MATHEMATICAL ANALYSIS OF SUCCESSIVE LINEAR APPROXIMATION FOR MOONEY-RIVLIN MATERIAL MODEL IN FINITE ELASTICITY

R. Cipolatti[1,†], I-Shih Liu[1,*], M. A. Rincon[2,*]

Abstract   For calculating large deformations of solid bodies, we have proposed a method of successive linear approximation, by considering the relative Lagrangian formulation. In this article we briefly describe this method, which is applied for nearly incompressible Mooney-Rivlin materials. We prove the existence and uniqueness of weak solutions for associated boundary value problems that arise in each step of the method. In our analysis we consider also a non-standard case, where the coefficients present in the constitutive function of Mooney-Rivlin materials do not satisfy the usual E-inequalities.

Keywords   Linearized constitutive equations, Mooney-Rivlin material, relative lagrangian formulation, existence and uniqueness.

MSC(2000)   74B15, 74B20, 74G20, 74G30, 74S30.

1. Introduction

The constitutive equation of a solid body is usually expressed relative to a preferred reference configuration which exhibits specific material symmetries such as isotropy. The constitutive functions are in general nonlinear and linearizations can be used as valid approximation only for small deformations (see Liu [4] and Temam & Miranville [11]). Therefore, the problem for large deformations leads to boundary value problems involving systems of nonlinear partial differential equations.

In order to circumvent the difficulties due to the nonlinearities, we have proposed a new method for solving numerically the boundary value problem for large deformations. It is based on a successive linear approximation by considering the relative Lagrangian formulation. Roughly speaking, the constitutive equations are calculated at each state which will be regarded as the reference configuration for the next state, and assuming that the deformation to the next state is small, the constitutive equations can be linearized at the updated present state.

Although the idea of successive linearisation is widely discussed in the literature (see for example Ciarlet [1] and Ogden [9]), our method presents an essential difference from those, which are usually formulated with domain in the initial reference configuration, i.e., total Lagrangian formulation.

[†]the corresponding author. Email address: cipolatti@im.ufrj.br (R. Cipolatti)
[1]Instituto de Matemática, Universidade Federal do Rio de Janeiro, CP. 68530, 21945-970, Rio de Janeiro, Brasil
[2]Dep. Ciência da Computação, Instituto de Matemática, Universidade Federal do Rio de Janeiro, Brasil

For validation of our method, numerical simulations were done (see Liu [5] and Liu etc. [7]) for two classical problems concerning Mooney-Rivlin (MR) materials, for which the exact solutions are known, namely, the pure shear of a square and the bending of a retangular block into a circular section. The comparison of the numerical results with the exact solutions of these two examples confirms the efficiency of our method. There is another reason to consider the classical MR model in our considerations; it is part of the constitutive function of a viscoelastic model that we are treating in the study of sediment-salt migration (salt diapirism), as can be seen in Liu etc. [8].

In the present paper we consider the mathematical analysis of the boundary value problem obtained by linearizing the constitutive equations of nearly incompressible MR materials relative to the present configuration and prove the existence and uniqueness of weak solutions. Our main result here (see Thm. 5.1) summarizes an essential step for the study of the viscoelastic model for salt diapirism. Importantly, according to Liu [6], we consider in our analysis also the case where the coefficients of the MR model satisfy the condition $0 < s_2 < s_1$ (see Remark 3.1), instead of the so-called E-inequalities, $s_2 \le 0 < s_1$.

We organize this paper as follows. In Section 2 we introduce briefly the notion of relative description and we formally deduce in Section 3 the linearization of the constitutive function of a nearly incompressible MR material. In Section 4 we consider a boundary value problem involving a system of partial differential equations, that are obtained by this linearization, and which corresponds to one of the steps of the successive linear approximation method. As mentioned above, the main result of this paper is contained in Section 5, where we prove the existence and uniqueness of weak solutions of this boundary value problem, by considering its variational formulation. For simplicity, we restrict our analysis to the two-dimensional case, but the arguments presented can be extended to three dimensions.

2. Relative description and successive linear approximation

In this section we introduce the notion of *relative description*. We formally obtain the linearization of a general constitutive equation of a solid body respect to this configuration and we describe the successive linear approximation method.

Let κ_0 be a reference configuration of a solid body $\mathcal{B}$, $\mathcal{B}_0 = \kappa_0(\mathcal{B})$, and let

$$\boldsymbol{x} = \chi(X, t), \quad X \in \mathcal{B}_0$$

be the parametrization of its deformation. Let κ_t be the deformed configuration at time t (which we shall always refer as the present time), $\mathcal{B}_t = \kappa_t(\mathcal{B})$, and

$$F(X, t) = \nabla_X \chi(X, t)$$

be the deformation gradient with respect to the configuration κ_0.

Let κ_τ be the deformed configuration at time $\tau > t$. We define the *relative deformation* from κ_t to κ_τ as the function $\chi_t : \mathcal{B}_t \to \mathcal{B}_\tau$ given by

$$\chi_t(\boldsymbol{x}, \tau) = \chi(X, \tau), \quad \boldsymbol{x} = \chi(X, t) \in \mathcal{B}_t \tag{2.1}$$

and the corresponding *relative displacement* as

$$\boldsymbol{u}_t(\boldsymbol{x}, \tau) = \mathcal{X}_t(\boldsymbol{x}, \tau) - \boldsymbol{x}. \tag{2.2}$$

Taking the gradient relative to $\boldsymbol{x}$ in both sides of (2.2), we obtain

$$H_t(\boldsymbol{x}, \tau) = F_t(\boldsymbol{x}, \tau) - I, \tag{2.3}$$

where I is the identity tensor and

$$H_t(\boldsymbol{x}, \tau) = \nabla_{\boldsymbol{x}} \boldsymbol{u}_t(\boldsymbol{x}, \tau), \quad F_t(\boldsymbol{x}, \tau) = \nabla_{\boldsymbol{x}} \mathcal{X}_t(\boldsymbol{x}, \tau)$$

are called the *displacement gradient* and the *deformation gradient* in the relative description, relative to the present configuration.
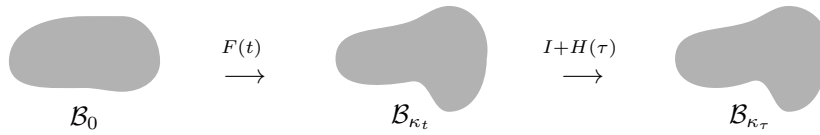
On the other hand, taking the gradient relative to X in both sides of (2.2), we obtain from (2.1) and the chain rule,

$$H_t(\boldsymbol{x}, \tau) F(X, t) = F(X, \tau) - F(X, t),$$

from which we get easily

$$F(X, \tau) = \big(I + H_t(\boldsymbol{x}, \tau)\big) F(X, t). \tag{2.4}$$

We can represent this situation by the following picture:

where and hereafter, for simplicity, sometimes only the time dependence is indicated. Position dependence is usually self-evident and will be indicated for clarity only if necessary.

By considering the time $\tau = t + \Delta t$ for small enough Δt, we can assume that the relative displacement gradient is small,

$$H(\tau) = H_t(\boldsymbol{x}, \tau), \quad \|H(\tau)\| \ll 1.$$

Let T be the Cauchy stress tensor given by the constitutive equation

$$T = \mathcal{F}_{\kappa_0}(F). \tag{2.5}$$

Assuming that the operator $\mathcal{F}_{\kappa_0}$ is differentiable, we can calculate the linearization of the constitutive equation (2.5) relative to the current configuration κ_t, and assuming that Δt is small enough, we have formally

$$T(\tau) \approx T(t) + \mathrm{d}\mathcal{F}_{\kappa_0}(F(t))[F(\tau) - F(t)] = T(t) + \mathrm{d}\mathcal{F}_{\kappa_0}(F(t))[H(\tau)F(t)], \tag{2.6}$$

where $\mathrm{d}\mathcal{F}_{\kappa_0}(F)$ denotes the Fréchet-differential of $\mathcal{F}_{\kappa_0}$ calculated at F. For convenience, we shall write (2.6) as

$$T(\tau) = T(t) + L(F(t))[H(\tau)], \tag{2.7}$$

where $L(F)[H] = \mathrm{d}\mathcal{F}_{\kappa_0}(F)[HF]$ defines a fourth order elasticity tensor $L(F)$ relative to the current configuration κ_t.

