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CONTROLLABILITY OF NONLINEAR THIRD ORDER DISPERSION EQUATION WITH DISTRIBUTED DELAY*

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Abstract This paper is concerned with the exact controllability of nonlinear third order dispersion equation with infinite distributed delay. Sufficient conditions are formulated and proved for the exact controllability of this system. Without imposing a compactness condition on the semigroup, we establish controllability results by using a fixed point analysis approach.

Keywords Controllability, semigroup theory, nonlinear dispersion equation, Korteweg-de Vries equation, distributed delay.

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1. Introduction

For a long time, the Korteweg-de Vries (KdV) equation has attracted much attention due to its significant nature in physical contexts, stratified internal waves, ion-acoustic wave, plasma physics (see [1,8]). The controllability problem of KdV equation has been studied extensively by the researchers as far as the linear system is concerned. George, Chalishajar and Nandakumaran [4] discussed the exact controllability of nonlinear third-order dispersion equation. They established the controllability results using two standard types of nonlinearities, namely Lipschitzian and monotone. Later on, Chalishajar [3] studied the exact controllability of nonlinear integro-differential third order dispersion system by using the Schaefer fixed-point theorem. Recently, Sakthivel, Mahmudov and Ren [11] focused on the approximate controllability for the nonlinear third-order dispersion equation. They discussed the approximate controllability under the assumption that the corresponding linear control system is approximately controllable.

It has been widely argued and accepted [6, 12] that for various reasons, time delay should be taken into consideration in modeling. Obviously, the KdV equation with time delay has more actual significance. Zhao and Xu [13] studied the existence of solitary waves for KdV equation with time delay. However, it is more likely there are multiple states, even infinite states affecting the current state. The purpose of this paper is to study the exact controllability of the following nonlinear third order

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dispersion equation with infinite distributed delay:

$$\frac{\partial w}{\partial t}(x,t)+\frac{\partial^3 w}{\partial x^3}(x,t)=(Gu)(x,t)+\int_{-\infty}^{0}P(\theta,w(x,t+\theta))d\theta \tag{1.1}$$

on the domain $t\in J,\ 0\le x\le 2\pi$, with periodic boundary conditions

$$\frac{\partial^k w}{\partial x^k}(0,t)=\frac{\partial^k w}{\partial x^k}(2\pi,t),\quad k=0,1,2, \tag{1.2}$$

and initial condition

$$w(x,\theta)=w_0(x,\theta),\ -\infty<\theta\le 0,\ 0\le x\le 2\pi, \tag{1.3}$$

where $J=[0,b],\ P:(-\infty,0]\times R\to R$ and $w_0:[0,2\pi]\times(-\infty,0]\to R$ are continuous functions. $w_t(x,\theta)=w(x,t+\theta),\ -\infty<\theta\le 0$. u is the control function and the operator G is defined by

$$(Gu)(x,t)=g(x)\big\{u(x,t)-\int_0^{2\pi}g(s)u(s,t)ds\big\}. \tag{1.4}$$

Then G is a bounded linear operator and $g(x)$ is a piece-wise continuous nonnegative function on $[0,2\pi]$ such that

$$[g]:=\int_0^{2\pi}g(s)ds=1. \tag{1.5}$$

Moreover, M_2 is a positive constant such that $\|G\|\le M_2$.

The state $w(\cdot,t)$ takes values in a Banach space $X=L^2(0,2\pi)$ with the norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. The control function $u(\cdot,t)$ is given in $L^2(J,U)$, a Banach space of all admissible control functions, with $U=L^2(0,2\pi)$ as a Banach space.

The system (1.1)-(1.3) arises from realistic models, such as a Boussinesq equation-based model for wave breaking, which depends on the whole histories (i.e., there is the effect of infinite distributed delay on state equations). To the author's knowledge the corresponding theory for controllability of nonlinear third order dispersion equation with time delay has not been explored. In the present paper we will concentrate on the case with infinite distributed delay, choose a function g on $(-\infty,0]$ in a way that in the *weighted* (or *friendly* in some literature) phase space B, and establish sufficient conditions for the controllability of systems (1.1)-(1.3) by using a fixed-point analysis approach.

