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PRINCIPAL AND NONPRINCIPAL SOLUTIONS OF IMPULSIVE DYNAMIC EQUATIONS WITH APPLICATIONS*

Weina Li, Qiaoluan Li†, Xiaojing Liu and Feifei Cui

Abstract In this paper, we introduce the concept of principal and nonprincipal solutions for second order impulsive dynamic equations on time scales. Polya and Trench Factorizations play an important role in this article. Firstly we establish these factorizations. Using these factorizations, we establish some new oscillation criteria for second impulsive dynamic equations on time scales.

Keywords Principal, nonprincipal, dynamic equation, impulse, oscillation.

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1. Introduction

The theory of time scales was introduced by Hilger [7] in his Ph.D thesis in 1988 in order to unify continuous and discrete analysis, where a time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. And the new theory of the so-called "dynamic equations" extends these classical cases to cases "in between", as e.g., to the so-called q-difference equations. Of course many other interesting time scales exist, and they give rise to plenty of applications. The theory of dynamic equations on time scales has been developing rapidly and has received much attention. We refer the reader to the book by Bohner & Peterson [2] and the references cited therein.

The concept of the principal solution was introduced in 1936 by Leighton & Morse [8]. Since then the principal and nonprincipal solutions have been used successfully in connection with oscillation, see Bohner & Peterson [2], Özbekler & Zafer [9] , Zafer [10] and the references cited therein.

In recent years, impulsive dynamic equations on time scales have been investigated by Belarbi et al. [1], Benchohra et al. [3], [4], [5], Huang & Feng [6] and so forth. Principal and nonprincipal solutions of impulsive differential equations with application have been investigated by few authors and they gained some results, see Özbekler & Zafer [9] and Zafer [10]. In the present work, using the similar method we continue our investigation to extend the work in Özbekler & Zafer [9] and Zafer [10] to second order impulsive dynamic equations on scale times.

†the corresponding author. Email address:qll71125@163.com(Q. Li)
College of Math. & Info. Sci., Hebei Normal University, Shijiazhuang, 050024, China

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Denote by $\mathbf{PLC}[t_0,\infty)_{\mathbb{T}}$ the set of functions $x:[t_0,\infty)_{\mathbb{T}}\to\mathbb{R}$ such that x is continuous on each interval (θ_i,θ_{i+1}), $h(\theta_i^{\pm})$ exist, and $h(\theta_i)=h(\theta_i^-)$ for $i\in\mathbb{N}$.

Throughout the remainder of the paper, we assume that for each $i=1,2,\dots,$ the points of impulses θ_i are right dense(rd for short). We let $\mathbb{T}$ be a time scale with $\sup\mathbb{T}=\infty$, fix $t_0\in\mathbb{T}$ and define $\mathbb{T}_{t_0}=\mathbb{T}\bigcap[t_0,\infty)$.

In this paper we are concerned with oscillation of solutions of second-order impulsive dynamic equations of the form

$$\begin{aligned} &(r(t)z^{\Delta}(t))^{\Delta}+q(t)z^{\sigma}(t)=0, && t\neq\theta_i,\\ &\Delta r(t)z^{\Delta}(t)+q_iz^{\sigma}(t)=0, && t=\theta_i, \end{aligned}\tag{1.1}$$

and

$$\begin{aligned} &(r(t)z^{\Delta}(t))^{\Delta}+q(t)z^{\sigma}(t)=f(t), && t\neq\theta_i,\\ &\Delta r(t)z^{\Delta}(t)+q_iz^{\sigma}(t)=f_i, && t=\theta_i, \end{aligned}\tag{1.2}$$

where $r(t)>0$ and r,q,f are rd-continuous. Here we introduce the space

$$\begin{aligned}\mathbb{D}:=\ &\{x\mid x:\mathbb{T}_{t_0}\to\mathbb{R}\text{ such that }x^{\Delta}:\mathbb{T}_{t_0}^{\kappa}\to\mathbb{R}\text{ is continuous and such that}\\ &(rx^{\Delta})^{\Delta}:\mathbb{T}_{t_0}^{\kappa^2}\to\mathbb{R}\text{ is rd-continuous and }x^{\Delta},\ (rx^{\Delta})^{\Delta}\in\mathbf{PLC}[t_0,\infty)_{\mathbb{T}}\}.\end{aligned}$$

Definition 1.1. A function $x(t)\in\mathbb{D}$ is said to be a solution of (1.1) or (1.2) provided $x(t)$ satisfies (1.1) or (1.2).

