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BIFURCATION OF LIMIT CYCLES IN SMALL PERTURBATIONS OF A HYPER-ELLIPTIC HAMILTONIAN SYSTEM WITH TWO NILPOTENT SADDLES*

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Abstract In this paper we study the first-order Melnikov function for a planar near-Hamiltonian system near a heteroclinic loop connecting two nilpotent saddles. The asymptotic expansion of this Melnikov function and formulas for the first seven coefficients are given. Next, we consider the bifurcation of limit cycles in a class of hyper-elliptic Hamiltonian systems which has a heteroclinic loop connecting two nilpotent saddles. It is shown that this system can undergo a degenerate Hopf bifurcation and Poincarè bifurcation, which emerges at most four limit cycles in the plane for sufficiently small positive ε. The number of limit cycles which appear near the heteroclinic loop is discussed by using the asymptotic expansion of the first-order Melnikov function. Further more we give all possible distribution of limit cycles bifurcated from the period annulus.

Keywords Melnikov function, Hilbert's 16th problem, limit cycles, heteroclinic loop, nilpotent saddle.

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1. Introduction and statements of results

The second part of the Hilbert's 16th problem asks to find an upper bound for the number of limit cycles and their relative locations in planar polynomial vector fields. Although the problem is still far from being completely solved, the research on this problem has made great progress with significant contributions to the development of modern mathematics. The recent developments of Hilbert's 16th problem were summarized in the survey articles by Li [15], Han & Li [11] and Yu [22]. A weaker version of this problem is proposed by Arnold to study the zeros of Abelian integrals obtained by integrating polynomial 1-forms over ovals of polynomial Hamiltonian, that is called the weak Hilbert's 16th problem or infinitesimal Hilbert's 16th problem [1], where oval is a smooth simple closed curve which is free of critical points of Hamiltonian function. To state the weak Hilbert's 16th problem more precisely, consider a perturbed planar Hamiltonian system

$$\begin{aligned} \dot{x} &= H_y + \varepsilon p(x, y, \varepsilon, \delta), \\ \dot{y} &= -H_x + \varepsilon q(x, y, \varepsilon, \delta), \end{aligned} \tag{1.1}$$

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where p, q and H are C^ω functions, ε is a small positive parameter and δ is a vector parameter where $\delta \in D \subset \mathbb{R}^m$ and D is a compact set. Suppose the unperturbed system

$$\dot{x} = H_y, \ \ \dot{y} = -H_x \tag{1.2}$$

has a family of periodic orbits L_h defined by the equation $H(x,y) = h$. Then, associated to a given perturbation of the system (1.1) there exist a so-called first-order Melnikov function (also known as Abelian integrals) of the following form

$$M(h,\delta) = \oint_{L_h} qdx - pdy|_{\varepsilon=0}, \tag{1.3}$$

which plays an important role in the study of bifurcation of limit cycles from system (1.1). We recall that a limit cycle of system (1.1) corresponds to an isolated zero of $M(h,\delta)$ [7,19].

When h is a critical value h_0, the graph of $H(x,y) = h_0$ contains a singular point. The study of the asymptotic expansion of the Melnikov function $M(h,\delta)$ about critical values is an interesting problem which is closely related to the weak Hilbert's 16th problem. Roussarie in [17] studied the asymptotic expansion of (1.3) about the level set $L_{h_0} := \{(x,y) : H(x,y) = h_0\}$, which is a homoclinic loop through a saddle point. Han et al studied the asymptotic expansion of the Melnikov function near the critical values corresponding to a cuspidal loop and a homoclinic loop through a nilpotent saddle [13, 23].

In this paper first we study the asymptotic expansion of the Melnikov function near a heteroclinic loop through two nilpotent saddles. Next, we consider a Liénard system of type $(7,6)$ that is a small perturbation of Hamiltonian vector field with a hyper-elliptic Hamiltonian of degree eight with $\mathbb{Z}_2$ symmetry. In the progress of solving the weakened Hilbert's 16th problem, the generalized Liénard system $\dot{x} = y, \ \ \dot{y} = Q_1(x) + yQ_2(x)$ of type (m,n) is of great interest, where Q_1 and Q_2 are polynomials of degree respectively m and n. See for instance works of Dumortier & Li [2–5] and Wang & Xiao [20].

Our system is as follows:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x(x^2-1)^3 + \varepsilon(a + bx^2 + cx^4 + x^6)y, \end{aligned} \tag{H_ε}$$

with Hamiltonian function

$$H(x,y) = y^2/2 + x^2/2 - 3x^4/4 + x^6/2 - x^8/8, \tag{1.4}$$

where $0 < \varepsilon \ll 1$ and a, b and c are real parameters. The level sets of Hamiltonian function (1.4) are sketched in Figure 1. The ovals $\gamma_h = \{(x,y) : \ H(x,y) = h, \ h \in (0, 1/8)\}$ are closed orbits of system (H_0) which form a unique period annulus in the plane. When $h = 0$, γ_0 is an elementary center $(0,0)$ of system (H_0), and $\gamma_{1/8}$ is the heteroclinic loop passing through nilpotent saddles $S_1(1,0)$ and $S_2(-1,0)$ of system (H_0). The Melnikov function of system (H_ε) is

$$M(h,\delta) = \oint_{\gamma_h} (a + bx^2 + cx^4 + x^6)ydx = aI_0(h) + bI_1(h) + cI_2(h) + I_3(h), \tag{1.5}$$

where $I_k(h) = \oint_{\gamma_h} x^{2k} ydx, \ k = 0, 1, 2, 3.$

We shall give a complete description of the number and the possible configurations of limit cycles for system (H_ε) in the plane. We study the Hopf bifurcation and Poincaré bifurcation of system (H_ε).

This paper is organized as follows. In Section 2, we study the Melnikov function near the heteroclinic loop connecting two nilpotent saddles point and give the asymptotic expansion of the first-order Melnikov function near the heteroclinic loop connecting two nilpotent saddle points for (1.1). In Section 3, we discuss the existence and number of limit cycles of (H_ε). In subsection 3.1, we give a general analysis on system (H_ε) such as the properties and bifurcations of equilibria, and we prove that there is no closed orbit surrounding two equilibria of system (H_ε) and system (H_ε) can undergo degenerated Hopf bifurcation which emerges at most three limit cycles in the plane. In subsection 3.2, we show that Melnikov function $M(h,\delta)$ of system (H_ε) has the Chebyshev property with accuracy one. In subsection 3.3 we calculate the asymptotic expansions of Melnikov function $M(h,\delta)$ at the end points of open interval $(0,1/8)$ for (H_ε), and conclude that system (H_ε) can have three limit cycles near the heteroclinic loop $\gamma_{1/8}$. In subsection 3.4 we will discuss all possible distribution of bifurcated limit cycles from the period annulus.

2. Asymptotic expansions of Melnikov function of (1.2) about a heteroclinic loop

In this section we consider the first-order Melnikov function (1.3) of a heteroclinic loop through two nilpotent saddles for general planar near-Hamiltonian systems. The asymptotic expansion of this Melnikov function and formulas for its first seven coefficients are given. For this we need the following result from the work of Zang et al [23]:

Suppose the unperturbed system (1.2) has a family of periodic orbits L_h defined by the equation $H(x,y)=h$. Let the boundary of the family $\{L_h\}$ be a homoclinic loop $L_0=\{(x,y):H(x,y)=0\}$ with one nilpotent saddle of order one at the origin. Then by [23] without loss of generality we can assume that $H(x,y)$ has the following formal expansion near the origin

$$H(x,y)=-\frac{1}{4}x^4+\sum_{j\geq 5}h_{j,0}x^j+y^2\sum_{i+j\geq 0}h_{i,j}x^iy^j. \tag{2.1}$$

In this case the following theorem due to Zang et al. [23] gives the asymptotic expansion of the Melnikov function about the homoclinic loop L_0 and an explicit formula for its first seven coefficients $(c_i, i=1,\dots,7)$.