The successive linear approximation method is the discrete construction of the parametrization $\chi(X, t)$ based on the previous arguments. More precisely, let $t_0 < \cdots < t_{n-1} < t_n < t_{n+1} < \cdots$ be a sequence of steps with small enough constant spacing Δt, where at each step we set $t = t_n$ and $\tau = t_{n+1}$. Let the deformation gradient $F(X, t_n)$ and the elastic Cauchy stress tensor $T(X, t_n)$, relative to the preferred configuration κ_{t_0}, assumed to be known. If in any way we calculate the relative displacement $\boldsymbol{u}_{t_n}(\boldsymbol{x}, t_{n+1})$, $\boldsymbol{x} \in \kappa_{t_n}(\mathcal{B})$, it allows us to update the new reference configuration $\kappa_{t_{n+1}}$ relative to the next step by using (2.1) and (2.2), i.e.,

$$\chi(X, t_{n+1}) = \boldsymbol{u}_{t_n}(\boldsymbol{x}, t_{n+1}) + \boldsymbol{x}, \quad \boldsymbol{x} \in \kappa_{t_n}(\mathcal{B}),$$

while the deformation gradient (2.4) and the Cauchy stress (2.7), relative to the preferred configuration κ_{t_0}, can be determined at instant t_{n+1} respectively by

$$F(X, t_{n+1}) = \big(I + H_{t_n}(\boldsymbol{x}, t_{n+1})\big)F(X, t_n),$$
$$T(X, t_{n+1}) = T(X, t_n) + L(F(t_n))[H_{t_n}(\boldsymbol{x}, t_{n+1})].$$

Therefore, after updating the boundary data and the eventual body forces acting on the body, we repeat the cycle from the updated reference configuration $\kappa_{t_{n+1}}$.

We remark that this method can easily be extended to constitutive equation $T = \mathcal{F}(F, \dot{F})$ for viscoelastic solid bodies in general. As an example, we have applied it to the study of salt migration, where the salt has been considered as a nearly incompressible viscoelastic material referred to as a *Mooney-Rivlin type isotropic viscoelastic solid* (see Liu etc. [8]).

3. Application to nearly incompressible Mooney-Rivlin materials

From now on we consider a Mooney-Rivlin material whose constitutive equation relative to the preferred reference configuration κ_0 is given by

$$T = \mathcal{F}_{\kappa_0}(F) = -pI + \widetilde{\mathcal{F}}(F), \quad \widetilde{\mathcal{F}}(F) = s_1 B + s_2 B^{-1},$$

where $B = FF^T$ is the left Cauchy-Green strain tensor and the material parameters s_1 and s_2 are constant satisfying

$$s_1 > 0 \quad \text{and} \quad s_2 < s_1. \tag{3.1}$$

Remark 3.1. It is usually assumed $s_2 \leq 0 < s_1$, the so-called E-inequalities (see Haupt [3], Truesdell & Noll [12]), based on the assumption that the free energy function is positive definite for any deformation. However, Liu [6] has pointed out that a thermodynamical stability analysis only requires $s_2 < s_1$. Therefore, we shall include the case $0 < s_2 < s_1$ in our analysis.

A direct calculation of the Fréchet-differential of $\widetilde{\mathcal{F}}$ at F gives

$$\mathrm{d}\widetilde{\mathcal{F}}(F)[H] = s_1\big(HB + BH^T\big) - s_2\big(B^{-1}H + H^TB^{-1}\big).$$

For compressible body in general, the pressure p may depend on the deformation gradient F. However, for compressible elastic bodies, we shall assume that the

pressure depends only on the determinant of the deformation gradient or, by the mass balance, depends only on the mass density ρ,

$$p = p(\rho), \quad \rho(t) = \frac{\rho_0}{\det F(t)},$$

where ρ_0 denotes the mass density in the referential configuration κ_0.

For time $\tau = t + \Delta t$ and from (2.4), we have

$$\rho(\tau) - \rho(t) = \rho_0\big(\det F(\tau)^{-1} - \det F(t)^{-1}\big) = \rho(t)\big(\det F(t)F(\tau)^{-1} - 1\big)$$
$$= \rho(t)\big(\det(I + H(\tau))^{-1} - 1\big) = -\rho(t)\operatorname{tr}H(\tau) + o(2),$$

where $\operatorname{tr}H$ means the trace of H and $o(2)$ denotes higher order terms in the small displacement gradient $H(\tau)$. Therefore, assuming that p is differentiable as function of ρ, we have

$$p(\tau) - p(t) = \frac{dp}{d\rho}(t)\big(\rho(\tau) - \rho(t)\big) + o(2) = -\rho(t)\frac{dp}{d\rho}(t)\operatorname{tr}H(\tau) + o(2),$$

or

$$p(\tau) = p(t) - \beta(t)\operatorname{tr}H(\tau) + o(2),$$

where $\beta = \rho\dfrac{dp}{d\rho}$ is a material parameter depending on the mass density ρ.

A body is called *nearly incompressible* if its density is nearly insensitive to change of pressure. Hence, if we regard the density as a function of pressure, $\rho = \rho(p)$, then its derivative with respect to the pressure is nearly zero. This means that, for nearly incompressible materials, the parameter β must be large, i.e.,

$$\beta = \beta(\boldsymbol{x}, t) \geq \beta_0, \quad \forall \boldsymbol{x} \in \mathcal{B}_t,$$

for some $\beta_0 > 0$.

Therefore, the Cauchy stress tensor relative to the current configuration κ_t is given by

$$T(\tau) = T(t) + L(F(t))[H(\tau)] + o(2),$$

where

$$L(F)[H] = \beta(\operatorname{tr}H)I + s_1\big(HB + BH^T\big) - s_2\big(B^{-1}H + H^T B^{-1}\big)$$

and the first Piola-Kirchhoff stress tensor at time τ relative to the current configuration κ_t is given by

$$\begin{aligned}
T_{\kappa_t}(\tau) &= \det F_t(\tau)T(\tau)F_t(\tau)^{-T} = \det(I + H)T(\tau)(I + H)^{-T} \\
&= \big[1 + \operatorname{tr}H + o(2)\big]\big[T(t) + L(F(t))[H] + o(2)\big]\big[I - H^T + o(2)\big] \quad (3.2) \\
&= T(t) + (\operatorname{tr}H)T(t) - T(t)H^T + L(F(t))[H] + o(2).
\end{aligned}$$

4. Linearized boundary value problem and its variational formulation

For simplicity, we denote by κ the current configuration κ_t, $\Omega = \kappa(\mathcal{B})$ be the bounded domain of $\mathbb{R}^3$ representing the interior of the region occupied by the body

at current configuration κ at the present time t, $T_0 = T(t)$ and $B_0 = B(t)$. Let $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, $\boldsymbol{n}_\kappa$ be the exterior unit normal to $\partial\Omega$ and $\boldsymbol{g}$ be the gravitational force (per unit mass).

We consider the following boundary value problem for the relative displacement $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, \tau)$,

$$\begin{cases} -\operatorname{\mathbf{div}} T_\kappa(\tau) = \rho(\tau)\boldsymbol{g} & \text{in } \Omega \times \mathbb{R}, \\ T_\kappa(\tau)\boldsymbol{n}_\kappa = \boldsymbol{f}(\tau) & \text{on } \Gamma_1, \\ \boldsymbol{u}(\tau) \cdot \boldsymbol{n}_\kappa = 0 & \text{on } \Gamma_2, \\ T_\kappa(\tau)\boldsymbol{n}_\kappa \times \boldsymbol{n}_\kappa = \boldsymbol{0} & \text{on } \Gamma_2, \\ \boldsymbol{u}(\tau) = \boldsymbol{0} & \text{on } \Gamma_3, \end{cases} \tag{4.1}$$

where $\operatorname{\mathbf{div}}$ is the divergence operator with respect to $\boldsymbol{x}$, $T_\kappa(\tau) = T_\kappa(\boldsymbol{x}, \tau)$ is the Piolla-Kirchhoff stress tensor at time τ relative to configuration κ at the present time t, which, up to linear terms in relative displacement gradient $H = H(\tau) = \nabla_{\boldsymbol{x}}\boldsymbol{u}(\tau)$, is given by (see (3.2))

$$T_\kappa = T_0 + (\operatorname{tr} H)(T_0 + \beta I) - T_0 H^T + s_1\left(HB_0 + B_0 H^T\right) - s_2\left(B_0^{-1}H + H^T B_0^{-1}\right),$$

and $\boldsymbol{f}(\tau)$ is the surface traction (per unit surface area).

