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ADOMIAN POLYNOMIALS: A POWERFUL TOOL FOR ITERATIVE METHODS OF SERIES SOLUTION OF NONLINEAR EQUATIONS

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Abstract In this article, we illustrate how the Adomian polynomials can be utilized with different types of iterative series solution methods for nonlinear equations. Two methods are considered here: the differential transform method that transforms a problem into a recurrence algebraic equation and the homotopy analysis method as a generalization of the methods that use inverse integral operator. The advantage of the proposed techniques is that equations with any analytic nonlinearity can be solved with less computational work due to the properties and available algorithms of the Adomian polynomials. Numerical examples of initial and boundary value problems for differential and integro-differential equations with different types of nonlinearities show good results.

Keywords Adomian polynomials, differential transform method, homotopy analysis method, nonlinear equations.

MSC(2000) 34L30, 47G20.

1. Introduction

The Adomian decomposition method (ADM) [2] and [3] has been used to give analytic approximate solution for a large class of linear and nonlinear functional equations. Consider the standard nonlinear operator equation

$$Pu + Ru + Qu = g, \tag{1.1}$$

where P is the highest order derivative which is assumed to be easily invertible, R is a linear differential operator of order less than P, Q is nonlinear operator, and g is the source term. The standard ADM defines the solution u by the series

$$u = \sum_{k=0}^{\infty} u_k, \tag{1.2}$$

and replaces the nonlinear term by the series

$$Qu = \sum_{n=0}^{\infty} A_n, \tag{1.3}$$

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where A_n are known as the Adomian polynomials determined formally from the relation

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}[Q(\sum_{i=0}^{\infty}\lambda^i u_i)]\right]_{\lambda=0}. \tag{1.4}$$

Then, the components $u_0,\ u_1,\ u_2,\ \ldots,$ are determined recursively by using the relation

$$\begin{cases} u_0 = v, \\ u_{k+1} = -P^{-1}Ru_k - P^{-1}A_k,\ k \geq 0. \end{cases} \tag{1.5}$$

A great deal of interest has been focused to develop practical techniques that calculate Adomian polynomials without any need for formula (1.4) introduced by Adomian. Now the Adomian polynomials are obtained via several fast algorithms (see [6]-[8] and the references within). Also, the Adomian polynomials have been used to approximate nonlinear terms with other iterative methods [10].

This work illustrates how the Adomian polynomials can be integrated in the well known series solution methods for nonlinear differential equations (DEs). This is accomplished by utilizing these polynomials with two methods that represent two categories of iterative series solution methods. The first category is represented by the differential transform method (DTM) [17] which yields the Taylor series of the solution via algebraic recurrence relation. The second category constitute of the methods that use inverse integral operator and we choose from them the homotopy analysis method (HAM) [13]-[15] as many other methods are considered special cases of it [1]. The main advantage of utilizing the Adomian polynomials with these techniques is that nonlinear DEs can be easily solved with less computational work for any analytic nonlinearity due to the properties and available algorithms of the Adomian polynomials.

2. DTM with Adomian polynomials

The differential transformation of an analytic function $u(t)$ is defined by

$$U(k) = \frac{1}{k!}[\frac{d^k u(t)}{dt^k}]_{t=t_0}, k = 1, 2, ... \ . \tag{2.1}$$

The inverse differential transformation of $U(k)$ is defined by

$$u(t) = \sum_{k=0}^{\infty} U(k)(t - t_0)^k. \tag{2.2}$$

In real applications, function $u(t)$ is expressed by the truncated finite series of the form

$$u(t) = \sum_{k=0}^{N} U(k)(t - t_0)^k. \tag{2.3}$$

Some basic properties of the differential transformation are as follows [17]. Let $u(t)$, $v(t)$ and $w(t)$ be three uncorrelated functions of time t and $U(k)$, $V(k)$, and $W(k)$ be their corresponding differential transformations. Then:

(1) If $u(t) = v(t) \pm w(t)$, then $U(k) = V(k) \pm W(k)$.

(2) If $u(t) = av(t)$, then $U(k) = aV(k)$, where a is a constant.

(3) If $u(t) = v(t)w(t)$, then $U(k) = \sum\limits_{k_i=0}^{k} V(k_i)W(k-k_i)$.

(4) If $u(t) = t^n$, then $U(k) = \delta(k-n)$ where $\delta(k-n) = \{\begin{smallmatrix} 1 & k=n \\ 0 & k\neq n \end{smallmatrix}$.

(5) If $u(t) = \frac{d^n}{dt^n}v(t)$, then $U(k) = \frac{(k+n)!}{k!}V(k+n)$.

(6) If $u(t) = \int\limits_0^t v(x)dx$, then $U(k) = \frac{V(k-1)}{k}$.