Theorem A. *Let (2.1) be satisfied. Then for system (1.1), near the value $h=0$ $(0<-h\ll 1)$ corresponding to nilpotent saddle loop L_0 through a nilpotent saddle point $(0,0)$ we have*

$$\begin{aligned} M(h)= &\left[c_1+c_4|h|+\textstyle\sum_{j\geq 2}A_j|h|^j\right]+|h|^{3/4}\left[c_2+c_6|h|+\textstyle\sum_{j\geq 2}B_j|h|^j\right]\\ &+\ln|h|\left[c_3h+c_7h^2+\textstyle\sum_{j\geq 3}C_j|h|^j\right]+|h|^{5/4}\left[c_5+\textstyle\sum_{j\geq 2}D_j|h|^j\right], \end{aligned} \tag{2.2}$$

where

$$\begin{aligned}
c_1 &= M(0), \quad c_2 = -\frac{4\sqrt{2}d_{0,0}\Delta_{0,2}}{3}, \quad c_3 = -\frac{\sqrt{2}d_{1,0}}{2}, \\
c_4 &= \oint_{L_0}(p_x+q_y)|_{\varepsilon=0}dt \;\; for \;\; c_2=c_3=0 \quad c_5 = -\frac{4\sqrt{2}d_{2,0}\Delta_{2,2}}{5}, \\
c_6 &= \frac{8\sqrt{2}(d_{0,2}-2d_{4,0})\Delta_{0,2}}{21}, \quad c_7 = \frac{\sqrt{2}(4d_{5,0}-d_{1,2})}{8},
\end{aligned} \tag{2.3}$$

$\Delta_{0,2} > 0$ *and* $\Delta_{2,2} < 0$ *are constants and* $d_{i,0}$ $(i = 1,2,4,5)$ *and* $d_{i,2}$ $(i = 1,2)$ *are some terms depending explicitly on the coefficients of the expansions of* $H(x,y)$, $p(x,y,0,\delta)$ *and* $q(x,y,0,\delta)$.

The coefficients c_2, c_3, c_5, c_6 and c_7 in Theorem A are called local coefficients of M at the nilpotent saddle O.

Now, inspired by the work of Sun et al [18] and using Theorem A we are ready to obtain the asymptotic expansion of the Melnikov function near a heteroclinic loop through two nilpotent saddles. Suppose system (1.1) has two nilpotent saddles S_1 and S_2. Moreover assume :

($\mathbf{A}_1$) The unperturbed system (1.2) has a heteroclinic loop denoted by $L_0 := \{(x,y) : H(x,y) = 0\} = L_1 \cup L_2 \cup \{S_1, S_2\}$, where L_1 and L_2 are heteroclinic orbits connecting saddle points S_1 and S_2 so that $\omega(L_1) = \alpha(L_2) = S_2$ and $\omega(L_2) = \alpha(L_1) = S_1$.

($\mathbf{A}_2$) In a neighborhood of L_0 there is a family of periodic orbit of (1.2) denoted by $L_h = \{(x,y) : H(x,y) = h\}$ for $0 < -h \ll 1$.

Then there exist two transformations of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = Q_k \begin{pmatrix} u \\ v \end{pmatrix} + S_k, \quad k = 1,2, \tag{2.4}$$

where Q_k is a 2×2 matrix satisfying $det(Q_k) = 1$, such that

$$\dot{u} = \frac{\partial H_k}{\partial v} + \varepsilon p_k(u,v,\varepsilon,\delta), \qquad \dot{v} = -\frac{\partial H_k}{\partial u} + \varepsilon q_k(u,v,\varepsilon,\delta), \tag{2.5}$$

where

$$\begin{aligned}
H_k(u,v) &= -\frac{1}{4}u^4 + \sum_{j\ge 5} h^k_{j,0}u^j + v^2 \sum_{i+j\ge 0} h^k_{i,j}u^i v^j, \; k = 1,2, \\
p_k(u,v,0,\delta) &= \sum_{i+j\ge 0} a^k_{i,j}u^i v^j, \quad q_k(u,v,0,\delta) = \sum_{i+j\ge 0} b^k_{i,j}u^i v^j, \; k = 1,2.
\end{aligned}$$

Then we can apply the formula for the local coefficients c_2, c_3, c_5, c_6 and c_7 in Theorem A to the new system (2.5) with $k = 1,2$, and obtain the corresponding values $c_i(S_k,\delta)$, $i = 2,3,5,6,7, k = 1,2$ for the nilpotent saddles $S_k, k = 1,2$. Now we are ready to state the following theorem.

Theorem 2.1. *Consider the* C^ω *system (1.1) and suppose (1.2) satisfy assumptions* (A_1) *and* (A_2). *Then near* $h = 0$ *corresponding to heteroclinic loop* L_0, *Melnikov function of system (1.1) has the following asymptotic expansion:*

$$\begin{aligned}
M(h) = \; & \left[\tilde{c}_1 + \tilde{c}_4|h| + \textstyle\sum_{j\ge 2}\tilde{A}_j|h|^j\right] + |h|^{3/4}\left[\tilde{c}_2 + \tilde{c}_6|h| + \textstyle\sum_{j\ge 2}\tilde{B}_j|h|^j\right] \\
& + \ln|h|\left[\tilde{c}_3 h + \tilde{c}_7 h^2 + \textstyle\sum_{j\ge 3}\tilde{C}_j|h|^j\right] + |h|^{5/4}\left[\tilde{c}_5 + \textstyle\sum_{j\ge 2}\tilde{D}_j|h|^j\right]
\end{aligned} \tag{2.6}$$

where

$$\tilde{c}_1 = \tilde{c}_1(\delta) = M(0,\delta) = \oint_{L_0} qdx - pdy|_{\varepsilon=0} = \sum_{k=1}^{2} \oint_{L_k} (qdx - pdy)|_{\varepsilon=0},$$
$$\tilde{c}_i = \tilde{c}_i(\delta) = c_i(S_1,\delta) + c_i(S_2,\delta), \quad i = 2,3,5,6,7$$

and if $c_2(S_1,\delta) = c_2(S_2,\delta) = c_3(S_1,\delta) = c_3(S_2,\delta) = 0$ *then*

$$\tilde{c}_4 = \tilde{c}_4(\delta) = \oint_{L_0} (p_x + q_y)|_{\varepsilon=0} dt = \sum_{k=1}^{2} \int_{L_k} (p_x + q_y)|_{\varepsilon=0} dt. \tag{2.7}$$

Proof. The idea of proof is borrowed from [18]. Let U_i denote a disk of diameter $\epsilon_0 \geq 0$ with centers at $S_k, k = 1,2$ respectively (see Figure 1). Then for ϵ_0 sufficiently small we can write

$$M(h,\delta) = I_1(h,\delta) + I_2(h,\delta) + I_3(h,\delta), \quad \text{for } 0 < -h \ll 1, \tag{2.8}$$

where

$$I_k(h,\delta) = \oint_{L_{h,k}} (qdx - pdy)|_{\varepsilon=0}, \qquad k = 1,2,3$$
$$L_{h,k} = L_h \cap U_k, \quad k = 1,2, \qquad L_{h,3} = \overline{(L_h \setminus \cup_{k=1}^{2} L_{h,k})}.$$

By Theorem (2) in [23] we can apply the formula for the local coefficients c_2, c_3, c_5, c_6 and c_7 in Theorem A to the system (2.5) with $k = 1,2$ and obtain the following expansion of I_k:

$$\begin{aligned} I_k(h) \;=\; & c_2(S_k,\delta)|h|^{3/4} + c_3(S_k,\delta)h\ln|h| + c_5(S_k,\delta)|h|^{5/4} + c_6(S_k,\delta)|h|^{7/4} \\ & + c_7(S_k,\delta)h^2\ln|h| + O(h^2) + \varphi_k(h,\delta) \end{aligned} \tag{2.9}$$

for $0 < -h \ll 1$ and $\varphi_k \in C^\omega$ at $h = 0$, with $\varphi_k(0,\delta) = O(\varepsilon_0)$. According to (2.8)-(2.9) we have

$$\begin{aligned} M(h,\delta) \;=\; & \tilde{c}_2|h|^{3/4} + \tilde{c}_3 h\ln|h| + \tilde{c}_5|h|^{5/4} + \tilde{c}_6|h|^{7/4} + \tilde{c}_7 h^2\ln|h| \\ & + O(h^2) + N(h,\delta) \end{aligned} \tag{2.10}$$

for $0 < -h \ll 1$ where $N(h,\delta) = \varphi_1(0,\delta) + \varphi_2(0,\delta) + I_3(h,\delta)$ and $\tilde{c}_i = \tilde{c}_i(\delta) = c_i(S_1,\delta) + c_i(S_2,\delta)$, $i = 2,3,5,6,7$. Let

$$N(h,\delta) = \tilde{c}_1(\delta) + \tilde{c}_4(\delta)h + O(h^2). \tag{2.11}$$