Remark 4.1. At every time step, the idea of formulating the boundary value problem in the form (4.1) is similar to the theory of small deformations superposed on finite deformations (see Green etc. [2] and Truesdell & Noll [12]). In this manner, either we are interested in the evolution of solutions with gradually changing boundary conditions resulting in large deformation, or, we can treat the boundary values of finite elasticity as the final value of a successive small incremental boundary values at each time step (see Ciarlet [1]).

The boundary value problem (4.1) can be formulated as a variational problem. Indeed, let Ω be a smooth enough bounded domain in $\mathbb{R}^3$ and define the space

$$\mathcal{V} = \left\{\boldsymbol{u} \in (H^1(\Omega))^3 \,;\, \boldsymbol{u} \cdot \boldsymbol{n}_\kappa = 0 \text{ on } \Gamma_2 \text{ and } \boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma_3\right\}.$$

Taking formally the inner product of both sides of the equation in (4.1) by $\boldsymbol{w} \in \mathcal{V}$ and integrating over Ω, we obtain after integration by parts,

$$\int_\Omega \operatorname{tr}\left(K[H]W^T\right) d\boldsymbol{x} = \int_{\Gamma_1} \boldsymbol{f}(\tau) \cdot \boldsymbol{w} \, d\Gamma - \int_\Omega \operatorname{tr}\left(T_0 W^T\right) d\boldsymbol{x},$$

where we are denoting $H = \nabla_{\boldsymbol{x}}\boldsymbol{u}$, $W = \nabla_{\boldsymbol{x}}\boldsymbol{w}$ and $K[H]$ is given by

$$K[H] = (\operatorname{tr} H)(T_0 + \beta I) - T_0 H^T + s_1\left(HB_0 + B_0 H^T\right) - s_2\left(B_0^{-1}H + H^T B_0^{-1}\right).$$

Therefore, for $\boldsymbol{u}, \boldsymbol{w} \in \mathcal{V}$ we consider respectively the bilinear and the linear forms:

$$\begin{aligned} \mathcal{L}(\boldsymbol{u}, \boldsymbol{w}) &= \int_\Omega \operatorname{tr}(K[H]W^T) \, d\boldsymbol{x}, \\ \mathcal{N}(\boldsymbol{w}) &= \int_{\Gamma_1} \boldsymbol{f}(\tau) \cdot \boldsymbol{w} \, d\Gamma - \int_\Omega \operatorname{tr}\left(T_0 W^T\right) d\boldsymbol{x} + \int_\Omega \rho(\tau)\boldsymbol{g} \cdot \boldsymbol{w} \, d\boldsymbol{x}. \end{aligned} \tag{4.2}$$

We notice that the forms $\mathcal{L}$ and $\mathcal{N}$ can be written in terms of coordinates by

$$\mathcal{L}(\boldsymbol{u}, \boldsymbol{w}) = \int_{\Omega} \frac{\partial u_k}{\partial x_k} \left([T_0]_{ij} + \beta \delta_{ij}\right) \frac{\partial w_i}{\partial x_j} \, dV - \int_{\Omega} [T_0]_{ik} \frac{\partial u_j}{\partial x_k} \frac{\partial w_i}{\partial x_j} \, dV$$
$$+ s_1 \int_{\Omega} \left(\frac{\partial u_i}{\partial x_k} [B_0]_{kj} + [B_0]_{ik} \frac{\partial u_j}{\partial x_k}\right) \frac{\partial w_i}{\partial x_j} \, dV$$
$$- s_2 \int_{\Omega} \left([B_0^{-1}]_{ik} \frac{\partial u_k}{\partial x_j} + \frac{\partial u_k}{\partial x_i} [B_0^{-1}]_{kj}\right) \frac{\partial w_i}{\partial x_j} \, dV,$$
$$\mathcal{N}(\boldsymbol{w}) = \int_{\Gamma_1} f_i w_i \, d\Gamma - \int_{\Omega} [T_0]_{ij} \frac{\partial w_i}{\partial x_j} \, d\boldsymbol{x} + \int_{\Omega} \rho g_i w_i \, d\boldsymbol{x},$$

where in the above formulas we have used the standard summation convention for repeated indices.

Then, the variational problem is to find the solution $\boldsymbol{u} \in \mathcal{V}$ such that

$$\mathcal{L}(\boldsymbol{u}, \boldsymbol{w}) = \mathcal{N}(\boldsymbol{w}), \quad \forall \boldsymbol{w} \in \mathcal{V}. \tag{4.3}$$

In order to prove that the solutions of (4.3) is a weak solution of (4.1), the following result concerning existence of a *normal trace* is useful (see Temam [10]).

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $\mathcal{C}^2$. If $\boldsymbol{F} \in L^2(\Omega)^N$ satisfies $\operatorname{div} \boldsymbol{F} \in L^2(\Omega)$, then $\boldsymbol{F} \cdot \boldsymbol{n}_\kappa$ can be defined as an element of $H^{-1/2}(\partial\Omega)$ and there exists a constant $C_1 > 0$ depending only on Ω such that*

$$\|\boldsymbol{F} \cdot \boldsymbol{n}_\kappa\|_{H^{-1/2}} \leq C_1 \left(\|\boldsymbol{F}\|_2 + \|\operatorname{div} \boldsymbol{F}\|_2\right).$$

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^3$ be a domain of class $\mathcal{C}^2$. We assume that $\beta, p_0 \in L^\infty(\Omega)$, $\rho \in L^2(\Omega)$ and $T_0, B_0 \in L^\infty(\Omega, M_3(\mathbb{R}))$, where $M_3(\mathbb{R})$ denotes the set of 3×3 real matrices. If $\boldsymbol{u}$ is a solution of (4.3), then $\boldsymbol{u}$ is a weak solution of (4.1).*

Proof. Let $u \in \mathcal{V}$ be a solution of (4.3). Then, $H = \nabla_{\boldsymbol{x}} \boldsymbol{u} \in L^2(\Omega, M_3(\mathbb{R}))$, which implies that $T_\kappa = T_0 + K[H] \in L^2(\Omega, M_3(\mathbb{R}))$. Since $C_0^\infty(\Omega)^3 \subset \mathcal{V}$, we have

$$-\int_{\Omega} \operatorname{\mathbf{div}} T_\kappa \cdot \boldsymbol{w} \, d\boldsymbol{x} = \int_{\Omega} \operatorname{tr}(T_\kappa W^T) \, d\boldsymbol{x} = \int_{\Omega} \rho \boldsymbol{g} \cdot \boldsymbol{w} \, d\boldsymbol{x}, \quad \forall \boldsymbol{w} \in C_0^\infty(\Omega)^3,$$

where $W = \nabla_{\boldsymbol{x}} \boldsymbol{w}$ and the partial derivatives in $\operatorname{\mathbf{div}}$ are taken in the sense of distributions in Ω. Hence, $\boldsymbol{u}$ satisfies

$$-\operatorname{\mathbf{div}} T_\kappa = \rho \boldsymbol{g}$$

in the sense of distributions. Moreover, since we are assuming that $\rho \in L^2(\Omega)$, it follows from the density of $C_0^\infty(\Omega)$ in $L^2(\Omega)$ that $\operatorname{\mathbf{div}} T_\kappa \in L^2(\Omega)^3$. From Lemma 4.1, (4.3) reduces to

$$\int_{\partial\Omega} T_\kappa \boldsymbol{n}_\kappa \cdot \boldsymbol{w} \, d\Gamma = \int_{\Gamma_1} \boldsymbol{f} \cdot \boldsymbol{w} \, d\Gamma, \tag{4.4}$$

where the above surface integral on $\partial\Omega$ are taken in the sense of the duality between $H^{-1/2}(\partial\Omega)^3$ and $H^{1/2}(\partial\Omega)^3$. In particular, for any $\boldsymbol{w} \in \mathcal{V}$ such that $\boldsymbol{w} = \boldsymbol{0}$ on Γ_2, we have

$$\int_{\Gamma_1} \left(T_\kappa \boldsymbol{n}_\kappa - \boldsymbol{f}\right) \cdot \boldsymbol{w} \, d\Gamma = 0,$$

which gives the Γ_1-boundary condition in (4.1). So, (4.4) reduces to

$$\int_{\Gamma_2} T_\kappa \boldsymbol{n}_\kappa \cdot \boldsymbol{w}\, d\Gamma = 0, \quad \forall \boldsymbol{w} \in \mathcal{V}. \tag{4.5}$$