Definition 1.2. We say that a solution $x(t)$ of (1.1) or (1.2) has a generalized zero at t if $x(t)=0$ or t is left-scattered and $r(\rho(t))x(\rho(t))x(t)<0$.

This paper is organized as follows. In Section 2, we give some preliminaries and lemmas. In section 3, the main result concerning the existence of principal and nonprincipal solution of (1.1) is given, the proof is based on Polya and Trench Factorizations. And the section also contains two important applications, namely Wong and Leighton-Wintner theorems. An example is given to illustrate the relevance of the results.

2. Preliminaries and lemmas

Consider the linear operators

$$\begin{aligned} &\mathbf{L}z:=(r(t)z^{\Delta}(t))^{\Delta}+q(t)z^{\sigma}(t)=0, && t\neq\theta_i,\\ &\mathbf{I}z:=\Delta r(t)z^{\Delta}(t)+q_iz^{\sigma}(t)=0, && t=\theta_i. \end{aligned}\tag{2.1}$$

Let $W(\mu,\eta)=\mu\eta^{\Delta}-\mu^{\Delta}\eta$ denote the Wronskian of $\mu,\eta\in\mathbb{D},t\in\mathbb{T}_{t_0}^{\kappa}$. Then for $t\in\mathbb{T}_{t_0}^{\kappa^2}$, we have

$$\begin{aligned}\mu^{\sigma}\mathbf{L}\eta-\eta^{\sigma}\mathbf{L}\mu\ &=\mu^{\sigma}\{(r\eta^{\Delta})^{\Delta}+q\eta^{\sigma}\}-\eta^{\sigma}\{(r\mu^{\Delta})^{\Delta}+q\mu^{\sigma}\}\\ &=\mu^{\sigma}(r\eta^{\Delta})^{\Delta}-\eta^{\sigma}(r\mu^{\Delta})^{\Delta}\\ &=(\mu r\eta^{\Delta}-\eta r\mu^{\Delta})^{\Delta}\\ &=\{rW(\mu,\eta)\}^{\Delta},\qquad t\neq\theta_i,\end{aligned}\tag{2.2}$$

and

$$\begin{aligned}\mu^\sigma \mathbf{I}\eta - \eta^\sigma \mathbf{I}\mu &= \mu^\sigma(\Delta r\eta^\Delta + q_i\eta^\sigma) - \eta^\sigma(\Delta r\mu^\Delta + q_i\mu^\sigma)\\ &= \mu^\sigma \Delta r\eta^\Delta - \eta^\sigma \Delta r\mu^\Delta\\ &= \Delta r(\mu^\sigma\eta^\Delta - \eta^\sigma\mu^\Delta)\\ &= \Delta r W(\mu,\eta), \qquad t=\theta_i.\end{aligned} \tag{2.3}$$

Lemma 2.1. *(Polya Factorization). If* (2.1) *has a solution* $v(t)$ *with no generalized zeros in* $\mathbb{T}_{t_0}$*, then for any* $\eta \in \mathbb{D}$ *we have*

$$\begin{aligned}\mathbf{L}\eta &= \rho_1^\sigma\{\rho_2(\rho_1\eta)^\Delta\}^\Delta, \quad t\neq\theta_i,\ \ t\in\mathbb{T}_{t_0}^{\kappa^2},\\ \mathbf{I}\eta &= \rho_1^\sigma \Delta\rho_2(\rho_1\eta)^\Delta, \qquad t=\theta_i,\ \ t\in\mathbb{T}_{t_0}^{\kappa^2},\end{aligned} \tag{2.4}$$

where $\rho_1(t) := 1/v(t)$ *and* $\rho_2(t) := r(t)v(t)v^\sigma(t)$*.*

Proof. Assume that $v(t)$ is a solution of (2.1) with no generalized zeros in $\mathbb{T}_{t_0}$. Then

$$\rho_2(t) = r(t)v(t)v^\sigma(t) > 0, \quad t\in\mathbb{T}_{t_0}^{\kappa}$$

and

$$\mathbf{L}v \equiv 0, \qquad t\neq\theta_i, \quad t\in\mathbb{T}_{t_0}^{\kappa^2},$$

$$\mathbf{I}v \equiv 0, \qquad t=\theta_i, \quad t\in\mathbb{T}_{t_0}^{\kappa^2}.$$