It is easy to see that

$$\begin{aligned} \tilde{c}_1(\delta) &= \varphi_1(0,\delta) + \varphi_2(0,\delta) + I_3(0,\delta) \\ &= \lim_{\varepsilon_0\to 0} [\varphi_1(0,\delta) + \varphi_2(0,\delta) + I_3(0,\delta)] \\ &= \lim_{\varepsilon_0\to 0} I_3(0,\delta) = \oint_{L_0} (qdx - pdy)|_{\varepsilon=0} \\ &= \sum_{i=1}^{2} \oint_{L_i} (qdx - pdy)|_{\varepsilon=0} = M(0,\delta) \end{aligned} \tag{2.12}$$

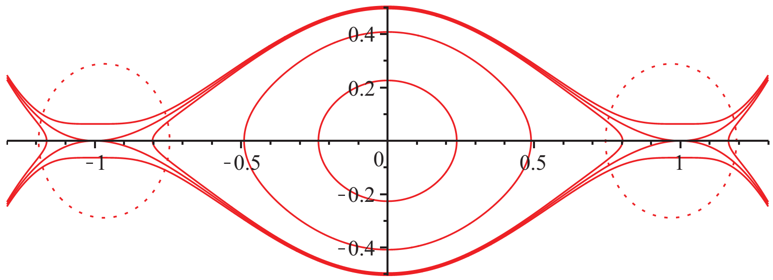

Figure 1. Level curves of equation (H_ε) for $0 \le h \le 1/8$

since $\varphi_k(0,\delta) = O(\varepsilon_0), k = 1,2$. We only need to give the formula for $c_4(\delta)$. By (2.10) and (2.11), we have

$$\tilde{c}_4(\delta) + O(h) = N_h(h,\delta) = M_h(h,\delta) + \frac{3}{4}\tilde{c}_2|h|^{-1/4} - \tilde{c}_3(1 + \ln|h|) + O(|h|^{1/4}),$$

according to [9] we know that $M_h(h,\delta) = \oint_{L_h}(p_x + q_y)|_{\varepsilon=0}dt$, then

$$\tilde{c}_4(\delta) = N_h(0,\delta) = \lim_{h\to 0}\left[\oint_{L_h}(p_x + q_y)|_{\varepsilon=0}dt + \frac{3}{4}\tilde{c}_2|h|^{-1/4} - \tilde{c}_3(1 + \ln|h|)\right].$$

If $c_2(S_1,\delta) = c_2(S_2,\delta) = 0$ and $c_3(S_1,\delta) = c_3(S_2,\delta) = 0$ then it is obvious that $\tilde{c}_2(\delta) = \tilde{c}_3(\delta) = 0$ and therefore

$$\tilde{c}_4(\delta) = \oint_{L_0}(p_x + q_y)|_{\varepsilon=0}dt = \sum_{k=1}^{2}\int_{L_k}(p_x + q_y)|_{\varepsilon=0}dt.$$

This completes the proof. □

3. Application

In this section we provide a complete description of the number and the possible configurations of limit cycles for system (H_ε) in the plane.

3.1. Local stability and Hopf bifurcation of system (H_ε)

In this subsection we study the topological classifications of the equilibria of system (H_ε), and show that the system can undergo a degenerate Hopf bifurcation from which at most three limit cycles emerge near the equilibrium $O(0,0)$. Moreover, we show that system (H_ε) does not have a closed orbit surrounding the equilibria $S_1(1,0)$ and $S_2(-1,0)$ of system (H_ε).

Clearly, system (H_ε) always has three equilibria $S_1(1,0)$, $O(0,0)$ and $S_2(-1,0)$ for each set of parameters (a,b,c). In the following lemma we give a detailed classification of all possible dynamics of these equilibria.

Lemma 3.1. *Suppose $0 < \varepsilon \ll 1$. Then equilibrium $O(0,0)$ is a focus and S_i, $i = 1,2$ are degenerated (or nilpotent) saddles. More precisely we have:*

(O_i) *if* $a \neq 0$ *and* $|\varepsilon a| < 2$*, then* $O(0,0)$ *is a hyperbolic focus. And it is stable (unstable) if* $a < 0$ $(a > 0)$*;*

(O_{ii}) *if* $a = 0$ *and* $b \neq 0$*, then* $O(0,0)$ *is a weak focus of order one. And it is stable (unstable) if* $b < 0$ $(b > 0)$*;*

(O_{iii}) *if* $a = 0,\ b = 0$ *and* $c \neq 0$*, then* $O(0,0)$ *is a weak focus of order two. It is stable (unstable) if* $c < 0$ $(c > 0)$*;*

(O_{iv}) *if* $a = 0,\ b = 0$ *and* $c = 0$*, then* $O(0,0)$ *is an unstable weak focus of order three;*

(S_i) *if* $1 + a + b + c \neq 0$*, then* $S_1(1,0)$ *and* $S_2(-1,0)$ *are degenerated saddles;*

(S_{ii}) *if* $1 + a + b + c = 0$*, then* $S_1(1,0)$ *and* $S_2(-1,0)$ *are nilpotent saddles.*

Proof. We first study the equilibrium $O(0,0)$. The statement (O_i) can be proved by a straightforward calculation of eigenvalues at $O(0,0)$ for (H_ε).

To prove the statements $(O_{ii})-(O_{iv})$, we use change of coordinate $X = x,\ Y = y - \varepsilon(bx^3/3 + cx^5/5 + x^7/7)$ to convert system (H_ε) to the following Liénard system

$$\dot{X} = Y - F(X), \quad \dot{Y} = -g(X), \tag{3.1}$$

where $F(X) = -\varepsilon(bX^3/3 + cX^5/5 + X^7/7)$, and $g(X) = -X(X^2 - 1)^3$. For convenience we still use x and y instead of X and Y, respectively. It is clear that $g(0) = F(0) = F'(0) = 0$ and $g'(0) > 0$. Let $G(x) = \int_0^x g(s)ds$. According to lemma 1 and lemma 2 in [10] if there exists a C^∞ function $\alpha(x)$ in a small neighborhood of zero with $\alpha(x) = -x + O(x^2)$, such that $G(\alpha(x)) \equiv G(x)$ and $F(\alpha(x)) - F(x) = \sum_{i\geq 1} B_i x^i$, then the equilibrium $O(0,0)$ of (3.1) is a weak focus of order k if $B_j = 0,\ j = 1, 2, \cdots, 2k$, and $B_{2k+1} \neq 0$. And it is stable (unstable) if $B_{2k+1} < 0$ $(B_{2k+1} > 0)$. By symmetry, we have $\alpha(x) = -x$ and

$$F(\alpha(x)) - F(x) \quad = \quad 2\varepsilon\left(bx^3/3 + cx^5/5 + x^7/7\right).$$

Now using the lemma 1 and lemma 2 in [10] we can obtain statements $(O_{ii})-(O_{iv})$. Now we consider the equilibrium $E_1(1,0)$. By moving $E_1(1,0)$ to the origin, the system (H_ε) becomes

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= (x+1)x^3(x+2)^3 + \varepsilon\left[a + b(x+1)^2 + c(x+1)^4 + (x+1)^6\right]y.\end{aligned} \tag{3.2}$$

If $1 + a + b + c \neq 0$, then the eigenvalues of system (3.2) at $(0,0)$ are zero and $\varepsilon(1 + a + b + c)$. Therefore, $(0,0)$ is a degenerate equilibrium for (3.2). In order to classify $(0,0)$ topologically let us denote $(1 + a + b + c)\varepsilon = \mu$ and

$$X = x - \mu^{-1}y, \quad Y = y, \quad \tau = \mu t.$$

Using this transformation system (3.2) becomes

$$\frac{dX}{d\tau} = p_2(X,Y), \quad \frac{dY}{d\tau} = Y + q_2(X,Y), \tag{3.3}$$

where

$$\begin{aligned}p_2(X,Y) &= -\varepsilon(6 + 4c + 2b)\mu^{-2}Y(X + \mu^{-1}Y) - 8\mu^{-2}(X + \mu Y)^3\\ &\quad -(15 + b + 6c)\varepsilon\mu^{-2}(X + \mu^{-1}Y)^2Y + O(|X,Y|^4),\\ q_2(X,Y) &= -\mu p_2(X,Y).\end{aligned}$$

By implicit function theorem, we know that, there exists a smooth function $Y = \varphi(X)$ and a small positive number δ such that $\varphi(X) + q_2(X, \varphi(X)) = 0$ for $|X| < \delta$, where

$$\varphi(X) = -8\mu^{-1}X^3 + O(X^4).$$

Therefore, $p_2(X, \varphi(X)) = -8\mu^{-2}X^3 + O(X^4)$. According to theorem 3.5 in [6], the equilibrium $(0, 0)$ is a saddle. This implies the statement (S_i) for $S_1(1, 0)$ and by symmetry this is true for $S_2(-1, 0)$ as well.