In order to show that (4.5) gives the Γ_2-boundary condition in (4.1), let $\varphi \in H_0^1(\Omega)$, $\varphi < 0$, be the first eigenfunction of $-\Delta$ and define

$$\boldsymbol{w}_0(\boldsymbol{x}) = \nabla_{\boldsymbol{x}} \varphi(\boldsymbol{x}) |\nabla_{\boldsymbol{x}} \varphi(\boldsymbol{x})|^{-1}, \quad \boldsymbol{x} \in \Omega.$$

Since Ω is of class C^2, we can extend $\boldsymbol{w}_0$ to the boundary $\partial\Omega$ and we have from the maximum principle that $\boldsymbol{w}_0(\boldsymbol{x}) = \boldsymbol{n}_\kappa(\boldsymbol{x})$, for almost all $\boldsymbol{x} \in \partial\Omega$. Let $\widetilde{\boldsymbol{w}} \in H^1(\Omega)^3$ be an arbitrary function which vanishes on Γ_3 and consider $\boldsymbol{w} = \boldsymbol{w}_0 \times \widetilde{\boldsymbol{w}}$. Then, it is clear that $\boldsymbol{w} \in \mathcal{V}$, since $\boldsymbol{w} = \boldsymbol{0}$ on Γ_3 and

$$\boldsymbol{w} \cdot \boldsymbol{n}_\kappa\,|_{\Gamma_2} = (\boldsymbol{n}_\kappa \times \widetilde{\boldsymbol{w}}) \cdot \boldsymbol{n}_\kappa\,|_{\Gamma_2} = -(\boldsymbol{n}_\kappa \times \boldsymbol{n}_\kappa) \cdot \widetilde{\boldsymbol{w}}|_{\Gamma_2} = 0.$$

Therefore, from (4.5),

$$0 = \int_{\Gamma_2} T_\kappa \boldsymbol{n}_\kappa \cdot \boldsymbol{w}\, d\Gamma = \int_{\Gamma_2} T_\kappa \boldsymbol{n}_\kappa \cdot (\boldsymbol{n}_\kappa \times \widetilde{\boldsymbol{w}})\, d\Gamma = \int_{\Gamma_2} (T_\kappa \boldsymbol{n}_\kappa \times \boldsymbol{n}_k) \cdot \widetilde{\boldsymbol{w}}\, d\Gamma$$

and the proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

5. Existence and uniqueness of solution in two-dimensions

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain of classe $\mathcal{C}^2$ whose boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, with $\mathrm{meas}(\Gamma_i) \neq 0$ for $i = 1, 2, 3$, and consider the space

$$\mathcal{V} = \big\{ \boldsymbol{u} = (u_1, u_2) \in H^1(\Omega)^2 \,;\, \boldsymbol{u} \cdot \boldsymbol{n}_\kappa = 0 \text{ on } \Gamma_2 \text{ and } \boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma_3 \big\}. \tag{5.1}$$

For $\boldsymbol{u}, \boldsymbol{w} \in \mathcal{V}$, we introduce

$$\langle \boldsymbol{u} | \boldsymbol{v} \rangle = \int_\Omega \big(\nabla u_1(\boldsymbol{x}) \cdot \nabla v_1(\boldsymbol{x}) + \nabla u_2(\boldsymbol{x}) \cdot \nabla v_2(\boldsymbol{x}) \big)\, d\boldsymbol{x},$$
$$\|\boldsymbol{u}\|_{\mathcal{V}}^2 = \|\nabla u_1\|_{L^2}^2 + \|\nabla u_2\|_{L^2}^2, \tag{5.2}$$

where $\|\ \|_{L^2}$ is the usual L^2-norm. It is well-known that the Poincaré inequality holds if $\mathrm{meas}(\Gamma_3) \neq 0$, i.e., there exists a constante C such that

$$\|\boldsymbol{u}\|_{\mathcal{V}}^2 \geq C \|\boldsymbol{u}\|_{L^2}^2, \quad \forall \boldsymbol{u} \in \mathcal{V}.$$

In this case, $\langle \cdot ; \cdot \rangle$ and $\|\ \|_{\mathcal{V}}$ define an inner product and a norm in $\mathcal{V}$, respectively.
From now on we assume that

$$\rho \in L^2(\Omega), \quad \beta, p_0 \in L^\infty(\Omega), \quad B_0 \in L^\infty\big(\Omega, S_2^+(\mathbb{R})\big), \tag{5.3}$$

where by $S_2^+(\mathbb{R})$ we denote the set of all symmetric and positive definite 2×2 matrix, and we set

$$T_0 = -p_0 I + s_1 B_0 + s_2 B_0^{-1}.$$

It is clear that the forms $\mathcal{L}$ and $\mathcal{N}$ defined in (4.2) are continuous in $\mathcal{V}$.

Recalling that H and W are 2×2 matrix whose entries are given by

$$[H]_{ij} = \frac{\partial u_i}{\partial x_j}, \quad [W]_{ij} = \frac{\partial w_i}{\partial x_j}, \quad \boldsymbol{u}, \boldsymbol{w} \in \mathcal{V},$$

the bilinear form $\mathcal{L}(\boldsymbol{u}, \boldsymbol{w})$ defined in (4.2) can be written as

$$\mathcal{L}(\boldsymbol{u}, \boldsymbol{w}) = \int_\Omega \mathcal{A}(\boldsymbol{x}; H(\boldsymbol{x}), W(\boldsymbol{x})) \, d\boldsymbol{x},$$

where

$$\begin{aligned} \mathcal{A}(\boldsymbol{x}; H, W) = {} & \operatorname{tr}(H) \operatorname{tr}\left[(T_0 + \beta I)W^T\right] - \operatorname{tr}(T_0 H^T W^T) \\ & + s_1 \operatorname{tr}\left[(HB_0 + B_0 H^T)W^T\right] - s_2 \operatorname{tr}\left[(B_0^{-1}H + H^T B_0^{-1})W^T\right]. \end{aligned}$$

In particular, for $W = H$ we have

$$\begin{aligned} \mathcal{A}(\boldsymbol{x}; H, H) = {} & \operatorname{tr}(H) \operatorname{tr}\left[(T_0 + \beta I)H^T\right] - \operatorname{tr}(T_0 H^T H^T) \\ & + s_1 \operatorname{tr}\left[(HB_0 + B_0 H^T)H^T\right] - s_2 \operatorname{tr}\left[(B_0^{-1}H + H^T B_0^{-1})H^T\right]. \end{aligned} \tag{5.4}$$