Taking $\mu(t) = v(t)$, we obtain

$$\begin{aligned}\mathbf{L}\eta &= \frac{1}{v^\sigma}\{rW(v,\eta)\}^\Delta = \frac{1}{v^\sigma}(rv\eta^\Delta - rv^\Delta\eta)^\Delta\\ &= \frac{1}{v^\sigma}\{rvv^\sigma(\frac{\eta}{v})^\Delta\}^\Delta = \rho_1^\sigma\{\rho_2(\rho_1\eta)^\Delta\}^\Delta,\end{aligned}$$

and

$$\begin{aligned}\mathbf{I}\eta &= \frac{1}{v^\sigma}\Delta r(v^\sigma\eta^\Delta - \eta^\sigma v^\Delta) = \frac{1}{v^\sigma}\Delta r(v\eta^\Delta - \eta v^\Delta)\\ &= \frac{1}{v^\sigma}\Delta rvv^\sigma(\frac{\eta}{v})^\Delta = \rho_1^\sigma\Delta\rho_2(\rho_1\eta)^\Delta,\end{aligned}$$

for all $\eta\in\mathbb{D}$. We complete the proof. □

Lemma 2.2. *(Trench Factorization). If* (2.1) *has a positive solution in* $\mathbb{T}_{t_0}$*, then for any* $\eta\in\mathbb{D}$ *we have*

$$\begin{aligned}\mathbf{L}\eta &= \gamma_1^\sigma\{\gamma_2(\gamma_1\eta)^\Delta\}^\Delta, \quad t\neq\theta_i,\ \ t\in\mathbb{T}_{t_0}^{\kappa^2},\\ \mathbf{I}\eta &= \gamma_1^\sigma \Delta\gamma_2(\gamma_1\eta)^\Delta, \qquad t=\theta_i,\ \ t\in\mathbb{T}_{t_0}^{\kappa^2},\end{aligned} \tag{2.5}$$

where $\gamma_1(t) > 0, \gamma_2(t) > 0$ *and* $\int_{t_0}^\infty \frac{\Delta t}{\gamma_2(t)} = \infty$*.*

Proof. Since (2.1) has a positive solution in $\mathbb{T}_{t_0}$, $\mathbf{L}\eta$, $\mathbf{I}\eta$ have Polya Factorization in $\mathbb{T}_{t_0}$ by Lemma 2.1.

If $\int_{t_0}^\infty \frac{\Delta t}{\rho_2(t)} = \infty$, then taking $\gamma_2(t) = \rho_2(t)$ and $\gamma_1(t) = \rho_1(t)$, we have what we want for $t\in\mathbb{T}_{t_0}^{\kappa^2}$.

If $\int_{t_0}^{\infty} \frac{\Delta t}{\rho_2(t)} < \infty$, we set

$$\gamma_1(t) = \rho_1(t)\Big\{\int_t^{\infty} \frac{\Delta s}{\rho_2(s)}\Big\}^{-1} > 0 \quad and \quad \gamma_2(t) = \rho_2(t)\int_t^{\infty} \frac{\Delta s}{\rho_2(s)}\int_{\sigma(t)}^{\infty} \frac{\Delta s}{\rho_2(s)} > 0$$

for $t \in \mathbb{T}_{t_0}^{\kappa^2}$.

For $t \neq \theta_i$, $t \in \mathbb{T}_{t_0}^{\kappa^2}$ and $\eta \in \mathbb{D}$, note that

$$(\gamma_1\eta)^{\Delta}(t) = \Big\{\frac{\rho_1(t)\eta(t)}{\int_t^{\infty} \frac{\Delta s}{\rho_2(s)}}\Big\}^{\Delta} = \frac{(\rho_1\eta)^{\Delta}(t)\int_t^{\infty} \frac{\Delta s}{\rho_2(s)} - \rho_1(t)\eta(t)\{-\frac{1}{\rho_2(t)}\}}{\int_t^{\infty} \frac{\Delta s}{\rho_2(s)}\int_{\sigma(t)}^{\infty} \frac{\Delta s}{\rho_2(s)}}.$$