If $1 + a + b + c = 0$, then both eigenvalues of system (3.2) are zeros and the linearized matrix is not zero matrix. Hence, in this case the equilibrium $(0, 0)$ is nilpotent and by Theorem 3.5 in [6], $(0, 0)$ is a nilpotent saddle for system (3.2). This implies statement (S_{ii}) for $S_1(1, 0)$ and by symmetry this is true for $S_2(-1, 0)$ as well. □

By Hopf bifurcation theorem and lemma 3.1, there are three surfaces for which as the parameters a, b and c pass through, the equilibrium $O(0, 0)$ can exprience a series of Hopf bifurcation for any given ε with $0 < \varepsilon \ll 1$. The Hopf bifurcation surface of codimension one is given by

$$H_1 = \{(a, b, c, \varepsilon) : \ a = 0, \ b \neq 0, \ 0 < \varepsilon \ll 1\}.$$

And on the closure of H_1 there is a curve

$$H_2 = \{(a, b, c, \varepsilon) : \ a = 0, \ b = 0, \ c \neq 0 < \varepsilon \ll 1\},$$

which is a degenerate Hopf bifurcation curve of codimension two. On the closure of this curve, there is point

$$H_3 = \{(a, b, c, \varepsilon) : \ a = 0, \ b = 0, \ c = 0 < \varepsilon \ll 1\},$$

which is degenerate Hopf point of codimension three. More precisely, we have the following theorem.

Theorem 3.1. *Suppose $0 < \varepsilon \ll 1$. A series of Hopf bifurcations occurs near equilibrium $O(0, 0)$ of system (H_ε) in a small neighborhood of bifurcation point $(a, b, c) = (0, 0, 0)$. In particular, a unique unstable limit cycle bifurcates from equilibrium $O(0, 0)$ of system (H_ε) for $a = 0, \ b = 0$ and as c decreases from zero; a unique stable limit cycles bifurcates from equilibrium $O(0, 0)$ of system (H_ε) for $a = 0, \ c < 0$ and as b increases from zero; and a unique unstable limit cycle bifurcates from equilibrium $O(0, 0)$ of system (H_ε) for $c < 0, \ b > 0$ and as a decreases from zero. Therefore, system (H_ε) has three limit cycles surrounding the equilibrium $O(0, 0)$ for $a < 0, \ b > 0$ and as $c < 0$, in which two of the limit cycles are unstable and the other is stable.*

Theorem 3.1 implies that the maximum number of small amplitude limit cycles which can bifurcate from equilibrium $O(0, 0)$ of system (H_ε) is three. In the following we study the existence of a closed orbit surrounding two equilibria of system (H_ε) and the maximum number of limit cycles (not only small amplitude) of system (H_ε) in a small neighborhood of bifurcation value $(a, b, c) = (0, 0, 0)$ for $0 < \varepsilon \ll 1$.

Theorem 3.2. *Suppose $0 < \varepsilon \ll 1$. Then*

(i) System (H_ε) has no closed orbits surrounding two equilibria $E_1(1, 0)$ or $E_2(-1, 0)$.

(ii) System (H_ε) does not have a closed orbit if $a = b = c = 0$.

Proof. We first prove the statement (i) by contradiction. Suppose that system (H_ε) has a closed orbit γ surrounding $E_1(1,0)$. Then γ crosses line $x = 1$ and positive x-axis respectively at $U(1, y_1^+)$, $D(1, y_1^-)$ and $R(x_r, 0)$, where $y_1^- < 0 < y_1^+$ and $x_r > 1$. Hence, the vector field of system (H_ε) at $P(1, y_1^+)$ is $(y_1^+, \varepsilon(1 + a + b + c)y_1^+)$, and vector field of system (H_ε) at $R(x^+, 0)$ is $(0, x_r(x_r^2 - 1)^3)$. Therefore the orientation of vector field on γ at R is counterclockwise while the orientation of vector field on γ at U is clockwise. This is a contradiction. By symmetry similarly, it can be shown that system (H_ε) has no closed orbits surrounding equilibrium $E_2(-1, 0)$. Thus statement (i) is proved.

Next, we prove the statement (ii). From (i) we know that there will be no periodic orbit surrounding E_1 and E_2. On the other hand the direction of vector field of (H_ε) along lines $x = \pm 1$ for $y > 0$ remain unchanged (similarly for $y < 0$). Using this and part (i) we conclude that the closed orbits encircling the origin can not intersect lines $x = \pm 1$. Therefore we only need to study existence of a closed orbit surrounding only the equilibrium $O(0,0)$ in the the region $D = \{(x, y) : -1 < x < 1, -\infty < y < +\infty\}$ when $a = b = c = 0$. Now we set $a = b = c = 0$, and convert the system (H_ε) to the following system

$$\dot{x} = y - F_1(x), \quad \dot{y} = -g_1(x), \tag{3.4}$$

where $F_1(x) = -\varepsilon x^7/7$, $g_1(x) = -x(x^2 - 1)^3$ and $-1 < x < 1$. It is clear that $xg_1(x) > 0$ for $-1 < x < 1$ and $x \neq 0$. Let

$$G_1(x) = \int_0^x g_1(s)ds = -\frac{1}{8}x^8 + \frac{1}{2}x^6 - \frac{3}{4}x^4 + \frac{1}{2}x^2.$$

Also by straightforward calculations we see that the system of equations

$$F_1(u) = F_1(x), \qquad G_1(u) = G_1(x), \tag{3.5}$$

has no solution (u, x) with $-1 < u < 0$ and $0 < x < 1$. Therefore Theorem 2.4 in [14] implies that system (3.4) does not have a closed orbit for $0 < x < 1$. This implies the statement (ii) and ends the proof of theorem. $\square$

Based on theorems 3.1 and 3.2 we have the following theorem.

Theorem 3.3. *Suppose $0 < \varepsilon \ll 1$. For (a, b, c) in a small neighborhood of $(0, 0, 0)$, system (H_ε) has at most three limit cycles in the plane. Moreover, there exists parameters values in this small neighborhood such that system (H_ε) has exactly three limit cycles surrounding the equilibrium $O(0, 0)$ and no limit cycles surrounding any two equilibria or all three equilibria.*

3.2. Bifurcation of limit cycles from the period annulus

In this subsection we study the maximum number of limit cycles which bifurcate from the period annulus of system (H_ε) for $0 < \varepsilon \ll 1$. We use an algebraic criterion developed in [16] to study the related Melnikov function $M(h, \delta)$ of system (H_ε). But first we give the following definition:

Definition 3.1. The base functions $\{I_i(h),\ i = 1, \ldots, n\}$ in the Melnikov function $M(h, \delta)$ is said to be a Chebyshev system with accuracy k, if number of zeros of any nontrivial linear combination

$$\alpha_0 I_0(h) + \alpha_1 I_1(h) + \cdots + \alpha_n I_n(h)$$

counted with multiplicity is at most $n + k - 1$.

Now consider a Hamiltonian function with the following special form

$$H(x, y) = A(x) + B(x)y^{2m},$$

which is analytic in some open subset of the plane and has a local minimum at the origin. Then there exist a punctured neighborhood $\mathcal{P}$ of the origin foliated by the ovals or period annulus $\gamma_h \subset \{H(x, y) = h\}$. We fix that $H(0, 0) = 0$. The period annulus can be parameterized by the energy levels $h \in (0, h_0)$ for some $h_0 \in (0, +\infty]$. In what follows, we shall denote the projection of $\mathcal{P}$ on the x-axis by (x_ℓ, x_r) with $x_l < 0 < x_r$. It is easy to verify that, under the above assumptions, $xA'(x) > 0$ for any $x \in (x_\ell, x_r) \setminus \{0\}$ and $B(x) > 0$ for all $x \in (x_\ell, x_r)$. Thus there exists a smooth invertible function $z(x)$ with $x_\ell < z(x) < 0$ such that $A(x) = A(z(x))$ for $0 < x < x_r$. The following theorem is Theorem A in [16].

