*) which satisfies the condition (3.1) from Lemma 3.1, we can show the existence of 4 small limit cycles for the generic case of Chen system. Since the proof is similar to the previous ones but rather tedious, it is omitted here.

Theorem 3.1. *There exist certain perturbed coefficients of system (2.6) such that there are 2 limit cycles in a small enough neighborhood of the origin for system (2.5).*

Moveover, we note that, due to the nondegenerate transformation: $\mathbf{x} = \mathrm{P}\,\mathbf{y}$ and the time re-scaling: $t \to t/\omega$, system (2.5) is topological equivalent to system (2.3) in the vicinity of the origin. In particular, if the system (2.5) has 2 small amplitude limit cycles in the vicinity of the origin, then the same is true for the origin of system (2.3) or for the equilibrium E_1 of system (1.1). At the same time, because of the symmetry, the system (1.1) also has 2 limit cycles in the vicinity of the equilibrium E_2 under the above assumption.

Therefore, we have the following conclusion.

Theorem 3.2. *For the Chen system (1.1), under the conditions of Theorem 3.1, there exist 4 small amplitude limit cycles by Hopf bifurcation.*

Remark 3.1. (i) If $\mu_1 \neq 0$ holds in (2.9), then E_i is a weak focus of order 1, we can also take a perturbation to guarantee that one small limit cycle is generated by Hopf bifurcation at the origin of system (2.5). Namely, when $v_3 = \mathbf{i}\pi\mu_1 > 0$, the limit cycle is subcritical for each $E_i\,(i = 1, 2)$ of system (1.1). For the contrary case, it is supercritical.

(ii) In fact, for the Chen system (1.1), the above Hopf bifurcation is investigated under generic condition whether or not the three parameters a, b and c are all confined to be positive.

4. Conclusions

In summary, we have investigated more deeply Hopf bifurcation of Chen system by a new method based on precise symbolic computation, and obtained its singular point quantities and at least 4 limit cycles. Our results and many previous work on Chen system are complementary.

In addition, when the first two singular point quantities disappear, the first all twenty ones of equilibrium E_i will disappear in the section 3. Thus we have reason to think that the highest order of the weak focus E_i is 2, which means there exist only 4 small limit cycles at most by Hopf bifurcation for Chen system. But it still need further proof. Another question is whether there exists a logical relation because the conclusion in Theorem 3.2 for Chen system is identical with the one for Lorenz system reported in a forthcoming paper.

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