Hence

$$\gamma_2(t)(\gamma_1\eta)^{\Delta}(t) = \rho_2(t)(\rho_1\eta)^{\Delta}(t)\int_t^{\infty} \frac{\Delta s}{\rho_2(s)} + \rho_1(t)\eta(t).$$

Taking the derivative of both sides we get

$$\begin{aligned}\{\gamma_2(\gamma_1\eta)^{\Delta}\}^{\Delta}(t) &= \{\rho_2(\rho_1\eta)^{\Delta}\}^{\Delta}(t)\int_{\sigma(t)}^{\infty} \tfrac{\Delta s}{\rho_2(s)} \\ &\quad + \rho_2(t)(\rho_1\eta)^{\Delta}(t)\{-\tfrac{1}{\rho_2(t)}\} + (\rho_1\eta)^{\Delta}(t) \\ &= \{\rho_2(\rho_1\eta)^{\Delta}\}^{\Delta}(t)\int_{\sigma(t)}^{\infty} \tfrac{\Delta s}{\rho_2(t)}.\end{aligned}$$

It follows that

$$\gamma_1^{\sigma}(t)\{\gamma_2(\gamma_1\eta)^{\Delta}\}^{\Delta}(t) = \rho_1^{\sigma}(t)\{\rho_2(\rho_1\eta)^{\Delta}\}^{\Delta}(t) = \mathbf{L}\eta(t).$$

Then

$$\mathbf{L}\eta = \gamma_1^{\sigma}\{\gamma_2(\gamma_1\eta)^{\Delta}\}^{\Delta}, \quad t \neq \theta_i, \;\; t \in \mathbb{T}_{t_0}^{\kappa^2}.$$

For $t = \theta_i$, $t \in \mathbb{T}_{t_0}^{\kappa^2}$ and $\eta \in \mathbb{D}$, we obtain

$$\begin{aligned}\Delta\gamma_2(t)(\gamma_1\eta)^{\Delta}(t) &= \Delta\rho_2(t)(\rho_1\eta)^{\Delta}(t)\int_t^{\infty} \tfrac{\Delta s}{\rho_2(s)} + \Delta\rho_1(t)\eta(t) \\ &= \int_t^{\infty} \tfrac{\Delta s}{\rho_2(s)}\Delta\rho_2(t)(\rho_1\eta)^{\Delta}(t).\end{aligned}$$

Since $t = \theta_i, \; i \in \mathbb{N}$ are right dense, it follows that

$$\begin{aligned}\gamma_1^{\sigma}(t)\Delta\gamma_2(t)(\gamma_1\eta)^{\Delta}(t) &= \frac{\int_t^{\infty} \frac{\Delta s}{\rho_2(s)}}{\int_{\sigma(t)}^{\infty} \frac{\Delta s}{\rho_2(s)}}\rho_1^{\sigma}(t)\Delta\rho_2(t)(\rho_1\eta)^{\Delta}(t) \\ &= \rho_1^{\sigma}(t)\Delta\rho_2(t)(\rho_1\eta)^{\Delta}(t). \\ &= \mathbf{I}\eta(t).\end{aligned}$$

Then

$$\mathbf{I}\eta = \gamma_1^{\sigma}\Delta\gamma_2(\gamma_1\eta)^{\Delta}, \qquad t = \theta_i, \;\; t \in \mathbb{T}_{t_0}^{\kappa^2}.$$

So $\gamma_1(t)$ and $\gamma_2(t)$ satisfy (2.5) and

$$\begin{aligned}\int_{t_0}^{\infty} \frac{\Delta t}{\gamma_2(t)} &= \int_{t_0}^{\infty} \frac{\Delta t}{\rho_2(t)\int_t^{\infty} \frac{\Delta s}{\rho_2(t)}\int_{\sigma(t)}^{\infty} \frac{\Delta s}{\rho_2(t)}} = \int_{t_0}^{\infty}\Big\{\frac{1}{\int_t^{\infty} \frac{\Delta s}{\rho_2(s)}}\Big\}^{\Delta}\Delta t \\ &= \frac{1}{\int_t^{\infty} \frac{\Delta s}{\rho_2(s)}}\Big|_{t=t_0}^{t=\infty} - \sum_{t_0 \le \theta_k} \Delta\Big\{\int_{\theta_k}^{\infty} \frac{\Delta s}{\rho_2(s)}\Big\}^{-1} = \infty.\end{aligned}$$