Hence, to prove that $\mathcal{L}$ is coercive, it is sufficient to show that there exists $\alpha > 0$ such that

$$\mathcal{A}(\boldsymbol{x}; H, H) \geq \alpha \|H\|^2, \ \forall \, \boldsymbol{x} \in \Omega,$$

i.e., it suffices to show that the bilinear form $\mathcal{A}(\boldsymbol{x}; H, W)$ is uniformly coercive as function of 2×2 matrices. Furthermore, a direct calculation (see the Appendix) gives that the coercivity of $\mathcal{A}(\boldsymbol{x}, H, W)$ is equivalent to the semipositivity of the matrix $A(\boldsymbol{x}) - \alpha I$, for all $\boldsymbol{x} \in \Omega$, with $A(\boldsymbol{x})$ given by

$$\begin{pmatrix} \beta + 2s_1\gamma_1 - 2s_2\gamma_1^{-1} & \beta + \frac{1}{2}\operatorname{tr} T_0 & 0 & 0 \\ \beta + \frac{1}{2}\operatorname{tr} T_0 & \beta + 2s_1\gamma_2 - 2s_2\gamma_2^{-1} & 0 & 0 \\ 0 & 0 & 2s_1 \operatorname{tr} B_0 - 2s_2 \operatorname{tr} B_0^{-1} - \operatorname{tr} T_0 & s_1(\gamma_2 - \gamma_1) - s_2(\gamma_1^{-1} - \gamma_2^{-1}) \\ 0 & 0 & s_1(\gamma_2 - \gamma_1) - s_2(\gamma_1^{-1} - \gamma_2^{-1}) & \operatorname{tr} T_0 \end{pmatrix} \tag{5.5}$$

where γ_1 e γ_2 are the eigenvalues of B_0 and I is the 4×4 identity matrix. Therefore, we can also write

$$\mathcal{L}(\boldsymbol{u}, \boldsymbol{u}) = \int_\Omega X(\boldsymbol{x})^T \cdot A(\boldsymbol{x})X(\boldsymbol{x}) \, d\boldsymbol{x},$$

where, following the notation introduced in the Appendix,

$$H(\boldsymbol{x}) = \begin{bmatrix} a & b+d \\ b-d & c \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} \\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} \end{bmatrix}$$

and

$$X(\boldsymbol{x})^T = (a, c, b, d) = \left(\frac{\partial u_1}{\partial x_1}, \frac{\partial u_2}{\partial x_2}, \frac{1}{2}\left[\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right], \frac{1}{2}\left[\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right] \right)$$

We notice that if $A(\boldsymbol{x}) - \alpha I$ is uniformly semipositive in Ω, then

$$X(\boldsymbol{x})^T \cdot A(\boldsymbol{x})X(\boldsymbol{x}) \geq \alpha \left(\left| \frac{\partial u_1}{\partial x_1} \right|^2 + \left| \frac{\partial u_2}{\partial x_2} \right|^2 + \frac{1}{2} \left| \frac{\partial u_1}{\partial x_2} \right|^2 + \frac{1}{2} \left| \frac{\partial u_2}{\partial x_1} \right|^2 \right),$$

and consequently,

$$\mathcal{L}(\boldsymbol{u}, \boldsymbol{u}) \geq \frac{\alpha}{2} \left(\|\nabla u_1\|_{L^2}^2 + \|\nabla u_2\|_{L^2}^2 \right) = \frac{\alpha}{2} \|\boldsymbol{u}\|_{\mathcal{V}}^2.$$

In order to analyze the matrix (5.5) and in view of the conditions (3.1), we must distinguish two cases: $s_2 < 0 < s_1$ and $0 \leq s_2 < s_1$. In both cases, we fix a constant $k > \max\{0, s_2 s_1^{-1}\}$ and take $\varepsilon = s_1 - s_2 k^{-1}$. Now, let $a_0 = a_0(\boldsymbol{x})$ and $b_0 = b_0(\boldsymbol{x})$ be the functions defined by

$$\begin{aligned} a_0 &= -2s_2 \operatorname{tr} B_0^{-1} - 2\left(s_1 \sqrt{\det B_0} - s_2 \sqrt{\det B_0^{-1}} \right), \\ b_0 &= -2s_2 \operatorname{tr} B_0^{-1} + 2\left(s_1 \sqrt{\det B_0} - s_2 \sqrt{\det B_0^{-1}} \right). \end{aligned} \tag{5.6}$$

Assuming that $\det B_0 \geq k$, we have the following inequalities

$$\begin{aligned} b_0 - a_0 &\geq 4\sqrt{-s_1 s_2} \quad \text{if } s_2 < 0, \\ b_0 - a_0 &= 4\left(s_1 - \frac{s_2}{\det B_0} \right) \sqrt{\det B_0} \geq 4\varepsilon\sqrt{k} \quad \text{if } s_2 \geq 0. \end{aligned} \tag{5.7}$$

Notice that the above inequalities indicate that the interval $[a_0(\boldsymbol{x}), b_0(\boldsymbol{x})]$ has nonempty interior for all $\boldsymbol{x} \in \Omega$ if $\det B_0 \geq k$, and this will be essential in proving the next theorem.

Finally, we set

$$\bar{d} = \sup_{\boldsymbol{x} \in \Omega} \left(\frac{\operatorname{tr} B_0}{\sqrt{\det B_0}} \right). \tag{5.8}$$

Theorem 5.1. *Let $k > \max\{0, s_2 s_1^{-1}\}$ and $\varepsilon = s_1 - s_2 k^{-1}$. Assume that $\det B_0 \geq k$. Let $\bar{d}$ be the constant defined by (5.8), $\alpha > 0$ such that*

$$\alpha \bar{d} < \begin{cases} 2\sqrt{-s_1 s_2} & \text{if } s_2 < 0, \\ 2\varepsilon\sqrt{k} & \text{if } s_2 \geq 0, \end{cases} \tag{5.9}$$

and assume that p_0 satisfies the condition

$$a_0(\boldsymbol{x}) + \alpha\bar{d} < -2p_0(\boldsymbol{x}) < b_0(\boldsymbol{x}) - \alpha\bar{d}, \quad \forall \boldsymbol{x} \in \Omega. \tag{5.10}$$

Then, there exists a constant $\beta_0 = \beta_0(s_1, s_2, \alpha) \geq 0$ such that the matrix $A(\boldsymbol{x}) - \alpha I$ is uniformly positive semidefinite in Ω, provided that $\beta(\boldsymbol{x}, \tau) \geq \beta_0$ for almost all $\boldsymbol{x} \in \Omega$.

Remark 5.1. Since $2\sqrt{\det B_0} \leq \operatorname{tr} B_0$, it follows that $\bar{d} \geq 2$. Hence, if α satisfies (5.9), we have necessarily

$$\alpha < \begin{cases} \sqrt{-s_1 s_2} & \text{if } s_2 < 0, \\ \varepsilon\sqrt{k} & \text{if } s_2 \geq 0. \end{cases} \tag{5.11}$$

Proof. The nonzero entries of the matrix A are

$$\begin{cases} A_{11} = \beta + 2s_1\gamma_1 - 2s_2\gamma_1^{-1}, \\ A_{22} = \beta + 2s_1\gamma_2 - 2s_2\gamma_2^{-1}, \\ A_{12} = \beta + \dfrac{1}{2}\operatorname{tr}T_0, \end{cases} \qquad \begin{cases} A_{33} = 2s_1\operatorname{tr}B_0 - 2s_2\operatorname{tr}B_0^{-1} - \operatorname{tr}T_0, \\ A_{44} = \operatorname{tr}T_0, \\ A_{34} = s_1(\gamma_2 - \gamma_1) - s_2(\gamma_1^{-1} - \gamma_2^{-1}), \end{cases}$$

where γ_1 e γ_2 are the eigenvalues of B_0. To simplify the notation, we introduce the functions $f, g : (0, +\infty) \to \mathbb{R}$, as

$$f(\gamma) = s_1\gamma - s_2\gamma^{-1} \quad \text{and} \quad g(\gamma) = s_1\gamma + s_2\gamma^{-1}.$$

A necessary and sufficient condition for the matrix $A - \alpha I$ be positive semidefinite is that, for all $\boldsymbol{x} \in \Omega$,

$$\min\Big\{A_{11} - \alpha, \ (A_{11} - \alpha)(A_{22} - \alpha) - A_{12}^2, \ A_{33} - \alpha, \ (A_{33} - \alpha)(A_{44} - \alpha) - A_{34}^2\Big\} \geq 0. \tag{5.12}$$

Since A is symmetric, it is clear that the condition (5.12) implies, in particular, $A_{22} - \alpha \geq 0$ and $A_{44} - \alpha \geq 0$.