To illustrate how the Adomian polynomials are utilized with the DTM, consider a nonlinear function $f(u)$. Then, the Adomian polynomials approximating $f(u)$ can be arranged in the form

$$\begin{aligned}
A_0 &= f(u_0) \\
A_1 &= u_1 f^{(1)}(u_0) \\
A_2 &= u_2 f^{(1)}(u_0) + \frac{1}{2!}u_1^2 f^{(2)}(u_0) \\
A_3 &= u_3 f^{(1)}(u_0) + u_1 u_2 f^{(2)}(u_0) + \frac{1}{3!}u_1^3 f^{(3)}(u_0) \\
A_4 &= u_4 f^{(1)}(u_0) + (u_1 u_3 + \frac{1}{2!}u_2^2) f^{(2)}(u_0) + \frac{1}{2!}u_1^2 u_2 f^{(3)}(u_0) + \frac{1}{4!}u_1^4 f^{(4)}(u_0) \\
&\vdots
\end{aligned}$$

The differential transform components of $f(u)$ can be written in the following form (for $t_0 = 0$)

$$\begin{aligned}
F(0) &= f(u(0)) \\
&= f(U(0)), \\
F(1) &= \frac{d}{dt} f(u(t))\Big|_{t=0} \\
&= u'(0) f^{(1)}(u(0)) \\
&= U(1) f^{(1)}(U(0)), \\
F(2) &= U(2) f^{(1)}(U(0)) + \frac{1}{2!}(U(1))^2 f^{(2)}(U(0)), \\
F(3) &= U(3) f^{(1)}(U(0)) + U(1)U(2) f^{(2)}(U(0)) + \frac{1}{3!}(U(1))^3 f^{(3)}(U(0)), \\
F(4) &= U(4) f^{(1)}(U(0)) + (U(1)U(3) + \frac{1}{2!}(U(2))^2) f^{(2)}(U(0)) \\
&\quad + \frac{1}{2!}(U(1))^2 U(2) f^{(3)}(U(0)) + \frac{1}{4!}(U(1))^4 f^{(4)}(U(0)), \\
&\vdots
\end{aligned}$$

Next, we consider the nonlinear function $g(u^{(n)})$ where $u^{(n)}$ denotes the nth derivative of u. Then, the Adomian polynomials of $g(u^{(n)})$ are given by

$$\begin{aligned}
A_0 &= g(u_0^{(n)}) \\
A_1 &= u_1^{(n)} g^{(1)}(u_0^{(n)}) \\
A_2 &= u_2^{(n)} g^{(1)}(u_0^{(n)}) + \frac{1}{2!}(u_1^{(n)})^2 g^{(2)}(u_0^{(n)})
\end{aligned}$$

$$\begin{aligned}
A_3 &= u_3^{(n)} g^{(1)}(u_0^{(n)}) + u_1^{(n)} u_2^{(n)} g^{(2)}(u_0^{(n)}) + \frac{1}{3!}(u_1^{(n)})^3 g^{(3)}(u_0) \\
A_4 &= u_4^{(n)} g^{(1)}(u_0^{(n)}) + (u_1^{(n)} u_3^{(n)} + \frac{1}{2!}(u_2^{(n)})^2) g^{(2)}(u_0^{(n)}) \\
&\quad + \frac{1}{2!}(u_1^{(n)})^2 u_2^{(n)} g^{(3)}(u_0^{(n)}) + \frac{1}{4!}(u_1^{(n)})^4 g^{(4)}(u_0^{(n)}) \\
&\quad \vdots
\end{aligned}$$

The differential transform components of the nonlinear function $g(u^{(n)})$ have the following form

$$\begin{aligned}
G(0) &= g(u^{(n)}(t))\Big|_{t=0}, \\
&= g(n!\, U(n)). \\
G(1) &= (n+1)!\, U(n+1) g^{(1)}(n!\, U(n)), \\
G(2) &= \frac{(n+2)!}{2!}\, U(n+2) g^{(1)}(n!\, U(n)) \\
&\quad + \frac{1}{2!}((n+1)!\, U(n+1))^2 g^{(2)}(n!\, U(n)), \\
G(3) &= \frac{(n+3)!}{3!}\, U(n+3) g^{(1)}(n!\, U(n)) \\
&\quad + (n+1)!\, U(n+1) \frac{(n+2)!}{2!}\, U(n+2) g^{(2)}(n!\, U(n)) \\
&\quad + \frac{1}{3!}((n+1)!\, U(n+1))^3 g^{(3)}(n!\, U(n)), \\
G(4) &= \frac{(n+4)!}{4!}\, U(n+4) g^{(1)}(n!\, U(n)) \\
&\quad + ((n+1)!\, U(n+1) \frac{(n+3)!}{3!}\, U(n+3) \\
&\quad + \frac{1}{2!}(\frac{(n+2)!}{2!}\, U(n+2))^2) g^{(2)}(n!\, U(n)) \\
&\quad + \frac{1}{2!}((n+1)!\, U(n+1))^2 (\frac{(n+2)!}{2!}\, U(n+2)) g^{(3)}(n!\, U(n)) \\
&\quad + \frac{1}{4!}((n+1)!\, U(n+1))^4 g^{(4)}(n!\, U(n)), \\
&\quad \vdots
\end{aligned}$$

Then, by comparing the differential transform components of the considered nonlinear functions with their Adomian polynomials, we propose the following algorithm. Instead of computing the differential transformation of nonlinear terms, it is directly substituted by $\tilde{A}_k$ which is obtained by replacing each u_k and $u_k^{(n)}$ in the Adomian polynomial component A_k by $U(k)$ and $\frac{(k+n)!}{k!}U(k+n)$, respectively. This idea was initiated with fractional differential transform method in [9].