The proof is complete. □

3. Main results

Theorem 3.1. *If* (1.1) *has a positive solution in* $\mathbb{T}_{t_0}$, *then there exist linearly independent solutions* $u(t)$ *and* $v(t)$ *of* (1.1) *such that*

$$\lim_{t\to\infty}\frac{u(t)}{v(t)}=0,\qquad \int_{t_0}^{\infty}\frac{\Delta t}{r(t)u(t)u^{\sigma}(t)}=\infty,\qquad \int_{t_0}^{\infty}\frac{\Delta t}{r(t)v(t)v^{\sigma}(t)}<\infty,$$

and

$$\frac{r(t)v^{\Delta}(t)}{v(t)}>\frac{r(t)u^{\Delta}(t)}{u(t)},$$

for t *sufficiently large.*

The solutions $u(t)$ *and* $v(t)$ *are called principal and nonprincipal solutions of* (1.1), *respectively.*

Proof. Since (1.1) has a positive solution in $\mathbb{T}_{t_0}$, $\mathbf{L}\eta, \mathbf{I}\eta$ have Trench Factorization in $\mathbb{T}_{t_0}$. We define

$$u(t)=\frac{1}{\gamma_1(t)},\qquad v_0(t)=\frac{1}{\gamma_1(t)}\int_{t_0}^{t}\frac{\Delta s}{\gamma_2(s)}.$$

It follows that

$$\begin{aligned}\mathbf{L}u=\mathbf{L}v_0=0,&\quad t\neq\theta_i,\\ \mathbf{I}u=\mathbf{I}v_0=0,&\quad t=\theta_i.\end{aligned}$$

Not that $v_0(t_0)=0$, $u(t)$ and $v_0(t)$ are two linearly independent solutions of (1.1) and

$$\lim_{t\to\infty}\frac{u(t)}{v_0(t)}=\lim_{t\to\infty}\frac{1}{\int_{t_0}^{t}\frac{\Delta s}{\gamma_2(s)}}=0.$$

Taking $\mu(t)=u(t)$ and $\eta(t)=v_0(t)$ in (2.2) and (2.3), we get

$$\begin{aligned}\{r(t)W(u,v_0)(t)\}^{\Delta}=0,&\qquad t\neq\theta_i,\\ \Delta r(t)W(u,v_0)(t)=0,&\qquad t=\theta_i.\end{aligned}\tag{3.1}$$

Integrating (3.1) from t_0 to t, we obtain

$$W(u,v_0)(t)=\frac{c_0}{r(t)},$$

where $c_0=r(t_0)W(u,v_0)(t_0)\neq 0$. Note that

$$(\frac{v_0}{u})^{\Delta}(t)=\frac{W(u,v_0)}{u(t)u^{\sigma}(t)}=\frac{c_0}{r(t)u(t)u^{\sigma}(t)},\quad t\neq\theta_i.\tag{3.2}$$

Integrating both sides of (3.2) from t_0 to ∞ we get

$$\begin{aligned}\int_{t_0}^{\infty}\frac{\Delta t}{r(t)u(t)u^{\sigma}(t)} &= \frac{1}{c_0}\lim_{\omega\to\infty}\int_{t_0}^{\omega}\{\frac{v_0(t)}{u(t)}\}^{\Delta}\Delta t\\ &= \frac{1}{c_0}\lim_{\omega\to\infty}\{\frac{v_0(t)}{u(t)}\Big|_{t=t_0}^{t=\omega}-\sum_{t_0\le\theta_i<\omega}\Delta(\frac{v_0}{u})(\theta_i)\}\\ &= \frac{1}{c_0}\lim_{\omega\to\infty}\frac{v_0(\omega)}{u(\omega)}-\frac{v_0(t_0)}{cu(t_0)}\\ &= \infty.\end{aligned}$$