Step 1: *Analysis of the first block of A:*

In the case $s_2 < 0$, we have $f(\gamma) \geq \sqrt{-s_1 s_2}$ for all $\gamma > 0$. So,

$$A_{11} - \alpha = \beta - \alpha + 2f(\gamma_1) \geq \beta - \alpha + 2\sqrt{-s_1 s_2} > \beta - \alpha.$$

In the case $s_2 \geq 0$, we can assume without loss of generality that $\gamma_1 \geq \gamma_2$. Then, as $s_1 - s_2/\det B_0 \geq \varepsilon$, we have

$$f(\gamma_1) \geq s_1\gamma_1 - \frac{s_2}{\gamma_2} \geq \varepsilon\gamma_1 > 0,$$

and

$$A_{11} - \alpha = \beta - \alpha + 2f(\gamma_1) \geq \beta - \alpha + 2\varepsilon\gamma_1 \geq \beta - \alpha.$$

Therefore, in the two cases, $A_{11} - \alpha \geq 0$ if $\beta \geq \alpha$. On the other hand, if we denote $f_i = f(\gamma_i)$ and $g_i = g(\gamma_i)$, $i = 1, 2$, we get

$$\begin{aligned} (A_{11} - \alpha)(A_{22} - \alpha) - A_{12}^2 &= (\beta - \alpha + 2f_1)(\beta - \alpha + 2f_2) - \Big[\beta + \frac{1}{2}\operatorname{tr}T_0\Big]^2 \\ &= (\beta - \alpha)\big[2(f_1 + f_2) - 2(\alpha - p_0) - (g_1 + g_2)\big] \\ &\quad + 4f_1 f_2 - \Big[(\alpha - p_0) + \frac{g_1 + g_2}{2}\Big]^2. \end{aligned}$$

Since

$$2(f_1 + f_2) - (g_1 + g_2) = s_1\operatorname{tr}B_0 - 3s_2\operatorname{tr}B_0^{-1},$$
$$\alpha - p_0 + \frac{g_1 + g_2}{2} = \alpha + \frac{1}{2}\operatorname{tr}T_0,$$

we have

$$\begin{aligned} (A_{11} - \alpha)(A_{22} - \alpha) - A_{12}^2 &= (\beta - \alpha)\big[s_1\operatorname{tr}B_0 - 3s_2\operatorname{tr}B_0^{-1} - 2(\alpha - p_0)\big] \\ &\quad + 4f_1 f_2 - \Big[\alpha + \frac{1}{2}\operatorname{tr}T_0\Big]^2. \end{aligned}$$

Therefore, if

$$- 2p_0 < -2\alpha + s_1 \operatorname{tr} B_0 - 3s_2 \operatorname{tr} B_0^{-1}, \quad \forall \boldsymbol{x} \in \Omega, \tag{5.13}$$

it follows that $(A_{11} - \alpha)(A_{22} - \alpha) - A_{12}^2 \geq 0$ for $\beta > 0$ large enough.

Step 2: *Analysis of the second block of A:*

By the definition of T_0, we have

$$A_{33} - \alpha = s_1 \operatorname{tr} B_0 - 3s_2 \operatorname{tr} B_0^{-1} + 2p_0 - \alpha.$$

Hence, $A_{33} - \alpha \geq 0$, if and only if

$$- 2p_0 \leq -\alpha + s_1 \operatorname{tr} B_0 - 3s_2 \operatorname{tr} B_0^{-1}. \tag{5.14}$$

We notice that $A_{44} - \alpha = -2p_0 s_1 \operatorname{tr} B_0 + s_2 \operatorname{tr} B_0^{-1} - \alpha$, which implies that $A_{44} - \alpha \geq 0$, if and only if

$$- 2p_0 \geq \alpha - s_1 \operatorname{tr} B_0 - s_2 \operatorname{tr} B_0^{-1}. \tag{5.15}$$

The conditions (5.14) and (5.15) can be expressed by

$$-s_1 \operatorname{tr} B_0 - s_2 \operatorname{tr} B_0^{-1} + \alpha \leq 0 \leq -\alpha + s_1 \operatorname{tr} B_0 - 3s_2 \operatorname{tr} B_0^{-1}.$$

It is noteworthy that (5.13) implies (5.14). Moreover, the interval

$$[-s_1 \operatorname{tr} B_0 - s_2 \operatorname{tr} B_0^{-1} + \alpha, \; -2\alpha + s_1 \operatorname{tr} B_0 - 3s_2 \operatorname{tr} B_0^{-1}] \tag{5.16}$$

is not empty, because if we denote

$$\begin{cases} a_* = -s_1 \operatorname{tr} B_0 - s_2 \operatorname{tr} B_0^{-1}, \\ b_* = s_1 \operatorname{tr} B_0 - 3s_2 \operatorname{tr} B_0^{-1}, \end{cases}$$

it follows that

$$b_* - a_* - 3\alpha = 2s_1 \operatorname{tr} B_0 - 2s_2 \operatorname{tr} B_0^{-1} - 3\alpha.$$

and hence, from (5.11):

(1) in the case $s_2 < 0$ we have $b_* - a_* - 3\alpha = 2f_1 + 2f_2 - 3\alpha \geq 4\sqrt{-s_1 s_2} - 3\alpha > 0$;

(2) in the case $s_2 \geq 0$ we have $b_* - a_* - 3\alpha = 2 \operatorname{tr} B_0 \left(s_1 - \frac{s_2}{\det B_0} \right) - 3\alpha \geq 4\varepsilon\sqrt{k} - 3\alpha > 0$,

which imply that the interval defined by (5.16) is not empty.

Now, according to the notation introduced above, we have

$$\begin{cases} A_{33} = f_1 + f_2 - 2s_2 \operatorname{tr} B_0^{-1} + 2p_0, \\ A_{44} = g_1 + g_2 - 2p_0, \\ A_{34} = g_2 - g_1. \end{cases}$$

So,

$$\begin{aligned} (A_{33} - \alpha)(A_{44} - \alpha) - A_{34}^2 &= \left[f_1 + f_2 - 2s_2 \operatorname{tr} B_0^{-1} + 2p_0 - \alpha \right] \left[g_1 + g_2 - 2p_0 - \alpha \right] \\ &\quad - [g_2 - g_1]^2 \\ &= \left[F + 2p_0 \right] \left[G - 2p_0 \right] - [g_2 - g_1]^2, \end{aligned}$$

where we are denoting

$$F = f_1 + f_2 - 2s_2 \operatorname{tr} B_0^{-1} - \alpha \quad \text{and} \quad G = g_1 + g_2 - \alpha.$$

Hence,

$$(A_{33} - \alpha)(A_{44} - \alpha) - A_{34}^2 = FG + 2p_0(G - F) - (g_2 - g_1)^2 - 4p_0^2.$$

For $X = -2p_0$, we have $(A_{33} - \alpha)(A_{44} - \alpha) - A_{34}^2 \geq 0$, if and only if

$$X^2 - (F - G)X + (g_2 - g_1)^2 - FG \leq 0. \tag{5.17}$$

The above inequality holds if the discriminant of the binomial (5.17) is positive. In fact, we have

$$
\begin{aligned}
&(F - G)^2 - 4\big[(g_2 - g_1)^2 - FG\big] \\
=&\big[f_1 + f_2 - 2s_2 \operatorname{tr} B_0^{-1} + g_1 + g_2 - 2\alpha\big]^2 - 4(g_2 - g_1)^2 \\
=&\big[2s_1 \operatorname{tr} B_0 - 2s_2 \operatorname{tr} B_0^{-1} - 2\alpha\big]^2 - 4\big[s_1(\gamma_2 - \gamma_1) + s_2(\gamma_2^{-1} - \gamma_1^{-1})\big]^2 \\
=&4\big[s_1(\gamma_1 + \gamma_2) - s_2(\gamma_1^{-1} + \gamma_2^{-1}) - \alpha\big]^2 - 4\big[s_1(\gamma_2 - \gamma_1) + s_2(\gamma_2^{-1} - \gamma_1^{-1})\big]^2 \\
=&4\big[2s_1\gamma_2 - 2s_2\gamma_1^{-1} - \alpha\big]\big[2s_1\gamma_1 - 2s_2\gamma_2^{-1} - \alpha\big] \\
=&16\big[s_1\gamma_2 - s_2\gamma_1^{-1} - \alpha/2\big]\big[s_1\gamma_1 - s_2\gamma_2^{-1} - \alpha/2\big] \\
=&16\Big[f\big(\sqrt{\det B_0}\big)^2 - \frac{\alpha}{2}\big(s_1 \operatorname{tr} B_0 - s_2 \operatorname{tr} B_0^{-1}\big) + \frac{\alpha^2}{4}\Big].
\end{aligned}
$$