3. HAM with Adomian polynomials

Consider the following equation

$$N[u(t)] = 0, \tag{3.1}$$

where N is a nonlinear operator, $u(t)$ is an unknown function and t denotes the independent variable. By generalizing the traditional homotopy method, Liao [14] constructs the so-called zero-order deformation equation

$$(1-p)L[\phi(t;p)-u_0(t)] = p\hbar H(t)N[\phi(t;p)], \tag{3.2}$$

where $p \in [0,1]$ is an embedding parameter, $\hbar$ is a nonzero auxiliary parameter, $H(t)$ is an auxiliary function, L is an auxiliary linear operator, $u_0(t)$ is an initial guess of $u(t)$ and $\phi(t;p)$ is an unknown function. It is important to note that we have great freedom to choose auxiliary objects such as $\hbar$ and L in HAM. Obviously, when $p=0$ and $p=1$, we have $\phi(t;0)=u_0(t)$, $\phi(t;1)=u(t)$, respectively. Thus, as p increases from 0 to 1, the solution $\phi(t;p)$ varies from the initial guess $u_0(t)$ to the solution $u(t)$. Expanding $\phi(t;p)$ in Taylor series with respect to p, one has

$$\phi(t;p) = u_0(t) + \sum_{m=1}^{\infty} u_m(t)p^m, \tag{3.3}$$

where

$$u_m(t) = \frac{1}{m!}\frac{\partial^m \phi(t;p)}{\partial p^m}\mid_{p=0}. \tag{3.4}$$

If the auxiliary linear operator, the initial guess and the auxiliary parameter $\hbar$ and the auxiliary function are so properly chosen, then, as proved by Liao [14], the series (3.3) converges at $p=1$ and one has

$$u(t) = u_0(t) + \sum_{m=1}^{\infty} u_m(t), \tag{3.5}$$

which must be one of solutions of the original nonlinear equation, as proved by Liao[14]. Using definition (3.4), the governing equation of the HAM can be deduced from the zero-order deformation equation (3.2) as follows. Define the vector

$$\overrightarrow{u}_n = \{u_0(t), u_1(t), u_2(t), ..., u_n(t)\}. \tag{3.6}$$

Differentiating equation (3.2) m times with respect to the embedding parameter p and then setting $p=0$ and finally dividing them by $m!$, we have the so-called m^{th}-order deformation equation

$$L[u_m(t) - \chi_m u_{m-1}(t)] = \hbar H(t)\Re_m[\overrightarrow{u}_{m-1}(t)], \tag{3.7}$$

where

$$\Re_m[\overrightarrow{u}_{m-1}] = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(t;p)]}{\partial p^{m-1}}\mid_{p=0} \tag{3.8}$$

and

$$\chi_m = \begin{cases} 0, m \leq 1, \\ 1, m > 1. \end{cases} \tag{3.9}$$

Applying the inverse operator L^{-1} to both sides of (3.7), $u_m(t)$ can be easily obtained.

In the suggested approach here, the nonlinear term in $\Re_{m+1}[\overrightarrow{u}_m]$ is replaced by the m^{th} term of the Adomian polynomials series that approximates it.

4. Examples

In this section, the proposed schemes are applied to solve initial and boundary value problems for ordinary differential and integro-differential equations with different types of nonlinearity. The Adomian polynomials are generated by using the algorithm based on corollary 3 in [6]. Symbolic computations are carried out using Mathematica.

Example 4.1. Consider the quadratic Riccati differential equation [4]

$$\begin{cases} \frac{du}{dt} = 2u - u^2 + 1, \\ u(0) = 0. \end{cases} \tag{4.1}$$

First we solve it using the DTM with Adomian polynomials. This yields the recurrence scheme

$$\begin{cases} (k+1)U(k+1) = 2U(k) - \tilde{A}_k + \delta(k),\ \ k = 0, 1, 2, ..., \\ U(0) = 0, \end{cases} \tag{4.2}$$

where $\tilde{A}_k$ are obtained from the Adomian polynomials of the nonlinear term u^2 as illustrated in Table (1)

Table 1.