Let $v(t)$ be any solution of (1.1) such that $v(t)$ and $u(t)$ are linearly independent. Then

$$v(t) = c_1 u(t) + c_2 v_0(t),$$

where c_2 and c_1 are constants with $c_2 \neq 0$. It follows that

$$\lim_{t\to\infty}\frac{u(t)}{v(t)} = \lim_{t\to\infty}\frac{u(t)}{c_1 u(t) + c_2 v_0(t)} = \lim_{t\to\infty}\frac{\frac{u(t)}{v_0(t)}}{c_1\frac{u(t)}{v_0(t)} + c_2} = 0$$

and

$$\mathbf{L}u = \mathbf{L}v = 0, \qquad t \neq \theta_i,$$

$$\mathbf{I}u = \mathbf{I}v = 0, \qquad t = \theta_i.$$

Similarly we get

$$\{\frac{u(t)}{v(t)}\}^{\Delta} = \frac{W(v,u)(t)}{v(t)v^{\sigma}(t)} = \frac{c_3}{r(t)v(t)v^{\sigma}(t)}, \tag{3.3}$$

where $c_3 = r(t_0)W(v,u)(t_0) \neq 0$. Integrating both sides of (3.3) from t_0 to ∞ we get

$$\int_{t_0}^{\infty}\frac{\Delta t}{r(t)v(t)v^{\sigma}(t)} = \frac{1}{c_3}\{\frac{u(t)}{v(t)}\Big|_{t=t_0}^{t=\infty} - \sum_{t_0\le\theta_i}\Delta\frac{u(\theta_i)}{v(\theta_i)}\} = -\frac{u(t_0)}{c_3 v(t_0)} < \infty.$$

Pick $t_1 \in \mathbb{T}_{t_0}$ so that $v(t)v^{\sigma}(t) > 0$ for $t \in \mathbb{T}_{t_1}$. If $v(t)$ is replaced by $-v(t)$, the expression $\frac{r(t)v^{\Delta}(t)}{v(t)}$ remains the same. So without loss of generality we assume $v(t) > 0$ for $t \in \mathbb{T}_{t_1}$. It is easy to see that for $t \in \mathbb{T}_{t_1}$,

$$\frac{r(t)v^{\Delta}(t)}{v(t)} - \frac{r(t)u^{\Delta}(t)}{u(t)} = \frac{r(t)W(u,v)(t)}{u(t)v(t)} = -\frac{c_3}{u(t)v(t)}, \qquad t \neq \theta_i.$$

Since the right side is continuous, by taking limit as $t \to \theta^{\pm}$ we can get

$$\frac{r(t)v^{\Delta}(t)}{v(t)} - \frac{r(t)u^{\Delta}(t)}{u(t)} = \frac{r(t)W(u,v)(t)}{u(t)v(t)} = -\frac{c_3}{u(t)v(t)}, \qquad t = \theta_i.$$

It remains to show that $c_3 < 0$. Since

$$\lim_{t\to\infty}\frac{v(t)}{u(t)} = \infty, \qquad \{\frac{v(t)}{u(t)}\}^{\Delta} = \frac{W(u,v)(t)}{u(t)u^{\sigma}(t)} = -\frac{c_3}{r(t)u(t)u^{\sigma}(t)},$$

and $\frac{v(t)}{u(t)}$ is continuous for $t \in \mathbb{T}_{t_1}$, we can get the desired result that $c_3 < 0$ if t is large enough. This proof is complete. □

Theorem 3.2. *(Leighton-Wintner Theorem). If* $r(t) > 0$ *and*

$$\int_{t_0}^{\infty}\frac{\Delta\tau}{r(\tau)} = \int_{t_0}^{\infty} q(\tau)\Delta\tau + \sum_{t_0\le\theta_i} q_i = \infty,$$

then (1.1) *is oscillatory in* $\mathbb{T}_{t_0}$.

Proof. Suppose that Eq. (1.1) is nonoscillatory. Then by Theorem 3.1, there is a solution $v(t)$ of (1.1) and a number $t_2 \in \mathbb{T}_{t_0}$ such that $v(t) > 0$ in $\mathbb{T}_{t_2}$ and

$$\int_{t_2}^{\infty} \frac{\Delta t}{r(t)v(t)v^{\sigma}(t)} < \infty.$$

Define

$$z(t) = \frac{r(t)v^{\Delta}(t)}{v(t)}, \quad t \neq \theta_i, \quad t \in \mathbb{T}_{t_2}.$$

It is not difficult to see that

$$\begin{aligned} r(t) + \mu(t)z(t) &= r(t) + \mu(t)\tfrac{r(t)v^{\Delta}(t)}{v(t)} = \tfrac{r(t)\{v(t)+\mu(t)v^{\Delta}(t)\}}{v(t)} \\ &= \tfrac{r(t)v^{\sigma}(t)}{v(t)} > 0, \qquad t \neq \theta_i. \end{aligned}$$