2. Preliminaries

Define an operator A on X with domain $D=D(A)$ given by

$$D(A)=\big\{w\in H^3(0,2\pi):\ \frac{\partial^k w}{\partial x^k}(0)=\frac{\partial^k w}{\partial x^k}(2\pi);\ k=0,1,2\big\}$$

such that

$$Aw=-\frac{\partial^3 w}{\partial x^3}.$$

It follows from Lemma 8.5.2 and Korteweg-de Vries equation of Pazy [9] that A is the infinitesimal generator of a C_0-group of isometry on X. Let $\{T(t)\}_{t\geq 0}$ be the C_0-group generated by A. Obviously, one can show for all $w \in D(A)$,

$$\langle Aw, w\rangle_{L^2(0,2\pi)} = 0.$$

Also, there exists a constant $M > 0$ such that

$$\sup\{\|T(t)\| :\ t \in J\} \leq M.$$

To study the system (1.1)-(1.3), we assume that the histories $x_t : (-\infty, 0] \to X,\ x_t(\theta) = x(t+\theta)$ belong to some abstract phase space B, which is defined axiomatically. In this artical, we will empoly an axiomatic definition of the phase space B introduced by Hale and Kato [5] and follow the terminology used in [7]. Thus, B will be a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\|\cdot\|_B$. We will assume that B satisfies the following axioms:

(A) If $x : (-\infty, \sigma + a) \to X,\ a > 0$, is continuous on $[\sigma, \sigma + a)$ and $x_\sigma \in B$, then for every $t \in [\sigma, \sigma + a)$ the following conditions hold:

(i) x_t is in B;

(ii) $\|x(t)\| \leq H\|x_t\|_B$;

(iii) $\|x_t\|_B \leq K(t-\sigma)\sup\{\|x(s)\| : \sigma \leq s \leq t\} + M(t-\sigma)\|x_\sigma\|_B$;

Here $H \geq 0$ is a constant, $K, M : [0, +\infty) \to [0, +\infty)$, K is continuous and M is locally bounded, and H, K, M are independent of $x(t)$.

(A_1) For the function $x(\cdot)$ in (A), x_t is a B-valued continuous function on $[\sigma, \sigma + a]$.

(B) The space B is complete.

Example 2.1. The phase space $C_r \times L^p(\rho, X)$.

Let $r \geq 0, 1 \leq p < \infty$ and let $\rho : (-\infty, -r) \to R$ be a non-negative measurable function which satisfies the conditions (g-5), (g-6) in the terminology of [7]. In other words, this means that ρ is locally integrable and there exists a non-negative, locally bounded function γ on $(-\infty, 0]$ such that $\rho(\xi + \theta) \leq \gamma(\xi)\rho(\theta)$, for all $\xi \leq 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set with Lebesgue measure zero. The space $C_r \times L^p(\rho, X)$ consists of all classes of functions $\phi : (-\infty, 0] \to X$ such that ϕ is continuous on $[-r, 0]$, Lebesgue-measurable, and $\rho\|\phi\|^p$ is Lebesgue integrable on $(-\infty, -r)$. The seminorm in $C_r \times L^p(\rho, X)$ is defined by

$$\|\phi\|_B = \sup\{\|\phi(\theta)\| : -r \leq \theta \leq 0\} + \Big(\int_{-\infty}^{-r} \rho(\theta)\|\phi(\theta)\|^p d\theta\Big)^{1/p}.$$

The space $C_r \times L^p(\rho, X)$ satisfies axioms $(A), (A_1), (B)$. More over, if $r = 0$ and $p = 2$, the phase space $C_r \times L^p(\rho, X)$ is reduced to $B = C_0 \times L^2(\rho, X)$. We can take $H = 1,\ M(t) = \gamma(-t)^{1/2}$, and $K(t) = 1 + \big(\int_{-t}^{0} \rho(\theta)d\theta\big)^{1/2}$, for $t \geq 0$ (see [7] for the details).

By the variation of constant formula, we can write a mild solution of (1.1)-(1.3) as

$$\begin{aligned} w(x,t) = T(t)w(x,0) + \int_0^t T(t-s)(Gu)(x,s)ds \\ + \int_0^t T(t-s)\int_{-\infty}^0 P(\theta, w(x, s+\theta))d\theta ds. \end{aligned} \tag{2.1}$$

Definition 2.1. The system (1.1)-(1.3) is said to be exactly controllable over a time interval J, if for any given $w_b \in X$ with $[w_b] = 0$, there exists a control $u \in L^2(0,b; L^2(0,2\pi)) = L^2(J,U)$ such that the corresponding solution w of (1.1)-(1.3) satisfies $w(.,b) = w_b$.

We define, for all $\theta \leq 0$, $x \in [0,2\pi]$ and $\phi \in B$,

$$F(\phi)(x) = \int_{-\infty}^0 P(\theta, \phi(\theta)(x))d\theta \quad \text{and} \quad \phi(\theta)(x) = \phi(x,\theta) = w_0(x,\theta). \tag{2.2}$$

Russell and Zhang [10] studied the exact controllability of a corresponding linear system (i.e. with $F \equiv 0$ in (1.1)-(1.3)). In their analysis, they considered controls which conserve quantity $[w(\cdot,t)]$, which corresponds to the *volume* (refer to Russell and Zhang [10]). The following is their controllability result for the linear system.