A_k and $\tilde{A}_k$ for the power nonlinearity u^2.

k	0	1	2	3
A_k	u_0^2	$2u_0u_1$	$u_1^2 + 2u_0u_2$	$2u_0u_3 + 2u_1u_2$
$\tilde{A}_k$	$[U(0)]^2$	$2U(0)U(1)$	$[U(1)]^2 + 2U(0)U(2)$	$2U(0)U(3) + 2U(1)U(2)$

and so on. The following terms of the solution series are obtained: $U(1) = 1, U(2) = 1, U(3) = \frac{1}{3}, U(4) = \frac{-1}{3}, U(5) = \frac{-7}{15}, U(6) = \frac{-7}{45}, \ldots$. Substitute these values in the inverse differential transform, we obtain

$$u(t) = t + t^2 + \frac{t^3}{3} - \frac{t^4}{3} - \frac{7}{15}t^5 - \frac{7}{45}t^6 + ..., \tag{4.3}$$

which is the Taylor series of the exact solution of problem (4.1) given by $u(t) = 1 + \sqrt{2}\tanh(\sqrt{2}t + \frac{1}{2}\log(\frac{\sqrt{2}-1}{\sqrt{2}+1}))$.

Now, we use the HAM with Adomian polynomials. Some straightforward choices for this problem are the following: the auxiliary linear operator is

$$L[\phi] = \frac{d}{dt}(\phi), \tag{4.4}$$

and operator N in equation (3.2) is chosen as

$$N[\phi] = \frac{d}{dt}(\phi) - 2\phi + \phi^2 - 1. \tag{4.5}$$

Then, the m^{th}-order deformation equation for this problem is given by

$$\frac{d}{dt}[u_m(t) - \chi_m u_{m-1}(t)] = \hbar H(t)\Re_m[\overrightarrow{u}_{m-1}(t)], \tag{4.6}$$

where $\Re_m[\overrightarrow{u}_{m-1}(t)]$ is given by

$$\Re_m[\overrightarrow{u}_{m-1}(t)] = \frac{d}{dt}[u_{m-1}(t)] - 2u_{m-1}(t) + A_{m-1} - (1-\chi_m), \tag{4.7}$$

where A_i are the Adomian polynomials of the nonlinear term u^2. We choose $H(t) = 1$ and $u_0(t) = t$. Then, by applying the inverse integral operator, the following terms are obtained

$$\begin{aligned} &u_0(t) = t, \\ &u_1(t) = \hbar(-t^2 + \tfrac{t^3}{3}), \\ &u_2(t) = \hbar(1+\hbar)(-t^2 + \tfrac{t^3}{3}) + \hbar^2(\tfrac{2t^3}{3} - \tfrac{2t^4}{3} + \tfrac{2t^5}{15}), \\ &\vdots \end{aligned}$$

and an approximate solution is obtained as a partial sum of these terms. Figure (1) shows the $\hbar$-curves of 10^{th} order HAM solution at different values of time and the horizontal line segment that denotes the valid region of $\hbar$ that guarantees convergence. We choose $\hbar = -0.25$ and graph it along with exact solution and $\hbar = -0.5$ in Figure (2).

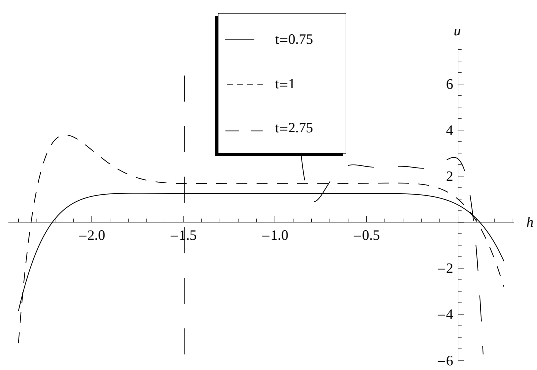

Figure 1. The $\hbar$-curves of 10^{th} order HAM solution of Example (1).

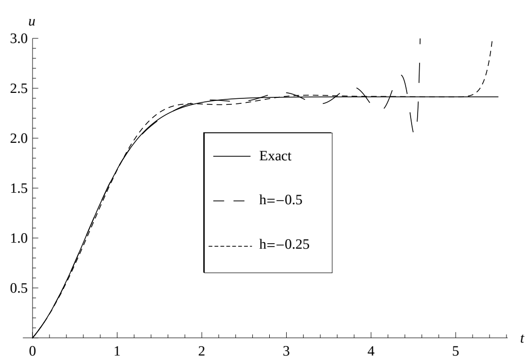

Figure 2. Exact and 10^{th} order HAM solution of Example (1).

Example 4.2. Consider the Lane–Emden type equation [16]

$$\begin{cases} u'' + \frac{2}{t}u' + 8e^{u} + 4e^{u/2} = 0, \\ u(0) = 0, \;\; u'(0) = 0. \end{cases} \tag{4.8}$$

For the DTM with Adomian polynomials, the recurrence scheme for this problem for non-negative integer k is given by

$$\begin{cases} \sum_{j=0}^{k} \delta(j-1)[W(k-j) + \tilde{A}_{k-j}] + 2(k+1)U(k+1) = 0, \\ U(0) = 0, \;\; U(1) = 0, \end{cases} \tag{4.9}$$

where $W(k) = (k+1)(k+2)U(k+2)$ and $\tilde{A}_k$ are obtained from the Adomian polynomials for the nonlinear term $8e^{u} + 4e^{u/2}$ as shown in Table (2)

Table 2.