Theorem B. *Let us consider the Abelian integrals*

$$I_i(h) = \int_{\gamma_h} f_i(x) y^{2s-1} dx, \quad i = 0, 1, \ldots, n-1,$$

where, for each $h \in (0, h_0)$, γ_h is the oval surrounding the origin inside the level curve $\{A(x) + B(x)y^{2m} = h\}$, f_i are analytic functions on (x_l, x_r) and $s \in \mathbb{N}$. We define

$$l_i(x) := \frac{f_i(x)}{A'(x)(B(x))^{\frac{2s-1}{2m}}} - \frac{f_i(z(x))}{A'(z(x))(B(z(x)))^{\frac{2s-1}{2m}}}.$$

If the following conditions are verified:

a) *$W[l_0, \ldots, l_i]$ is non-vanishing on $(0, x_r)$ for $i = 0, 1, \ldots, n-2$,*

b) *$W[l_0, \ldots, l_{n-1}]$ has k zeros on $(0, x_r)$ counted with multiplicities, and*

c) *$s > m(n + k - 2)$*

then the base functions $\{I_i(h),\ i = 1 \ldots, n-1\}$ form a Chebyshev system with accuracy k on $(0, h_0)$ where $W[l_0, l_1, \ldots, l_k]$ denotes the Wronskian of the functions $\{l_0, l_1, \ldots, l_k\}$ at $x \in (0, x_r)$.

The applicability of this theorem comes from the fact that finding an upper bound for the Melnikov function $M(h, \delta)$ follows just from some pure algebraic expression. Now we use Theorem B to show that that $\{I_i(h),\ i = 1, 2, 3, 4\}$ in (1.5) has Chebyshev property with accuracy 1 and therefore the number of zeros of the Melnikov function $M(h, \delta)$ in the open interval $(0, 1/8)$ is at most four.

Let us consider Melnikov function (1.5) with Hamiltonian function (1.4), which is a linear combination of $\{I_0(h), I_1(h), I_2(h), I_3(h)\}$, where $I_i(h) = \oint_{\gamma_h} x^{2i} y dx,\ i = 0, 1, 2, 3$ and

$$\gamma_h := \{(x, y) :\ A(x) + B(x)y^2 = h,\ 0 < h < 1/8\},$$

with $A(x) = -x^8/8 + x^6/2 - 3x^4/4 + x^2/2$, and $B(x) = 1/2$. The projection of the period annulus on the x-axis is $(-1, 1)$. Note that $xA'(x) > 0$ for all $x \in (-1, 1)\backslash\{0\}$.

Therefore, there exists an invertible function $z(x)$ with $-1 < z(x) < 0$ such that $A(x) = A(z(x))$ for $0 < x < 1$. In our case $z(x) = -x$. To apply Theorem B, we notice that in this case, $I_i(h) = \int_{\gamma_h} x^{2i} y dx$ and hence $m = 1,\ n = 4$ and $s = 1$. Therefore the hypothesis (c) ($s > m(n + k - 2)$) in Theorem B is not satisfied. However it is possible to overcome this problem using Lemma 4.1 in [8], and obtain some new Abelian integrals for which the corresponding s is large enough to verify the inequality. Here we need to promote the power s to four such that the condition $s > n - 1$ hold. On the oval γ_h, since $2h = 2A(x) + y^2$, we have

$$I_i(h) = \oint_{\gamma_h} x^{2i} y dx = \frac{1}{2h}\left(\oint_{\gamma_h} 2x^{2i}A(x) y dx + \oint_{\gamma_h} x^{2i} y^3 dx\right),\ i = 0, 1, 2, 3. \quad (3.6)$$

Now we apply Lemma 4.1 in [8] with $k = 3$ and $F(x) = 2x^{2i}A(x)$ to get

$$\oint_{\gamma_h} 2x^{2i}A(x) y dx = \oint_{\gamma_h} G_i(x) y^3 dx,$$

where $G_i(x) = \frac{d}{3dx}\left(\frac{2x^{2i}A(x)}{A'(x)}\right) = \frac{g_i}{12(x^2-1)^4}$, and

$$g_i = [(2i + 1)x^8 - (10i + 3)x^6 + (20i + 2)x^4 - (20i - 2)x^2 + (8i + 4)]x^{2i}.$$

By (3.6) we obtain

$$\begin{aligned} I_i(h) &= \frac{1}{2h}\oint_{\gamma_h}\left(x^{2i} + G_i(x)\right) y^3 dx = \frac{1}{4h^2}\oint_{\gamma_h}(2A(x) + y^2)(x^{2i} + G_i(x))y^3 dx \\ &= \frac{1}{4h^2}\left(\oint_{\gamma_h} 2(x^{2i} + G_i(x))A(x)y^3 dx + \oint_{\gamma_h}(x^{2i} + G_i(x))y^5 dx\right). \end{aligned} \quad (3.7)$$

Again we apply Lemma 4.1 in [8] with $k = 5$ and $F(x) = 2(x^{2i} + G_i(x))A(x)$ to get

$$\oint_{\gamma_h} 2(x^{2i} + G_i(x))A(x)y^3 dx = \oint_{\gamma_h} \tilde{G}_i(x) y^5 dx,$$

where $\tilde{G}_i(x) = \frac{d}{5dx}\left(\frac{2(x^{2i}+G_i(x))A(x)}{A'(x)}\right) = \frac{\tilde{g}_i}{-240\,(x^2-1)^8}$, and

$$\begin{aligned} \tilde{g}_i = \ & x^{2i+12}[(4i^2 + 28i + 13)x^4 - (40i^2 + 244i + 92)x^2 + (180i^2 + 936i + 271)] \\ & -x^{2i+6}[(480i^2 + 2064i + 416)x^4 - (832i^2 + 2872i + 364)x^2 \\ & +(960i^2 + 2616i + 212)] \\ & +x^{2i}[(720i^2 + 1584i + 100)x^4 - (320i^2 + 656i + 8)x^2 + (64i^2 + 160i + 64)]. \end{aligned}$$

By (3.7) we obtain

$$\begin{aligned} I_i(h) &= \frac{1}{4h^2}\oint_{\gamma_h}\left(x^{2i} + G_i(x) + \tilde{G}_i(x)\right) y^5 dx \\ &= \frac{1}{8h^3}\oint_{\gamma_h}(2A(x) + y^2)(x^{2i} + G_i(x) + \tilde{G}_i(x))y^5 dx \\ &= \frac{1}{8h^3}\oint_{\gamma_h} 2(x^{2i} + G_i(x) + \tilde{G}_i(x))A(x)y^5 dx \\ &\quad + \frac{1}{8h^3}\oint_{\gamma_h}(x^{2i} + G_i(x) + \tilde{G}_i(x))y^7 dx. \end{aligned} \quad (3.8)$$

Again we apply Lemma 4.1 in [8] with $k = 7$ and $F(x) = 2(x^{2i}+G_i(x)+\tilde{G}_i(x))A(x)$ to get

$$\oint_{\gamma_h} 2(x^{2i} + G_i(x) + \tilde{G}_i(x))A(x)y^5 dx = \oint_{\gamma_h} \bar{G}_i(x)y^7 dx,$$

where $\bar{G}_i(x) = \frac{d}{7dx}(\frac{2(x^{2i}+G_i(x)+\tilde{G}_i(x))A(x)}{A'(x)}) = \frac{\bar{g}_i}{6720\,(x^2-1)^{12}}$, and

$$\begin{aligned}
\bar{g}_i \quad = \quad & (8i^3 + 140i^2 + 614i + 273)x^{2i+24} \\
& -(120i^3 + 1924i^2 + 7722i + 3035)x^{2i+22} \\
& +(840i^3 + 12236i^2 + 44558i + 15213)x^{2i+20} \\
& -(3640i^3 + 47684i^2 + 155922i + 45183)x^{2i+18} \\
& +(10896i^3 + 126936i^2 + 368700i + 88314)x^{2i+16} \\
& -(23760i^3 + 243352i^2 + 621772i + 119898)x^{2i+14} \\
& +(38720i^3 + 344928i^2 + 769648i + 116808)x^{2i+12} \\
& -(47520i^3 + 365232i^2 + 709272i + 83028)x^{2i+10} \\
& +(43584i^3 + 288608i^2 + 488144i + 43224)x^{2i+8} \\
& -(29120i^3 + 168224i^2 + 248816i + 17160)x^{2i+6} \\
& +(13440i^3 + 70336i^2 + 92064i + 5936)x^{2i+4} \\
& -(3840i^3 + 19584i^2 + 24576i + 1152)x^{2i+2} \\
& +(512i^3 + 2816i^2 + 4352i + 1536)x^{2i}.
\end{aligned}$$

From (3.8) we obtain

$$8h^3 I_i(h) = \oint_{\gamma_h} f_i(x)y^7 dx \equiv \tilde{I}_i(h), \tag{3.9}$$

where $f_i(x) = x^{2i} + G_i(x) + \tilde{G}_i(x) + \bar{G}_i(x)$. It is clear that $\{\tilde{I}_0, \tilde{I}_1, \tilde{I}_2, \tilde{I}_3\}$ is an Chebyshev system with accuracy one on $(0, 1/8)$ if and only if $\{I_0, I_1, I_2, I_3\}$ is as well. Now we can apply Theorem B with

$$l_i(x) = \left(\frac{f_i}{A'}\right)(x) - \left(\frac{f_i}{A'}\right)(-x),$$

and $s = 4$, since the condition $s > m(n + k - 2)$ holds. We need to prove that $\{l_0, l_1, l_2, l_3\}$ satisfy hypothesis $((i) - (iii))$ in Theorem B with $k = 1$. To do this we prove the following lemma.