Theorem 2.1. *(Russell-Zhang). Let $b > 0$ be given and let $g \in C^0[0,2\pi]$ be associated with G in* (1.4)*. Given any final state $w_b \in X$ with $[w_b] = 0$, there exists a control $u \in L^2(J,U)$ such that the solution w of*

$$\frac{\partial w}{\partial t}(x,t) + \frac{\partial^3 w}{\partial x^3}(x,t) = (Gu)(x,t) \tag{2.3}$$

together with the initial and boundary conditions (1.2)-(1.3) *satisfies the terminal condition $w(\cdot,b) = w_b$ in $L^2(0,2\pi)$. Moreover, there exist a positive constant C_1 independent of w_b such that*

$$\|w\|_{L^2(J,X)} \leq C_1 \|w_b\|. \tag{2.4}$$

The main purpose of this paper is to obtain sufficient conditions on the perturbed nonlinear term F which will preserve the exact controllability. Without imposing a compactness condition on the semigroup, we establish controllability results.

3. Controllability result

We assume the following conditions hold:

(H_1) For each $\theta \leq 0$ and $\zeta_1, \zeta_2 \in R$, $|P(\theta,\zeta_1) - P(\theta,\zeta_2)| < k(\theta)|\zeta_1 - \zeta_2|$, where k is a measurable nonnegative function on $(-\infty, 0]$ such that

$$L_F := \Big(\int_{-\infty}^0 \frac{k^2(\theta)}{\rho(\theta)} d\theta\Big)^{1/2} < \infty.$$

Assumption (H_1) implies that, for $\phi_1, \phi_2 \in B$,

$$\begin{aligned}
&\|F(\phi_1) - F(\phi_2)\| \\
=&\Big[\int_0^{2\pi} |\int_{-\infty}^0 P(\theta, \phi_1(\theta)(x))d\theta - \int_{-\infty}^0 P(\theta, \phi_2(\theta)(x))d\theta|^2 dx\Big]^{1/2} \\
\le&\Big[\int_0^{2\pi} \Big(\int_{-\infty}^0 k(\theta)|\phi_1(\theta)(x) - \phi_2(\theta)(x)|d\theta\Big)^2 dx\Big]^{1/2} \\
\le&\Big(\int_{-\infty}^0 \frac{k^2(\theta)}{\rho(\theta)} d\theta\Big)^{1/2} \Big(\int_{-\infty}^0 \rho(\theta) \int_0^{2\pi} |\phi_1(\theta)(x) - \phi_2(\theta)(x)|^2 dx d\theta\Big)^{1/2} \\
\le&\Big(\int_{-\infty}^0 \frac{k^2(\theta)}{\rho(\theta)} d\theta\Big)^{1/2} \|\phi_1 - \phi_2\|_B \\
:=&L_F \|\phi_1 - \phi_2\|_B.
\end{aligned} \tag{3.1}$$

(H_2) $p = (1 + bMM_2M_3)bML_FK_b < 1$.

Theorem 3.1. *If the conditions* $(H_1) - (H_2)$ *and* $[w_b] = 0$ *are satisfied, then the nonlinear third order dispersion equation* (1.1)-(1.3) *is exactly controllable.*

Proof. Consider the space $C = C(J, X)$, the Banach space of all continuous functions from J into X with sup norm.

Define the linear operator $\widetilde{W} : L^2(J, U) \to X$ by

$$\widetilde{W}u = \int_0^b T(b - s)(Gu)(x, s)ds.$$

By Theorem 2.1 and the assumption $[w_b] = 0$, we obtain that the linear system (2.3) is exactly controllable. Then the operator $\widetilde{W}$ has an inverse operator $\widetilde{W}^{-1}$ which takes the values in $L^2(J, U)/ker\widetilde{W}$ and there exist a positive constant M_3 such that $\|\widetilde{W}^{-1}\| \le M_3$ (see [2]).