A_k and $\tilde{A}_k$ for the exponential nonlinearity $8e^{u} + 4e^{u/2}$.

k	0	1
A_k	$4e^{\frac{u_0}{2}} + 8e^{u_0}$	$u_1(2e^{\frac{u_0}{2}} + 8e^{u_0})$
$\tilde{A}_k$	$4e^{\frac{U(0)}{2}} + 8e^{U(0)}$	$U(1)(2e^{\frac{U(0)}{2}} + 8e^{U(0)})$

Table 2. ctd.

k	2
A_k	$2u_2(4e^{u_0} + e^{\frac{u_0}{2}}) + u_1^2(4e^{u_0} + \frac{1}{2}e^{\frac{u_0}{2}})$
$\tilde{A}_k$	$2U(2)(4e^{U(0)} + e^{\frac{U(0)}{2}}) + [U(1)]^2(4e^{U(0)} + \frac{1}{2}e^{\frac{U(0)}{2}})$

and so on. The following terms are obtained: $U(2) = -2,\ U(3) = 0,\ U(4) = 1,\ U(5) = 0,\ U(6) = -\frac{2}{3},\ U(7) = 0,\ U(8) = \frac{1}{2}, \ldots$ which yields the series solution

$$u(t) = -2t^2 + t^4 - \frac{2}{3}t^6 + \frac{1}{2}t^8 ..., \tag{4.10}$$

which is the Taylor series of problem (4.8) exact solution given by $u(t) = 2\ln[1+t^2]$.

To solve the problem using the HAM with Adomian polynomials, we choose

$$L[\phi] = \frac{d^2}{dt^2}(\phi) \tag{4.11}$$

and

$$N[\phi] = \frac{d^2}{dt^2}(\phi) + \frac{2}{t}\frac{d}{dt}\phi + 8e^{\phi} + 4e^{\phi/2}. \tag{4.12}$$

Then, the m^{th}-order deformation equation for this problem is given by

$$\frac{d^2}{dt^2}[u_m(t) - \chi_m u_{m-1}(t)] = \hbar H(t)\Re_m[\overrightarrow{u}_{m-1}(t)], \tag{4.13}$$

where $\Re_m[\overrightarrow{u}_{m-1}(t)]$ is given by

$$\Re_m[\overrightarrow{u}_{m-1}(t)] = \frac{d^2}{dt^2}[u_{m-1}(t)] + \frac{2}{t}u_{m-1}(t) + A_{m-1}, \tag{4.14}$$

where A_i are the Adomian polynomials for the nonlinear term $8e^{u} + 4e^{u/2}$. Take $H(t) = 1$ and $u_0(t) = 0$. Then, by applying the inverse double integral operator, the following terms are obtained

$$
\begin{aligned}
&u_0(t) = 0,\\
&u_1(t) = 6\hbar t^2,\\
&u_2(t) = 6\hbar(1+\hbar)t^2 + \hbar^2\left(12t^2 + 5t^4\right),\\
&\vdots
\end{aligned}
$$

Figure (3) shows the $\hbar$-curves of 10^{th} order HAM solution at different values of time. We choose $\hbar = -0.15$ and graph it along with exact solution and $\hbar = -0.25$ in Figure (4).

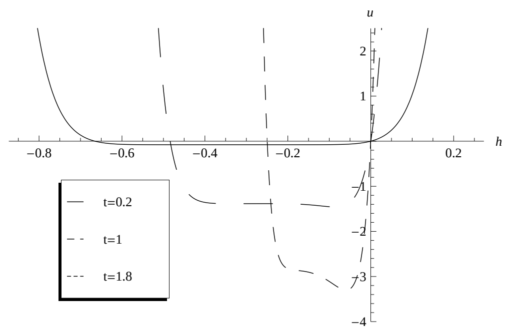

Figure 3. The $\hbar$-curves of 10^{th} order HAM solution of Example (2).

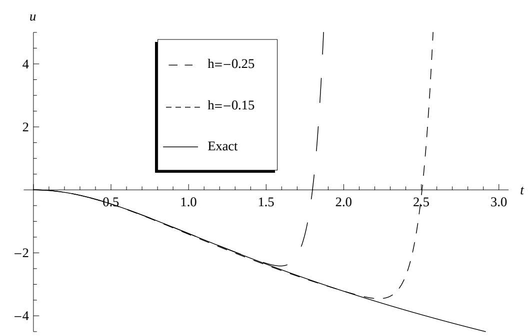

Figure 4. Exact and 10^{th} order HAM solution of Example (2).