Lemma 3.2.

(i) $W[l_0](x) \neq 0$ *for all* $x \in (0, 1)$;

(ii) $W[l_0, l_1](x) \neq 0$ *for all* $x \in (0, 1)$;

(iii) $W[l_0, l_1, l_2](x) \neq 0$ *for all* $x \in (0, 1)$;

(iv) $W[l_0, l_1, l_2, l_3](x)$ *has one zero on* $(0, 1)$ *counted with multiplicities.*

Proof. Using Maple we compute the above four Wronskians. We find out that

$$W[l_0](x) = \frac{q_0(x)}{x(x^2-1)^{15}}, \quad W[l_0, l_1](x) = \frac{q_1(x)}{x(x^2-1)^{30}},$$

$$W[l_0, l_1, l_2](x) = \frac{q_2(x)}{(x^2-1)^{45}}, \quad W[l_0, l_1, l_2, l_3](x) = \frac{q_3(x)}{(x^2-1)^{59}},$$

where $q_0(x)$ is a polynomial of degree 24, $q_1(x)$ is a polynomial of degree 48, $q_2(x)$ is a polynomial of degree 72 and $q_3(x)$ is a polynomial of degree 94 in x. By applying Sturm's theorem [8], we find that $q_0(x), q_1(x)$ and $q_2(x)$ are nonzero for all $x \in (0,1)$ while $q_3(x)$ has a unique root in the interval $(0,1)$ at $x^* \approx 0.7325481003$. This completes the proof. □

Thereby we have proved the following theorem:

Theorem 3.4. *If the Melnikov function $M(h,\delta)$ is not identically zero then it has at most four zeros, counting multiplicities, in any compact subinterval of $(0,1/8)$ and for all values of parameters (a,b,c). And the number of limit cycles bifurcating from the period annulus is at most four.*

3.3. Asymptotic expansion of Melnikov function $M(h,\delta)$

In this subsection we study the asymptotic expansion of Melnikov function $M(h,\delta)$ at the end points $h=0$ and $h=1/8$, respectively. Using these asymptotic expansions we prove the following theorem:

Theorem 3.5. *There exist some parameter values such that the Melnikov function $M(h,\delta)$ has three isolated zeros in $(0,1/8)$.*

Proof. To obtain the asymptotic expansion of Melnikov function $M(h,\delta)$ as $h \to 0^+$, we compute $M(h,\delta)$ near the elementary center $O(0,0)$. Let $x = r\cos\theta,\ y = r\sin\theta$. Then the oval

$$\gamma_h : (x^2+y^2)/2 - 3x^4/4 + x^6/2 - x^8/8 = h,$$

is transformed into

$$r\left(1 - 3r^2\cos^4\theta/2 + r^4\cos^6\theta - r^6\cos^8\theta/4\right)^{\frac{1}{2}} - \sqrt{2h} = 0,$$

for $0 < h \ll 1$. Let $\rho = \sqrt{2h}$ and define $F(r,\rho)$ to be the left hand expression of the above equality. Applying the Implicit Function Theorem to $F(r,\rho)=0$ at $(r,\rho)=(0,0)$, we obtain that there exists a smooth function $r=\varphi(\rho)$ and a small positive number $0<\delta\ll 1$ such that $F(\varphi(\rho),\rho)\equiv 0$ for $0<\rho<\delta$. It can be easily verified that $\varphi(\rho)$ has the following expansion

$$\begin{aligned}\varphi(\rho) &= \rho + \left(\frac{3}{4}\cos^4\theta\right)\rho^3 + \left(\frac{63}{32}\cos^8\theta - \frac{1}{2}\cos^6\theta\right)\rho^5 \\ &\quad + \left(\frac{891}{128}\cos^{12}\theta - \frac{27}{8}\cos^{10}\theta + \frac{1}{8}\cos^8\theta\right)\rho^7 + O(\rho^9). \end{aligned} \tag{3.10}$$

Now we compute Melnikov function $M(h,\delta)$ in the coordinate system (r,θ). From (3.10) we have

$$\begin{aligned} M(h,\delta) &= \oint_{\gamma_h} (a+bx^2+cx^4+x^6)y\,dx = \iint_{\mathrm{int}\gamma_h} (a+bx^2+cx^4+x^6)\,dx\,dy \\ &= \int_0^{2\pi} d\theta \int_0^{\varphi(\rho)} \left(a + br^2\cos^2\theta + cr^4\cos^4\theta + r^6\cos^6\theta\right) r\,dr. \end{aligned} \tag{3.11}$$

Note that $h = \frac{\rho^2}{2}$. With the help of Maple we use (3.11) to obtain the asymptotic expansion of $M(h,\delta)$ as $h \to 0^+$,

$$\begin{aligned} M(h,\delta) &= \pi h \left[2a + \left(\frac{9}{4}a + b\right) h + \left(\frac{235}{32}a + \frac{15}{4}b + c\right) h^2 \right. \\ &\quad \left. + \left(\frac{16625}{512}a + \frac{2275}{128}b + \frac{105}{16}c + \frac{5}{4}\right) h^3 + O(h^4) \right]. \end{aligned} \tag{3.12}$$

We set

$$\begin{aligned} b_0 &= 2a\pi, \qquad b_1 = (\frac{9}{4}a + b)\pi, \qquad b_2 = (\frac{235}{32}a + \frac{15}{4}b + c)\pi \\ b_3 &= (\frac{16625}{512}a + \frac{2275}{128}b + \frac{105}{16}c + \frac{5}{4})\pi. \end{aligned}$$

If $b_0 = b_1 = b_2 = 0$ then $a = b = c = 0$ and $b_3 = 5/4 \neq 0$. By theorem 2 in [12] we can see that $M(h,\delta)$ has three isolated zeros for $0 < h \ll 1$ in the neighborhood of $(a,b,c) = (0,0,0)$ which coincides with Hopf bifurcation values of system (H_ε) in subsection 3.1. Now let us apply Theorem 2.1 to system (H_ε) and obtain the asymptotic expansion of Melnikov function (1.5) as $h \to (1/8)^-$. It is clear that on the loop $\gamma_{1/8}$ we have $H(x,y) = 1/8$, which implies that $y^{\pm} = \pm\frac{1}{2}(x^2-1)^2$, thus

$$\begin{aligned} \tilde{c}_1(\delta) &= I(0,\delta) = 2\int_{-1}^{1} (a + bx^2 + cx^4 + x^6) y^+ dx \\ &= 16(1/693 + c/315 + b/105 + a/15). \end{aligned}$$

In order to find c_2 and c_3, we have to move saddles S_1 and S_2 to the origin separately. For $S_2 = (-1,0)$ let $X = \sqrt[4]{8}(x+1)$, $Y = y$, and $T = \sqrt[4]{8}t$ and still denote X, Y and T by x, y and t , respectively. Then system (H_ε) becomes

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x^7/64 - 7\sqrt[4]{8}x^6/64 + 9\sqrt{2}x^5/16 - 5\sqrt[4]{2}x^4/4 + x^3 + \varepsilon y q_1(x), \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} q_1(x) &= \sqrt[4]{8}x^6/64 - 3\sqrt{2}x^5/16 + \sqrt[4]{2}\,(120 + 8\,c)\,x^4/128 - (c+5)\,x^3/2 \\ &\quad + \sqrt[4]{8}\,(b + 6c + 15)\,x^2/8 - \sqrt{2}\,(b + 2c + 3)\,x/2 + \sqrt[4]{2}(a + b + c + 1)/2. \end{aligned}$$

For $\varepsilon = 0$ the Hamiltonian function is

$$H(x,y) = -x^8/512 + 2^{3/4}x^7/64 - 3\sqrt{2}x^6/32 + \sqrt[4]{2}x^5/4 - x^4/4.$$