For an arbitrary function $w(\cdot, t)$, define the control function

$$u(x, t) = \widetilde{W}^{-1}[w_b - T(b)w(x, 0) - \int_0^b T(b - s) \int_{-\infty}^0 P(\theta, w(x, s + \theta))d\theta ds](t). \tag{3.2}$$

Let $\widehat{w}(x, t) \in C((-\infty, b]; X)$ be the function defined by

$$\widehat{w}(x, t) = \begin{cases} w_0(x, t), & t \in (-\infty, 0], \\ T(t)w(x, 0), & t \in J. \end{cases}$$

Let $w(x, t) = v(x, t) + \widehat{w}(x, t),\ t \in (-\infty, b]$. It is easy to see that v satisfies $v(x, t) = 0,\ t \in (-\infty, 0]$, and

$$\begin{aligned}
v(x, t) =& \int_0^t T(t - s)(Gu)(x, s)ds \\
&+ \int_0^t T(t - s) \int_{-\infty}^0 P(\theta, v(x, s + \theta) + \widehat{w}(x, s + \theta))d\theta ds,\ t \in J.
\end{aligned}$$

Let $Z_b = \{v(x, t) \in C((-\infty, b]; X) : v(x, t) = 0, t \in (-\infty, 0]\}$. For any $v(x, t) \in Z_b$, $\|v(x, t)\|_b = \sup_{s \in J} \|v(x, s)\|$, thus $(Z_b, \|\cdot\|_b)$ is a Banach space.

On the space Z_b, we define the nonliear operator $\Phi : Z_b \to Z_b$ by $(\Phi v)(x,t) = 0, t \in (-\infty, 0]$, and

$$\begin{aligned}(\Phi v)(x,t) =& \int_0^t T(t-s)(Gu)(x,s)ds \\ &+ \int_0^t T(t-s)\int_{-\infty}^0 P(\theta, v(x,s+\theta)+\widetilde{w}(x,s+\theta))d\theta ds \\ =& \int_0^t T(t-s)\int_{-\infty}^0 P(\theta, v(x,s+\theta)+\widetilde{w}(x,s+\theta))d\theta ds \\ &+ \int_0^t T(t-\eta)G\widetilde{W}^{-1}[w_b - T(b)w(x,0) \\ &- \int_0^b T(b-s)\int_{-\infty}^0 P(\theta, v(x,s+\theta)+\widetilde{w}(x,s+\theta))d\theta ds](\eta)d\eta, \ t \in J.\end{aligned}$$

Note that the control (3.2) transfers the system (1.1)-(1.3) from the initial state to the final state provided that the operator Φ has a fixed point. So if the operator Φ has a fixed point then the system (1.1)-(1.3) is exactly controllable. To prove the exact controllability, it is enough to show that the operator Φ has a fixed point in Z_b. The proof is based on the classical fixed point theorem for contractions. It follows from the assumptions that Φ is well defined and continuous. In order to prove that Φ is a contraction mapping on Z_b, we take v and μ in Z_b.

From the conditions $(H_1)-(H_2)$ and (3.1), we get

$$\begin{aligned}&\|(\Phi\nu)(x,t)-(\Phi\mu)(x,t)\| \\ =&\|\int_0^t T(t-s)\int_{-\infty}^0 P(\theta, \nu(x,s+\theta)+\widetilde{w}(x,s+\theta))d\theta ds \\ &-\int_0^t T(t-s)\int_{-\infty}^0 P(\theta, \mu(x,s+\theta)+\widetilde{w}(x,s+\theta))d\theta ds \\ &+\int_0^t T(t-\eta)G\widetilde{W}^{-1}[w_b - T(b)w(x,0) \\ &\quad -\int_0^b T(b-s)\int_{-\infty}^0 P(\theta, \nu(x,s+\theta)+\widetilde{w}(x,s+\theta))d\theta ds](\eta)d\eta \\ &-\int_0^t T(t-\eta)G\widetilde{W}^{-1}[w_b - T(b)w(x,0) \\ &\quad -\int_0^b T(b-s)\int_{-\infty}^0 P(\theta, \mu(x,s+\theta)+\widetilde{w}(x,s+\theta))d\theta ds](\eta)d\eta\| \\ \le&\int_0^b ML_F\|\nu_s-\mu_s\|_B ds + bM\|G\|\|\widetilde{W}^{-1}\|\int_0^b ML_F\|\nu_s-\mu_s\|_B ds \\ \le&(1+bMM_2M_3)\int_0^b ML_F\|\nu_s-\mu_s\|_B ds \\ \le&(1+bMM_2M_3)bML_FK_b\|\nu-\mu\|_b \\ =&p\|v-\mu\|_b.\end{aligned} \tag{3.3}$$

Therefore, the above inequality (3.3) imply that Φ is a contraction mapping. Hence there exists a unique fixed point v in Z_b. Let $w(x,t) = v(x,t) + \widehat{w}(x,t), \ t \in$

$(-\infty, b]$. Then w is the mild solution of problem (1.1)-(1.3). Thus the system (1.1)-(1.3) is exactly controllable.

□

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