Then

$$\begin{aligned} z^{\Delta}(t) &= \frac{v(t)\{r(t)v^{\Delta}(t)\}^{\Delta} - r(t)\{v^{\Delta}(t)\}^2}{v(t)v^{\sigma}(t)} \\ &= -q(t) - \frac{v(t)}{r(t)v^{\sigma}(t)} z^2(t) \\ &= -q(t) - \frac{z^2(t)}{r(t) + \mu(t)z(t)} \\ &\leq -q(t), \qquad t \neq \theta_i, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \Delta z(t) &= \frac{r(t^+)v^{\Delta}(t^+)}{v(t^+)} - \frac{r(t^-)v^{\Delta}(t^-)}{v(t^-)} \\ &= \frac{\Delta r(t)v^{\Delta}(t)}{v(t)} \\ &= -\frac{q_i v^{\sigma}(t)}{v(t)} \\ &= -q_i, \qquad t = \theta_i. \end{aligned} \tag{3.5}$$

Integrating both sides of (3.4) from T to t, we obtain

$$z(t) - z(T) - \sum_{T \leq \theta_i < t} \Delta z(\theta_i) \leq -\int_T^t q(\tau)\Delta\tau. \tag{3.6}$$

Using (3.5) in (3.6), we get

$$z(t) \leq z(T) - \Big\{ \sum_{T \leq \theta_i < t} q_i + \int_T^t q(\tau)\Delta\tau \Big\}.$$

Then

$$\lim_{t\to\infty} z(t) = -\infty.$$

Let $t_3 \in \mathbb{T}_{t_2}$ where t_3 is sufficiently large such that

$$z(t) = \frac{r(t)v^{\Delta}(t)}{v(t)} < 0, \quad t \neq \theta_i, \quad t \in \mathbb{T}_{t_3}.$$

It follows that

$$v^{\Delta}(t) < 0, \quad t \neq \theta_i, \quad t \in \mathbb{T}_{t_3}.$$

Therefore $v(t)$ is decreasing on each interval $\mathbf{I}_i = (\theta_i, \theta_{i+1}]_{\mathbb{T}},\ i \in \mathbb{N}$. And since $v(t)$ is continuous, we get $v(t)$ is decreasing on $\mathbb{T}_{t_3}$. It follows

$$\int_{t_3}^{\infty} \frac{\Delta t}{r(t)v(t)v^{\sigma}(t)} \geq \frac{1}{v^2(t_3)} \int_{t_3}^{\infty} \frac{\Delta t}{r(t)} = \infty,$$

which implies that

$$\int_{t_3}^{\infty} \frac{\Delta t}{r(t)v(t)v^{\sigma}(t)} = \infty,$$

which contradicts. The proof is complete. □

Example 3.1. Let $\mathbb{T} = \mathbf{P}_{22} = \bigcup\limits_{i=0}^{\infty} [4i, 4i+2],\ \theta_i = 4i+1,\ i \in \mathbb{N}$. Consider the dynamic equation with impulse

$$\begin{aligned} &\Big\{\frac{1}{t^2} x^{\Delta}(t)\Big\}^{\Delta} + t^3 x^{\sigma}(t) = 0, && t \neq 4i+1, \\ &\Delta\Big\{\frac{1}{t^2} x^{\Delta}(t)\Big\} + (4i+1)^2 x^{\sigma}(t) = 0, && t = 4i+1. \end{aligned} \tag{3.7}$$

It is not difficult to see that

$$\int_4^{\infty} t^2 \Delta t = \int_4^{\infty} t^3 \Delta t + \sum_{4 \leq 4i+1} (4i+1)^2 = \infty.$$

Therefore Eq. (3.7) is oscillatory on $[4, \infty)_{\mathbb{T}}$ by Theorem 3.2.

Theorem 3.3. *(Wong's Theorem) Suppose that* (1.1) *is nonoscillatory and z(t) is a nonprincipal solution of* (1.1). *If*

$$\limsup_{t\to\infty} H(t) = -\liminf_{t\to\infty} H(t) = \infty,$$

then (1.2) *is oscillatory, where*

$$H(t) = \int_{t_0}^{t} \frac{1}{r(s)z(s)z^{\sigma}(s)} \Big\{ \int_{t_o}^{s} f(\tau) z^{\sigma}(\tau) \Delta\tau + \sum_{t_0 \leq \theta_i < s} f_i z^{\sigma}(\theta_i) \Big\} \Delta s,$$

the function $f(t)$ *and the sequence* $\{f_i\}$ *are as in* (1.2).