Example 4.3. Consider the nonlinear integro-differential equation [11]

$$u' = 1 + \int_0^x u(t)u'(t)dt, 0 \le x \le 1. \tag{4.15}$$

The recurrence scheme of the DTM is given by

$$\begin{cases} (k+1)U(k+1) = \delta(k) + \frac{\tilde{A}_{k-1}}{k}, \ k = 1, 2, ..., \\ U(0) = 0, U(1) = 1, \end{cases} \tag{4.16}$$

where $\tilde{A}_k$ are obtained from the Adomian polynomials of the nonlinear term uu' by replacing each $u_k^{(n)}$ by $\frac{(k+n)!}{k!}U(k+n)$ as illustrated in Table (3)

Table 3.			
A_k and $\tilde{A}_k$ for the product nonlinearity uu'.			
k	0	1	2
A_k	$u_0'u_0$	$u_1'u_0 + u_0'u_1$	$u_2'u_0 + u_1'u_1 + +u_0'u_2$
$\tilde{A}_k$	$U(0)U(1)$	$U(0)U(2) + [U(1)]^2$	$3U(0)U(3) + 3U(2)U(1)$

and so on. The following terms are obtained: $U(3) = \frac{1}{6}$, $U(5) = \frac{1}{30}$, $U(7) = \frac{17}{2520}$, $U(9) = \frac{31}{22680} \cdots$, and zero for even index terms. Substitute these values in the inverse differential transform, we obtain

$$u(x) = x + \frac{x^3}{3} + \frac{x^5}{30} + \frac{17x^7}{2520} + \frac{31x^9}{22680} + ..., \tag{4.17}$$

which is the Taylor series of the exact solution of problem (4.15) given by $u(x) = \sqrt{2}\tan(\frac{x}{\sqrt{2}})$.

Now using the HAM with Adomian polynomials, we choose

$$L[\phi] = \frac{d}{dx}\phi(x), \tag{4.18}$$

$$N[\phi] = \frac{d}{dx}\phi(x) - 1 - \int_0^x \phi(t)\frac{d}{dt}\phi(t)dt. \tag{4.19}$$

Then, the m^{th}-order deformation equation for this problem is given by

$$\frac{d}{dx}[u_m(x) - \chi_m u_{m-1}(x)] = \hbar H(x)\Re_m[\overrightarrow{u}_{m-1}(x)], \tag{4.20}$$

where $\Re_m[\overrightarrow{u}_{m-1}(x)]$ is given by

$$\Re_m[\overrightarrow{u}_{m-1}(x)] = \frac{d}{dx}[u_{m-1}(x)] - \int_0^x A_{m-1}(t)dt - (1-\chi_m), \tag{4.21}$$

where A_i are the Adomian polynomials for the nonlinear term uu'. Take $H(x) = 1$ and $u_0(x) = x$. Then, by applying the inverse integral operator, the following terms are obtained

$$\begin{aligned} &u_0(x) = x, \\ &u_1(x) = -\tfrac{\hbar}{6}x^3, \\ &u_2(x) = \tfrac{-\hbar(1+\hbar)}{6}x^3 + \tfrac{\hbar^2}{30}x^5, \\ &\vdots \end{aligned}$$

Figure (5) shows the $\hbar$-curves of 20^{th} order HAM solution at different values in the domain of the variable x which illustrates a wide valid region of $\hbar$ approximately between -0.2 $< \hbar < -2$. The graphs of the exact and the HAM approximate solutions at different values of $\hbar$ of this problem are close. Thus we show in Table (4) the relative error of the approximate solution at some values of $\hbar$ near the middle point of the valid region of $\hbar$.

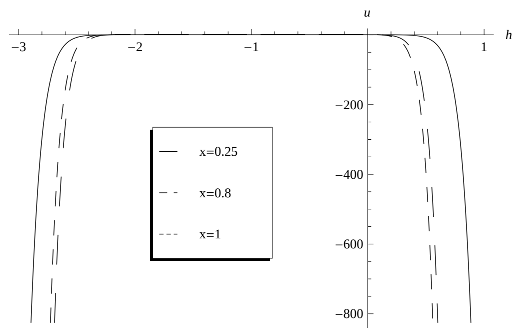

Figure 5. The $\hbar$-curves of 20^{th} order HAM solution of Example (3).

Table 4.			
The relative error of the 20^{th} order HAM solution of Example (3) at some values of $\hbar$			
x	$\hbar=-0.85$	$\hbar=-1$	$\hbar=-1.15$
0.1	$3.78*10^{-7}$	$3.97*10^{-9}$	$-3.70*10^{-7}$
0.2	$6.23*10^{-6}$	$2.55*10^{-7}$	$-5.72*10^{-6}$
0.3	0.00003	$2.92*10^{-6}$	-0.00002
0.4	0.00011	0.00001	-0.00007
0.5	0.00029	0.00006	-0.00016
0.6	0.00066	0.00019	-0.00028
0.7	0.00136	0.00049	-0.00037
0.8	0.00258	0.00111	-0.00035
0.9	0.00463	0.00231	$-5.88*10^{-6}$
1	0.00794	0.00446	0.000980

Example 4.4. Consider the nonlinear boundary value problem

$$\begin{cases} u''=e^{-2t}(u')^2,\ 0<t<1, \\ u(0)=1,\ u(1)=e^2. \end{cases} \tag{4.22}$$

When this type of problems is solved using DTM or HAM, a well known approach is to assume that $u'(0)=\alpha$, where α is a constant that is computed using the given boundary condition.