Thus from Theorem A we see that

$$\begin{aligned} c_2(S_2,\delta) &= -2\sqrt[4]{8}\,(a + 1 + b + c)\,/3, \\ c_3(S_2,\delta) &= \left(1/8 - \sqrt{2}\right) c + \left(1/8 - \sqrt{2}/2\right) b + 1/8\,a + 1/8 - 3/2\,\sqrt{2}. \end{aligned}$$

For the nilpotent saddle $S_1 = (1,0)$, we make the transformations $X = \sqrt[4]{8}(1 - x)$, $Y = y$ and $T = -\sqrt[4]{8}t$ and still denote X, Y and T by x, y and t , respectively. Then by $\mathbb{Z}_2$ symmetry system (H_ε) becomes exactly as system (3.13) and therefore $c_2(S_1,\delta) = c_2(S_2,\delta)$ and $c_3(S_1,\delta) = c_3(S_2,\delta)$. Then $\tilde{c}_2(\delta) = -4\sqrt[4]{8}\,(a + 1 + b + c)\,/3$

and $\tilde{c}_3(\delta) = \left(1/4 - 2\sqrt{2}\right)c + \left(1/4 - \sqrt{2}\right)b + 1/4\,a + 1/4 - 3\sqrt{2}$. If $c_2(S_1, \delta) = c_2(S_2, \delta) = 0$ and $c_3(S_1, \delta) = c_3(S_2, \delta) = 0$ then $\tilde{c_2}(\delta) = \tilde{c_3}(\delta) = 0$ and we have $(a, b, c) = (2 + c, -2c - 3, c)$. At this parameters value we obtain

$$\tilde{c}_4(\delta) = \oint_{\gamma_{1/8}} (a + bx^2 + cx^4 + x^6)dt = 2\int_{-1}^{1} (a + bx^2 + cx^4 + x^6)\frac{dx}{y^+} = 8c + \frac{56}{3}.$$

Then the equations $\tilde{c}_1 = \tilde{c}_2 = \tilde{c}_3 = 0$ have a unique solution

$$(a, b, c) = (-1/11, 13/11, -23/11),$$

substituting into $\tilde{c}_4$ we have $\tilde{c}_4 = 64/33 \neq 0$ and

$$\text{rank}\frac{\partial(\tilde{c}_1, \tilde{c}_2, \tilde{c}_3)}{\partial(a, b, c)} = 3,$$

by Theorem 4 in [23] we know that system (H_ε) can have three limit cycles near the heteroclinic loop $\gamma_{1/8}$. □

3.4. Distribution of bifurcated limit cycles

To obtain more limit cycles we consider the limit cycles bifurcated from the annulus not only near the center $O(0, 0)$ but also near the heteroclinic loop γ_0, based on the following discussion.

The Melnikov function $M(h, \delta)$ near the elementary center $O(0, 0)$ has the following expansion (see [12]):

$$M(h, \delta) = \sum_{j \geq 0} b_j(\delta)h^{j+1}, \quad 0 < h \ll 1. \tag{3.14}$$

Also by the result of section 2 the Melnikov function $M(h, \delta)$ near the heteroclinic loop $\gamma_{1/8}$ has the following expansion:

$$\begin{aligned} M(h, \delta) = \quad & \tilde{c}_1(\delta) + \tilde{c}_2(\delta)|h|^{\frac{3}{4}} + \tilde{c}_3(\delta)(h)\ln|h| + \tilde{c}_4(\delta)|h| \\ & + \tilde{c}_5(\delta)|h|^{\frac{5}{4}} + \tilde{c}_6(\delta)|h|^{\frac{7}{4}} + \tilde{c}_7(\delta)h^2\ln|h| + O(h^2), \quad 0 < -h \ll 1. \end{aligned} \tag{3.15}$$

Now we are ready to state the following theorem:

Theorem 3.6. *Consider system (1.1) and suppose there exists $\delta_0 \in D \subset \mathbb{R}^m$ such that*

$$\begin{aligned} & \tilde{c}_1(\delta_0) = \tilde{c}_2(\delta_0) = \cdots = \tilde{c}_m(\delta_0) = 0, \ \tilde{c}_{m+1}(\delta_0) \neq 0, \\ & b_0(\delta_0) = b_1(\delta_0) = \cdots = b_{k-1}(\delta_0) = 0, b_k(\delta_0) \neq 0 \end{aligned} \tag{3.16}$$

and

$$\text{rank}\frac{\partial(\tilde{c}_1, \tilde{c}_2, \cdots, \tilde{c}_m, b_0, b_1, \cdots, b_{k-1})}{\partial\delta} = m + k. \tag{3.17}$$

Then system (1.1) can have $m + k + \frac{1 - sgn(M(h_1,\delta_0)M(h_2,\delta_0))}{2}$ limit cycles for some (ε, δ) near $(0, \delta_0)$ from which m limit cycles are near the heteroclinic loop $\gamma_{1/8}$, k limit cycles are near the center $O(0, 0)$ and $\frac{1 - sgn(M(h_1,\delta_0)M(h_2,\delta_0))}{2}$ limit cycle are surrounding the center $O(0, 0)$, where $h_1 = 0 - \varepsilon_1$, $h_2 = 0 + \varepsilon_2$ with ε_1 and ε_2 are positive and very small.

Proof of Theorem 3.6, is similar to that of Theorem 2.1 in [21] by using implicit function theorem. Here we omit the proof for the sake of brevity.

In our case we have:

$$
\begin{aligned}
b_0(\delta) &= 2a\pi, \qquad b_1(\delta) = (\frac{9}{4}a + b)\pi, \qquad b_2(\delta) = (\frac{235}{32}a + \frac{15}{4}b + c)\pi, \\
b_3(\delta) &= (\frac{16625}{512}a + \frac{2275}{128}b + \frac{105}{16}c + \frac{5}{4})\pi, \\
\tilde{c}_1(\delta) &= 16(\frac{1}{693} + \frac{1}{315}c + \frac{1}{105}b + \frac{1}{15}a), \qquad \tilde{c}_2(\delta) = -\frac{4}{3}\sqrt[4]{8}(a + 1 + b + c), \\
\tilde{c}_3(\delta) &= (\frac{1}{4} - 2\sqrt{2})c + (\frac{1}{4} - \sqrt{2})b + \frac{1}{4}a + \frac{1}{4} - 3\sqrt{2}, \qquad \tilde{c}_4(\delta) = 8c + \frac{56}{3}.
\end{aligned}
$$

Now, we will use the above coefficients and apply Theorem 3.6, to discuss other distributions of limit cycles of system (H_ε).