Note that

$$s_1 \operatorname{tr} B_0 - s_2 \operatorname{tr} B_0^{-1} = s_1 \operatorname{tr} B_0 - s_2 \left(\frac{\operatorname{tr} B_0}{\det B_0}\right) = \operatorname{tr} B_0 \left(s_1 - \frac{s_2}{\det B_0}\right)$$

$$= \left(\frac{\operatorname{tr} B_0}{\sqrt{\det B_0}}\right) f\big(\sqrt{\det B_0}\big).$$

To simplify the notation, consider

$$C = f\big(\sqrt{\det B_0}\big) \quad \text{and} \quad D = \frac{\operatorname{tr} B_0}{\sqrt{\det B_0}}.$$

Then,

$$C^2 - \frac{\alpha}{2}CD + \frac{\alpha^2}{4} \geq C^2 - \frac{\alpha}{2}CD = C^2 \left(1 - \frac{\alpha D}{2C}\right).$$

Note also that, from (5.9) we have:

(1) if $s_2 < 0$, $\alpha D \leq \alpha\bar{d} < 2\sqrt{-s_1 s_2} \leq 2f\big(\sqrt{\det B_0}\big) = 2C$;

(2) if $s_2 \geq 0$, $\alpha D \leq \alpha\bar{d} < 2\varepsilon\sqrt{k} \leq 2\sqrt{\det B_0}\left(s_1 - \frac{s_2}{\det B_0}\right) = 2f\big(\sqrt{\det B_0}\big) = 2C$,

which implies that, in both cases, $0 < \alpha D/2C < 1$. So, by calculating the roots a_α and b_α of the binomial (5.17), we get

$$
\begin{cases}
b_\alpha = -2s_2 \operatorname{tr} B_0^{-1} + 2\sqrt{\Big[f\big(\sqrt{\det B_0}\big)\Big]^2 - \frac{\alpha}{2}\big(s_1 \operatorname{tr} B_0 - s_2 \operatorname{tr} B_0^{-1}\big) + \frac{\alpha^2}{4}}, \\[2ex]
a_\alpha = -2s_2 \operatorname{tr} B_0^{-1} - 2\sqrt{\Big[f\big(\sqrt{\det B_0}\big)\Big]^2 - \frac{\alpha}{2}\big(s_1 \operatorname{tr} B_0 - s_2 \operatorname{tr} B_0^{-1}\big) + \frac{\alpha^2}{4}},
\end{cases}
\tag{5.18}
$$

and the condition $(A_{33} - \alpha)(A_{44} - \alpha) - A_{34}^2 \geq 0$ is equivalent to $a_\alpha \leq -2p_0 \leq b_\alpha$.

We can rewrite (5.18) as

$$
\begin{cases}
b_\alpha = -2s_2 \operatorname{tr} B_0^{-1} + 2\sqrt{C^2 - \dfrac{\alpha}{2}DC + \dfrac{\alpha^2}{4}}, \\[2mm]
a_\alpha = -2s_2 \operatorname{tr} B_0^{-1} - 2\sqrt{C^2 - \dfrac{\alpha}{2}DC + \dfrac{\alpha^2}{4}},
\end{cases}
$$

so that

$$
\sqrt{C^2 - \frac{\alpha}{2}CD + \frac{\alpha^2}{4}} \geq \sqrt{C^2 - \frac{\alpha}{2}CD} = C\sqrt{1 - \frac{\alpha}{2}\frac{D}{C}} > C - \frac{\alpha}{2}D.
$$

Therefore, from (5.9) and (5.6), we have

$$
b_\alpha \geq b_0 - \alpha\overline{d} \quad \text{and} \quad a_\alpha \leq a_0 + \alpha\overline{d}.
$$

Notice also that, from (5.7) and (5.9), we have $b_\alpha - a_\alpha \geq b_0 - a_0 - 2\alpha\overline{d} > 0$. Thus, to conclude the proof, it suffices to show that, under the hypothesis (5.9), the following inequalities hold:

$$
a_* + \alpha \leq a_0 + \alpha\overline{d} \quad \text{and} \quad b_0 - \alpha\overline{d} \leq b_* - 2\alpha.
$$

Indeed, first note that

$$
2\sqrt{\det B_0} \leq \operatorname{tr} B_0 \quad \text{and} \quad 2\sqrt{\det B_0^{-1}} \leq \operatorname{tr} B_0^{-1}.
$$

Therefore, in the case $s_2 < 0$, we have

$$
s_1 \operatorname{tr} B_0 - s_2 \operatorname{tr} B_0^{-1} \geq 2s_1\sqrt{\det B_0} - 2s_2\sqrt{\det B_0^{-1}} = 2f\left(\sqrt{\det B_0}\right),
$$

from which we conclude that

$$
s_1 \operatorname{tr} B_0 - 3s_2 \operatorname{tr} B_0^{-1} \geq -2s_2 \operatorname{tr} B_0^{-1} + 2f\left(\sqrt{\det B_0}\right)
$$

and so, $b_* \geq b_0$. On the other hand, it is easy to see that

$$
-s_1 \operatorname{tr} B_0 + s_2 \operatorname{tr} B_0^{-1} \leq -2s_1\sqrt{\det B_0} + 2s_2\sqrt{\det B_0^{-1}} = -2f\left(\sqrt{\det B_0}\right),
$$

which implies that,

$$
-s_1 \operatorname{tr} B_0 - s_2 \operatorname{tr} B_0^{-1} \leq -2s_2 \operatorname{tr} B_0^{-1} - 2f\left(\sqrt{\det B_0}\right),
$$

and so, $a_* \leq a_0$.

In the case $s_2 \geq 0$, we have

$$
b_* - 2\alpha \geq b_0 - \alpha\overline{d} \Leftrightarrow \operatorname{tr} B_0\left(s_1 - \frac{s_2}{\det B_0}\right) - 2\alpha \geq 2f\left(\sqrt{\det B_0}\right) - \alpha\overline{d}
$$

$$
\Leftrightarrow \frac{\operatorname{tr} B_0}{\sqrt{\det B_0}}f\left(\sqrt{\det B_0}\right) - 2\alpha \geq 2f\left(\sqrt{\det B_0}\right) - \alpha\overline{d}
$$

$$
\Leftrightarrow \left(\frac{\operatorname{tr} B_0}{\sqrt{\det B_0}} - 2\right)f\left(\sqrt{\det B_0}\right) \geq (2 - \overline{d})\alpha.
$$

Since $\operatorname{tr} B_0 \geq 2\sqrt{\det B_0}$ and $\overline{d} \geq 2$, it follows that $b_* - 2\alpha \geq b_0 - \alpha\overline{d}$ holds for all $\alpha > 0$. Likewise,

$$a_* + \alpha \leq a_0 + \alpha\overline{d} \Leftrightarrow -s_1 \operatorname{tr} B_0 + s_2 \operatorname{tr} B_0^{-1} + \alpha \leq -2f\left(\sqrt{\det B_0}\right) + \alpha\overline{d}$$

$$\Leftrightarrow s_1\left(2\sqrt{\det B_0} - \operatorname{tr} B_0\right) - s_2\left(\frac{2}{\sqrt{\det B_0}} - \operatorname{tr} B_0^{-1}\right) \leq (\overline{d} - 1)\alpha$$

$$\Leftrightarrow \left(s_1 - \frac{s_2}{\det B_0}\right)\left(2\sqrt{\det B_0} - \operatorname{tr} B_0\right) \leq (\overline{d} - 1)\alpha$$

and $a_* + \alpha \leq a_0 + \alpha\overline{d}$ holds for all $\alpha > 0$. This finishes the proof.