First we solve problem (4.22) using the DTM with Adomian polynomials. This yields the following recurrence scheme

$$\begin{cases} (k+1)(k+2)U(k+2)=\sum\limits_{l=0}^{k} b(l)\tilde{A}_{k-l},\ k=0,1,2,..., \\ U(0)=0,\ U(1)=\alpha, \end{cases} \tag{4.23}$$

where $b(k)$ are the differential transform components of e^{-2t} and $\tilde{A}_k$ are obtained from the Adomian polynomials of the nonlinear term $(u')^2$ as illustrated in Table (5)

Table 5.				
A_k and $\tilde{A}_k$ for the nonlinearity term $(u')^2$.				
k	0	1	2	3
A_k	$(u'_0)^2$	$2u'_0u'_1$	$(u'_1)^2+2u'_0u'_2$	$2u'_0u'_3+2u'_1u'_2$
$\tilde{A}_k$	$[U(1)]^2$	$4U(1)U(2)$	$[2U(2)]^2+6U(1)U(3)$	$8U(1)U(4)+12U(2)U(3)$

and so on. The following terms of the solution series are obtained: $U(2) = \frac{1}{2}\alpha^2$, $U(3) = \frac{1}{3}\alpha^2(\alpha-1)$, $U(4) = \frac{1}{4}\alpha^2(\alpha-1.57735)(\alpha-0.42265), \ldots$. We compute the inverse differential transform with nine components and substitute the boundary condition to obtain a nonlinear equation in α which yields $\alpha = 2.000026$. Substitute this value of α in the inverse differential transform series solution, we obtain

$$\begin{aligned} u(t) = \quad & 1 + 2.00003t + 2.00005t^2 + 1.3334t^3 + 0.666736t^4 \\ & +0.266722t^5 + 0.0889257t^6 + ..., \end{aligned} \tag{4.24}$$

which is an approximation to the Taylor series of the exact solution of problem (4.1) given by $u(t) = e^{2t}$. Figure (6) shows a comparison between the exact solution and the DTM solution with 9 components.

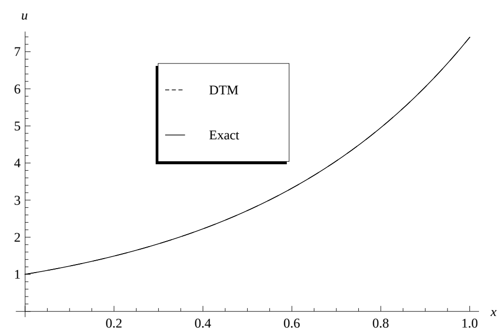

Figure 6. The exact and the 9 components DTM solution of Example (4).

Now, we use HAM with Adomian polynomials. The auxiliary linear operator is

$$L[\phi] = \frac{d^2}{dt^2}(\phi), \tag{4.25}$$

and the operator N in equation (3.2) is chosen as

$$N[\phi] = \frac{d^2}{dt^2}(\phi) - e^{-2t}(\phi')^2. \tag{4.26}$$

Then, the m^{th}-order deformation equation for this problem is given by

$$\frac{d^2}{dt^2}[u_m(t) - \chi_m u_{m-1}(t)] = \hbar H(t)\Re_m[\overrightarrow{u}_{m-1}(t)], \tag{4.27}$$

where $\Re_m[\overrightarrow{u}_{m-1}(t)]$ is given by

$$\Re_m[\overrightarrow{u}_{m-1}(t)] = \frac{d^2}{dt^2}[u_{m-1}(t)] - e^{-2t}A_{m-1}, \tag{4.28}$$

where A_i are the Adomian polynomials of the nonlinear term $(u')^2$. We choose $H(t) = 1$ and $u_0(t) = 1+\alpha t$. Then, by applying the inverse double-integral operator, the following terms are obtained

$$\begin{aligned}
u_0(t) = &\ 1+\alpha t,\\
u_1(t) = &\ \tfrac{1}{4}\hbar\alpha^2(-2t-2sinh^2(t)+sinh(2t)),\\
u_2(t) = &\ \tfrac{1}{16}e^{-4t}\hbar^2\alpha^3(-1+4e^{2t}+e^{4t}(4t-3))+\tfrac{1}{4}\hbar(1+\hbar)\alpha^2(-2t-2sinh^2(t)\\
&+sinh(2t)),\\
&\ \vdots
\end{aligned}$$

The boundary condition is substituted in the HAM series solution with 7 terms to obtain a nonlinear equation in α and $\hbar$. An approach to find the appropriate values of these two parameters is to draw an $\hbar$ curve for the unknown parameter α [12]. Figure (7) shows the horizontal segment that denotes the valid region of $\hbar$ that guarantees convergence. We choose $\hbar = -2.4$ for which $\alpha = 2.00062$. Figure (8) shows the 7^{th} order HAM solution for these values and some other values of $\hbar$ and α along with the exact solution of this problem.