1. By solving $b_0(\delta) = b_1(\delta) = b_2(\delta) = 0$ we obtain $a = b = c = 0$. If we take $\delta_0 = (0,0,0)$, we obtain $b_3(\delta_0) = \frac{5}{4}\pi, c_1(\delta_0) = \frac{16}{693}$ then $b_3(\delta_0)c_1(\delta_0) > 0$, and $\frac{1-sgn(M(h_1,\delta_0)M(h_2,\delta_0))}{2} = 0$ for $h_1 = \varepsilon_1, h_2 = -\varepsilon_2$ with ε_1 and ε_2 positive and very small. Note that $rank\left(\frac{\partial(b_0,b_1,b_2)}{\partial(a,b,c)}\right) = 3$ and by Theorem 3.6 there exists some (a,b,c,ε) near $(0,0,0,0)$ such that system (H_ε) has 3 limit cycles near the center γ_0, see Fig. 2(a).
2. By solving $b_0(\delta) = b_1(\delta) = c_1(\delta) = 0$ we obtain $a = b = 0, c = -\frac{5}{11}$. Then if we take $\delta_0 = (0,0,-\frac{5}{11})$, we obtain $b_2(\delta_0) = -\frac{5}{11}\pi, c_2(\delta_0) = -\frac{8}{11}\sqrt[4]{8}$ then $b_2(\delta_0)c_2(\delta_0) > 0$, and $\frac{1-sgn(M(h_1,\delta_0)M(h_2,\delta_0))}{2} = 0$ for $h_1 = \varepsilon_1, h_2 = -\varepsilon_2$ with ε_1 and ε_2 positive and very small. Note that $rank\left(\frac{\partial(b_0,b_1,c_1)}{\partial(a,b,c)}\right) = 3$ and by Theorem 3.6 there exists some (a,b,c,ε) near $(0,0,-\frac{5}{11},0)$ such that system (H_ε) has 3 limit cycles, 2 limit cycles are near the center γ_0 and 1 limit cycle is near the heteroclinic loop $\gamma_{\frac{1}{8}}$, see Fig. 2(b).
3. By solving $b_0(\delta) = c_1(\delta) = c_2(\delta) = 0$ we obtain $a = 0, b = \frac{3}{11}, c = -\frac{14}{11}$. Then if we take $\delta_0 = (0,\frac{3}{11},-\frac{14}{11})$, we obtain $b_1(\delta_0) = \frac{3}{11}\pi, c_3(\delta_0) = -\frac{8}{11}\sqrt{2}$ then $b_1(\delta_0)c_3(\delta_0) < 0$, and $\frac{1-sgn(M(h_1,\delta_0)M(h_2,\delta_0))}{2} = 1$ for $h_1 = \varepsilon_1, h_2 = -\varepsilon_2$ with ε_1 and ε_2 positive and very small. Note that $rank\left(\frac{\partial(b_0,c_1,c_2)}{\partial(a,b,c)}\right) = 3$ and by Theorem 3.6 there exists some (a,b,c,ε) near $(0,\frac{3}{11},-\frac{14}{11},0)$ such that system (H_ε) has 4 limit cycles, 1 limit cycle is near the center γ_0, 2 limit cycle are near the heteroclinic loop $\gamma_{\frac{1}{8}}$ and 1 limit cycle is surrounding the center γ_0, (between the center γ_0 and the heteroclinic loop $\gamma_{\frac{1}{8}}$), see Fig. 2(c).
4. By solving $c_1(\delta) = c_2(\delta) = c_3(\delta) = 0$ we obtain $a = -\frac{1}{11}, b = \frac{13}{11}, c = -\frac{23}{11}$. Then if we take $\delta_0 = (-\frac{1}{11},\frac{13}{11},-\frac{23}{11})$, we obtain $b_0(\delta_0) = -\frac{2}{11}\pi, c_4(\delta_0) = \frac{64}{33}$ then $b_1(\delta_0)c_3(\delta_0) < 0$, and $\frac{1-sgn(M(h_1,\delta_0)M(h_2,\delta_0))}{2} = 1$ for $h_1 = \varepsilon_1, h_2 = -\varepsilon_2$ with ε_1 and ε_2 positive and very small. Note that $rank\left(\frac{\partial(c_1,c_2,c_3)}{\partial(a,b,c)}\right) = 3$ and by Theorem 3.6 there exists some (a,b,c,ε) near $(-\frac{1}{11},\frac{13}{11},-\frac{23}{11},0)$ such that system (H_ε) has 4 limit cycles, 3 limit cycles are near the heteroclinic loop $\gamma_{\frac{1}{8}}$ and 1 limit cycle is surrounding the center γ_0, see Fig. 2(d).
5. By solving $b_0(\delta) = b_1(\delta) = 0$, we obtain $a = 0, b = 0$. Then if we take $\delta_0 = (0,0,c)$, we have $b_2(\delta_0) = c\pi, c_1(\delta_0) = \frac{16}{693} + \frac{16}{315}$. If we fix $c \in (-\frac{5}{11}, 0)$, then

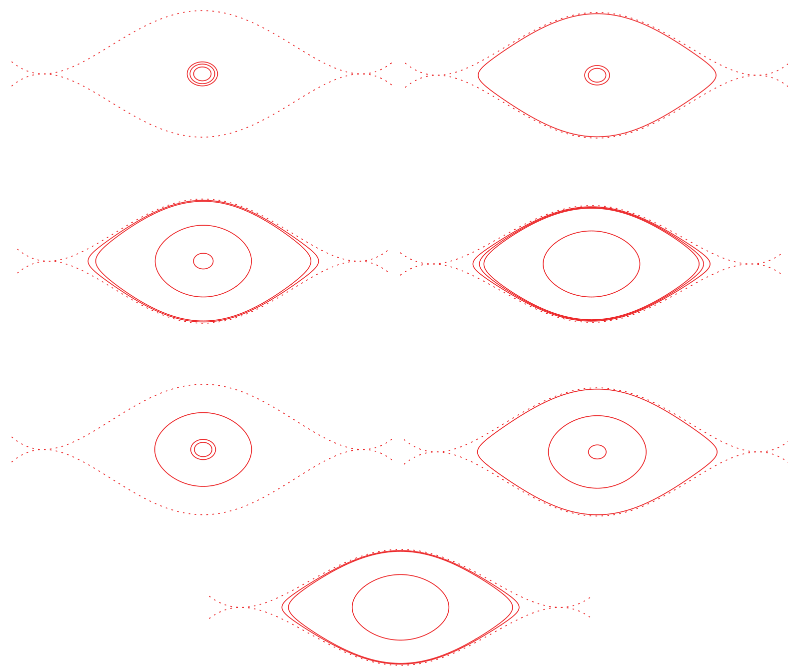

Figure 2. Distribution of limit cycles bifurcated from the period annulus of system (H_ε).

$b_2(\delta_0)c_1(\delta_0) < 0$, and $\frac{1-sgn(M(h_1,\delta_0)M(h_2,\delta_0))}{2} = 1$ for $h_1 = \varepsilon_1$ and $h_2 = -\varepsilon_2$ with ε_1 and ε_2 positive and very small. Note that $rank\left(\frac{\partial(b_0,b_1)}{\partial(a,b,c)}\right) = 2$ and by Theorem 3.6, there exists some (a, b, c, ε) near $(0, 0, c, 0)$ for $c \in (-\frac{5}{11}, 0)$, such that system (H_ε) has 3 limit cycles, 2 limit cycles are near the center γ_0, 1 limit cycle is surrounding the center γ_0, see Fig. 2(e).

6. By solving $b_0(\delta) = c_1(\delta) = 0$, we obtain $a = 0, b = -\frac{5}{33} - \frac{1}{3}c$. Then if we take $\delta_0 = (0, -\frac{5}{33} - \frac{1}{3}c, c)$, we have $b_1(\delta_0) = -(\frac{5}{33} + \frac{1}{3}c)\pi, c_2(\delta_0) = -\frac{4}{3}\sqrt[4]{8}(\frac{28}{33} + \frac{2}{3}c)$. If we fix $c \in (-\infty, -\frac{14}{11}) \cup (-\frac{5}{11}, \infty)$, then $b_1(\delta_0)c_2(\delta_0) < 0$, and $\frac{1-sgn(M(h_1,\delta_0)M(h_2,\delta_0))}{2} = 1$ for $h_1 = \varepsilon_1$ and $h_2 = -\varepsilon_2$ with ε_1 and ε_2 positive and very small. Note that $rank\left(\frac{\partial(b_0,b_1)}{\partial(a,b,c)}\right) = 2$ and by Theorem 3.6, there exists some (a, b, c, ε) near $(0, -\frac{5}{33} - \frac{1}{3}c, c, 0)$ for $c \in (-\infty, -\frac{14}{11}) \cup (-\frac{5}{11}, \infty)$ and ε positive and very small, such that system (H_ε) has 3 limit cycles, 1 limit cycles is near the center γ_0, 1 limit cycles is near the heteroclinic loop $\gamma_{\frac{1}{8}}$ and 1 limit cycle is surrounding the center γ_0, see Fig. 2(f).

7. By solving $c_1(\delta) = c_2(\delta) = 0$, we obtain $a = \frac{14}{99} + \frac{1}{9}c, b = -\frac{113}{99} - \frac{10}{9}c$. Then if we take $\delta_0 = (\frac{14}{99} + \frac{1}{9}c, -\frac{113}{99} - \frac{10}{9}c, c)$, we have $b_0(\delta_0) = 2(\frac{14}{99} + \frac{1}{9}c)\pi, c_3(\delta_0) = -\sqrt{2}(\frac{8}{9}c + \frac{184}{99})$. If we fix $c \in (-\infty, -\frac{23}{11}) \cup (-\frac{14}{11}, \infty)$, then $b_0(\delta_0)c_3(\delta_0) < 0$,and $\frac{1-sgn(M(h_1,\delta_0)M(h_2,\delta_0))}{2} = 1$ for $h_1 = \varepsilon_1$ and $h_2 = -\varepsilon_2$

with ε_1 and ε_2 positive and very small. Note that $rank\left(\frac{\partial(b_0,b_1)}{\partial(a,b,c)}\right) = 2$ and by Theorem 3.6, there exists some (a,b,c,ε) near $(\frac{14}{99}+\frac{1}{9}c, -\frac{113}{99}-\frac{10}{9}c, c, 0)$ for $c \in (-\infty, -\frac{23}{11}) \cup (-\frac{14}{11}, \infty)$, such that system (H_ε) has 3 limit cycles, 2 limit cycles are near the heteroclinic loop $\gamma_{\frac{1}{8}}$ and 1 limit cycle is surrounding the center γ_0, see Fig. 2(g).

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