$\square$

Remark 5.2. The above considerations permit us to conclude (by Lax-Milgram Lemma) that the boundary value problem (4.1) admits a unique solution. In fact,

Corollary 5.1. *Under the hypothesis of Theorem 5.1, the variational problem* (4.3) *admits a unique solution* $\boldsymbol{u} \in \mathcal{V}$.

Remark 5.3. Theorem 5.1 gives a sufficient condition for the existence of a unique weak solution of the boundary value problem (4.1) corresponding to each time step of the successive approximation. As we are supposing that the material is nearly incompressible, it is reasonable to expect that $\det B_0 \approx 1$. This implies that, if γ_1 and γ_2 are the eigenvalues of B_0, $\gamma_1 \approx 1/\gamma_2$ and $\operatorname{tr} B_0^{-1} \approx \gamma_1 + 1/\gamma_1$. So, the hypothesis (5.10) does not mean that we are assuming that p_0 is small. On the other hand, numerical experiments show that the hypothesis (5.10) can be very restrictive in the presence of gravitational body forces. In this case, it is possible to incorporate the potential of the gravitational force into the pressure, and analyze the re-formulated problem.

Remark 5.4. The previous results hold if we assume that $\Gamma_3 = \emptyset$. In fact, unlike the space $\mathcal{V}$ introduced in (5.1), we must consider

$$\mathcal{V} = \left\{\boldsymbol{u} \in (H^1(\Omega))^2 \, ; \, \boldsymbol{u} \cdot \boldsymbol{n}_\kappa = 0 \text{ on } \Gamma_2\right\}. \tag{5.19}$$

However, in this case, it is necessary to assume that the domain Ω satisfies a geometric property to ensure that (5.2) is a norm. This can be done by supposing that Ω has the following property: *There is no constant vector* $\boldsymbol{c} \in \mathbb{R}^2$ *such that* $\boldsymbol{c} \cdot \boldsymbol{n}_\kappa(\boldsymbol{x}) = 0$, $\forall \boldsymbol{x} \in \Gamma_2$.

Appendix

Without loss of generality, we can assume that B_0 is a diagonal matrix, given by

$$B_0 = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}$$

and, in this case,

$$T_0 = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} = \begin{pmatrix} -p_0 + f(\gamma_1) & 0 \\ 0 & -p_0 + f(\gamma_2) \end{pmatrix},$$

where $f(\gamma) = s_1\gamma - s_2\gamma^{-1}$.

Writing the quadratic form (5.4) as

$$\mathcal{A}(\boldsymbol{x}, H, H) = \mathcal{A}_1(\boldsymbol{x}, H, H) + \mathcal{A}_2(\boldsymbol{x}, H, H) + \mathcal{A}_3(\boldsymbol{x}, H, H) + \mathcal{A}_4(\boldsymbol{x}, H, H),$$

where

$$\mathcal{A}_1(\boldsymbol{x}, H, H) = \operatorname{tr}(H) \operatorname{tr}\Big[(T_0 + \beta I)H^T\Big],$$

$$\mathcal{A}_2(\boldsymbol{x}, H, H) = -\operatorname{tr}(T_0 H^T H^T),$$

$$\mathcal{A}_3(\boldsymbol{x}, H, H) = s_1 \operatorname{tr}\Big[(HB_0 + B_0 H^T)H^T\Big],$$

$$\mathcal{A}_4(\boldsymbol{x}, H, H) = -s_2 \operatorname{tr}\Big[(B_0^{-1}H + H^T B_0^{-1})H^T\Big].$$

and the matrix H as $H = E + R$, where $E = \frac{1}{2}(H + H^T)$ and $R = \frac{1}{2}(H - H^T)$, with

$$E = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix},$$

we obtain,

(1) $\operatorname{tr}(T_0 H^T + \beta H^T) = \operatorname{tr}(T_0 E + \beta E) = at_1 + ct_2 + \beta(a + c)$, which gives

$$\mathcal{A}_1(\boldsymbol{x}, H, H) = (a + c)(at_1 + ct_2) + \beta(a + c)^2. \tag{5.20}$$

(2)   Since $T_0 H^T H^T = T_0(E^2 + R^2) - T_0(ER + RE)$, we have

$$\operatorname{tr}(T_0 H^T H^T) = \operatorname{tr}\Big[T_0(E^2 + R^2)\Big]$$

and a direct calculation gives

$$\mathcal{A}_2(\boldsymbol{x}, H, H) = -t_1(a^2 + b^2 - d^2) - t_2(b^2 + c^2 - d^2). \tag{5.21}$$

(3)   We notice that

$$B_0 H^T H^T = B_0(E^2 + R^2) - B_0(ER + RE),$$
$$HB_0 H^T = (EB_0 E - RB_0 R) + (RB_0 E - EB_0 R).$$

Since $B_0(ER + RE)$ and $RB_0 E - EB_0 R$ are skew symmetric, we have

$$\operatorname{tr}\Big[(HB_0 + B_0 H^T)H^T\Big] = \operatorname{tr}\Big[B_0(E^2 + R^2) + (EB_0 E - RB_0 R)\Big],$$

and a direct calculation gives

$$\mathcal{A}_3(\boldsymbol{x}, H, H) = 2s_1\Big[\gamma_1 a^2 + \gamma_2 c^2 + (\gamma_1 + \gamma_2)b^2 + (\gamma_2 - \gamma_1)bd\Big]. \tag{5.22}$$

(4)   As before,

$$B_0^{-1} H H^T = B_0^{-1}(E^2 - R^2) + B_0(RE - ER + RE),$$
$$H^T B_0 H^T = (EB_0^{-1}E + RB_0^{-1}R) - (EB_0^{-1}R + RB_0^{-1}E),$$

a direct calculation gives

$$\mathcal{A}_4(\boldsymbol{x}, H, H) = -2s_2\Big[\gamma_1^{-1}a^2 + \gamma_2^{-1}c^2 + (\gamma_1^{-1} + \gamma_2^{-1})b^2 + (\gamma_1^{-1} - \gamma_2^{-1})bd\Big]. \tag{5.23}$$

Therefore, by denoting $X = (a, c, b, d)^T$ and considering (5.20)-(5.23), we can express the quadratic form (5.4) as

$$\mathcal{A}(\boldsymbol{x}, H, H) = X^T \cdot A(\boldsymbol{x})X,$$

where $A(\boldsymbol{x})$ is the matrix (5.5).

References

[1] P.G. Ciarlet, *Mathematical elasticity, Volume 1: Three-dimensional elasticity*, North-Holland, Amsterdam, 1988.

[2] A.E. Green, R.S. Rivlin and R.T. Shield, *General theory of small elastic deformations*, Pro. Roy. Soc. London, Ser. A, 211, (1952), 128–154.

[3] P. Haupt, *Continuum mechanics and theory of materials*, Second Edition, Springer, 2002.

[4] I-S. Liu, *Continuum mechanics*, Springer, Berlin Heidelberg, 2002.

[5] I-S. Liu, *Successive linear approximation for boundary value problems of nonlinear elasticity in relative-descriptional formulation*, Int. J. Eng. Sci., doi:10.1016/j.ijengsci.2011.02.006, 2011.

[6] I-S. Liu, *A note on Mooney-Rivlin material model, continuum mechanics and thermodynamics*, doi: 10.1007/s000161-011-0197-6, 2011.

[7] I-S. Liu, R. Cipolatti and M.A. Rincon, *Successive linear approximation for finite elasticity*, Computational and Applied Mathematics, 3(2010), 465–478.

[8] I-S. Liu, R. Cipolatti, M.A. Rincon and L.A. Palermo, *Numerical simulation of salt migration – Large deformation in viscoelastic solid bodies*, (submitted) arXiv:1108.2295.

[9] R.W. Ogden, *Non-linear elastic deformations*, Ellis Horwood, New York, 1984.

[10] R. Temam, *Navier-Stokes equations, theory and numerical analysis*, AMS Chelsea Publishing, 2000.

[11] R. Temam and A. Miranville, *Mathematical modeling in continuum mechanics*, Cambridge University Press, 2001.

[12] C. Truesdell and W. Noll, *The non-linear field theories of mechanics*, third ed., Springer, Berlin, 2004.