Proof. We suppose that y(t) is a nonoscillatory solution of Eq.(1.2). Define $y(t) = z(t)w(t)$, then

$$\begin{aligned} z^{\sigma}(t)\mathbf{L}y - y^{\sigma}(t)\mathbf{L}z &= \{r(t)z(t)y^{\Delta}(t) - r(t)z^{\Delta}(t)y(t)\}^{\Delta} \\ &= \{r(t)(\frac{y}{z})^{\Delta}(t)z(t)z^{\sigma}(t)\}^{\Delta} \\ &= (rw^{\Delta}zz^{\sigma})^{\Delta}(t), \quad t \neq \theta_i, \end{aligned}$$

and

$$\begin{aligned} z^{\sigma}(t)\mathbf{I}y - y^{\sigma}(t)\mathbf{I}z &= \Delta r(t)\{z^{\sigma}(t)y^{\Delta}(t) - z^{\Delta}(t)y^{\sigma}(t)\} \\ &= \Delta r(t)z(t)z^{\sigma}(t)\frac{y^{\Delta}(t)z(t) - z^{\Delta}(t)y(t)}{z(t)z^{\sigma}(t)} \\ &= \Delta r(t)w^{\Delta}(t)z(t)z^{\sigma}(t), \quad t = \theta_i, \end{aligned}$$

i.e.

$$(rw^{\Delta}zz^{\sigma})^{\Delta}(t) = z^{\sigma}(t)f(t), \qquad t \neq \theta_i, \tag{3.8}$$

$$\Delta r(t)w^{\Delta}(t)z(t)z^{\sigma}(t) = f_i z^{\sigma}(t), \quad t = \theta_i. \tag{3.9}$$

Since $y(t)$ and $z(t)$ are continuous, it follows that

$$\Delta w(t) = 0.$$

Integrating both sides of (3.8) from t_0 to s, we obtain

$$w^{\Delta}(s) = \frac{c_5}{r(s)z(s)z^{\sigma}(s)} + \frac{1}{r(s)z(s)z^{\sigma}(s)}\Big\{\int_{t_0}^{s} f(\tau)z^{\sigma}(\tau)\Delta\tau + \sum_{t_0\leq\theta_i<s} f_i z^{\sigma}(\theta_i)\Big\}. \tag{3.10}$$

Integrating both sides of (3.10) from t_0 to t, we obtain

$$\begin{aligned} w(t) &= c_4 + c_5\int_{t_0}^{t}\frac{\Delta s}{r(s)z(s)z^{\sigma}(s)} \\ &\quad + \int_{t_0}^{t}\frac{1}{r(s)z(s)z^{\sigma}(s)}\Big\{\int_{t_0}^{s} f(\tau)z^{\sigma}(\tau)\Delta\tau + \sum_{t_0\leq\theta_i<s} f_i z^{\sigma}(\theta_i)\Big\}\Delta s, \end{aligned}$$

where $c_4 = w(t_0)$, $c_5 = r(t_0)z(t_0)z^{\sigma}(t_0)w^{\Delta}(t_0)$. Which implies that

$$\limsup_{t\to\infty} w(t) = -\liminf_{t\to\infty} w(t) = \infty.$$

Then $w(t)$ is oscillatory. Because $z(t)$ is nonoscillatory, it follows that $y(t) = z(t)w(t)$ is oscillatory. □

Remark 3.1. If $q_i \equiv 0$, i.e, if there is no impulse condition, Lemma 2.1, Lemma 2.2, Theorem 3.1 and Theorem 3.2 reduce to Theorem 4.59, Theorem 4.60, Theorem 4.61 and Theorem 4.64 in Bohner & Peterson [2] respectively.

Remark 3.2. If we take $q_i \equiv f_i \equiv 0$, Theorem 3.3 recovers Theorem 3.2 in Zafer [10].

Remark 3.3. If we take $\mathbb{T} = \mathbb{R}$, then we recover all the results obtained by A.Özbekler & Zafer [9].

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