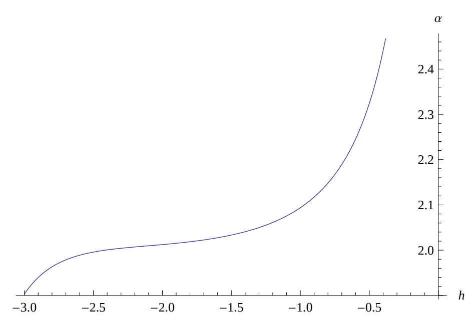

Figure 7. The $\hbar$-curves of 7^{th} order HAM solution with the parameter α for Example (4).

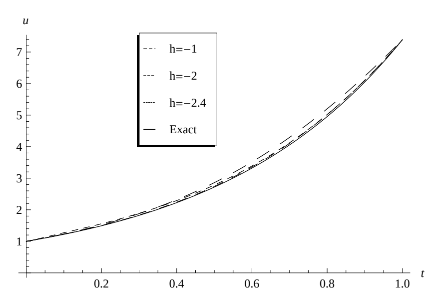

Figure 8. Exact and 7^{th} order HAM solution of Example (4).

5. Conclusion

The use of Adomian polynomials to approximate nonlinear terms in iterative techniques is illustrated. Two approaches are proposed to utilize these polynomials with DTM and HAM. This widens the applications of these methods to deal with different types of nonlinearities and benefits from the fact that Adomian polynomials are already computed via various fast algorithms. The numerical simulations carried out for nonlinear differential and integro-differential equations show good results.

References

[1] F. Abidi and K. Omrani, *The homotopy analysis method for solving the Fornberg Whitham equation and comparison with Adomian's decomposition method*, Comput. and Math. with Appl., 59(2010), 2743-2750.

[2] G. Adomian, *Solving frontier problems of physics: The decomposition method*, Kluwer Academic Publishers, Boston, MA, (1994).

[3] G. Adomian and R. Rach, *Inversion of nonlinear stochastic operators*, J. Math. Anal. Appl., 91(1983), 39-46.

[4] S.H. Behiry, H. Hashish, I.L. El-Kalla and A. Elsaid, *A new algorithm for the decomposition solution of nonlinear differential equations*, Comput. and Math. with Appl., 54(2007), 459-466.

[5] S.H. Chang and I.L. Chang, *A new algorithm for calculating one-dimensional differential transform of nonlinear functions*, Appl. Math. Comput., 195(2008), 799-808.

[6] J.S. Duan, *Convenient analytic recurrence algorithms for the Adomian polynomials*, Appl. Math. Comput., 217(2011), 6337-6348.

[7] J.S. Duan, *An efficient algorithm for the multivariable Adomian polynomials*, Appl. Math. Comput., 217(2010), 2456-2467.

[8] J.S. Duan, *Recurrence triangle for Adomian polynomials,* Appl. Math. Comput., 216(2010), 1235-1241.

[9] A. Elsaid, *Fractional differential transform method combined with the Adomian polynomials*, Appl. Math. Comput., 218(2012), 6899-6911.

[10] S.A. El-Wakil and M.A. Abdou, *New applications of variational iteration method using Adomian polynomials,* Nonlin. Dyn., 52(2008), 41-49.

[11] M. Ghasemi, M.T. Kajani and E. Babolian, *Application of He's homotopy perturbation method to nonlinear integro-differential equations*, Appl. Math. Comput., 188(2007), 538-548.

[12] H.N. Hassan and M.A. El-Tawil, *An efficient analytic approach for solving two-point nonlinear boundary value problems by homotopy analysis method*, Math. Meth. in the Appl. Sci., 34(2011), 977-989.

[13] S.J. Liao, *The proposed homotopy analysis technique for the solution of nonlinear problems*, Ph.D. Thesis, Shanghai Jiao Tong University, (1992).

[14] S.J. Liao, *Beyond perturbation: introduction to the homotopy analysis method*, Chapman and Hall/CRC Press, Boca Raton, (2003).

[15] S.J. Liao, *On the homotopy analysis method for nonlinear problems*, Appl. Math. Comput., 147(2004), 499-513.

[16] A. Yıldırım and T. Ozis, *Solutions of singular IVPs of Lane–Emden type by the variational iteration method*, Nonlin. Anal., 70(2009), 2480-2484.

[17] J.K. Zhou, *Differential transformation and its applications for electrical circuits*, Huazhong University Press, Wuhan, China, 1986 (in Chinese).