ASYMPTOTIC BEHAVIOR OF AN AGE-STRUCTURED POPULATION MODEL WITH DIFFUSION

S.M. Bouguima[1,†] and F.Z. Mehlia[1]

Abstract A reaction-diffusion system with stage-structure is studied. We provide well-posedness of the model and prove that time-dependent solutions evolve either towards a positive equilibrium or to the trivial one. Under suitable conditions, a branch of positive equilibrium is shown to exist.

Keywords Adult-juvenile, parabolic equations, asymptotic behavior.

MSC(2000) 05C38, 15A15, 05A15, 15A18.

1. Introduction

If diffusion is ignored in modeling, there are criteria like Bendixon and Dulac Theorems that allow if a system of autonomous differential equations does or does not admit periodic solutions. Hence, in the planar case, the long time behavior of trajectories is simple. If diffusion is present, asymptotic behavior is difficult to study. In this case, we establish global behavior of the trajectories. This paper deals with the system

$$
\begin{cases}
\frac{\partial}{\partial t}u - d_1\Delta u = \sigma v - eu - cu(u+v) & \text{in } \Omega \times [0,T], \\
\frac{\partial}{\partial t}v - d_2\Delta v = bu - fv - dv(u+v) & \text{in } \Omega \times [0,T], \\
u(0,x) = u_0, v(0,x) = v_0 & \text{in } \Omega, \\
\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega \times [0,T],
\end{cases}
\tag{1.1}
$$

where Ω is a bounded and regular open subset of $\mathbb{R}^n$. The parameters σ, e, c, b, f and d are positive. The model (1.1) describes the dynamics of a population with stage-strucure. In this system, u and v denote the density of adult and juvenile respectively. For further details, see for instance Bouguima etc. [3] and the references therein.

Our goal in this paper is to perform an analytic study of (1.1). Two aspects have to be considered: existence of solutions of (1.1) and their asymptotic behavior. It is important to understand under what conditions the system (1.1), evolves towards a stationary solution. The corresponding steady state system has been studied in Canada etc. [4], Brown & Zhang [2] and Bouguima etc. [3]. Usually one thinks of solutions of system (1.1) as a couple of functions $(u,v) \in \left[C^2\left(\Omega\right) \cap C^1\left(\overline{\Omega}\right) \right]^2$.

†the corresponding author. Email address: bouguima@yahoo.fr
  (S.M. Bouguima)
[1]Department of Mathematics, Faculty of Sciences, University of Tlemcen,
  B.P.119, Tlemcen 13000, Algeria

But often, it is better to start with mild solution $(u, v) \in \left[C \left(\overline{\Omega} \right) \right]^2$ and obtain more regularity using classical theory of parabolic systems.

The organization of the paper is as follows: in section 2 we study the spatially homogeneous model. We deal with existence and global behavior. We consider the spatially inhomogeneous model in section 3, where global solution is established. Section 4 is devoted to the study of local stability of steady state solutions. In section 5, we consider bifurcation of solution from the trivial one in suitable Sobolev spaces. Section 6 contains the main result namely, if the trivial solution is unstable, then the system converges to a positive equilibrium.

2. Spatially homogeneous model

In absence of spatial effects, the model (1.1) reads as follows:

$$\begin{cases} \dot{u} = \sigma v - eu - cu(u + v), \\ \dot{v} = bu - fv - dv(u + v), \\ u(0) = u_0, \quad v(0) = v_0. \end{cases} \tag{2.1}$$

Its steady-state are solutions of the following system

$$\begin{cases} \sigma v - eu - cu(u + v) = 0, \\ bu - fv - dv(u + v) = 0. \end{cases}$$

It is easy to see that there are two equilibriums, the trivial one $(0, 0)$ and a positive steady state (u^*, v^*) with $0 \leq u^* \leq \frac{\sigma}{c}$ and $0 \leq v^* \geq \frac{b}{d}$, provided that $b\sigma > ef$. The Jacobian matrix at $(0, 0)$ takes the form of

$$J = \begin{pmatrix} -e & \sigma \\ b & -f \end{pmatrix}.$$

Hence $(0, 0)$ is locally unstable if $b\sigma > ef$. We conclude that the positive equilibrium exists whenever $(0, 0)$ is unstable. Standard arguments show that the solution of (2.1) always exists and stay positive. In addition, we have

Proposition 2.1. *If u_0, and $v_0 \geq 0$, then*

$$\limsup_{t \to \infty} u(t) \leq K = \frac{\max(\sigma, b)}{\min(c, d)} \quad and \quad \limsup_{t \to \infty} v(t) \leq K = \frac{\max(\sigma, b)}{\min(c, d)}.$$

Proof. Let

$$w = u + v.$$

Then

$$w' \leq \max(\sigma, b)w - \min(c, d)w^2.$$

Standard comparison arguments gives:

$$w \leq z,$$

where z is the solution of the logistic equation

$$\begin{cases} z' = \max(\sigma, b)z - \min(c, d)z^2, \\ z(0) = w(0). \end{cases}$$

We know that:

$$\limsup_{t\to\infty} z(t) \le K = \frac{\max(\sigma, b)}{\min(c, d)}.$$

This proves the desired result.   □

We shall show that the system (2.1) has no positive periodic solutions. Our method involves an application of the criterion of Bendixon (see th.4.1 in Verhulst [8]). As a consequence, we establish the global behavior of solutions.

Theorem 2.1. *System (2.1) has no positive periodic solutions. In addition, if $b\sigma < ef$, then $(u(t), v(t))$ tends to $(0,0)$. If $b\sigma > ef$, then $(u(t), v(t))$ tends to (u^*, v^*).*

Proof.   Let

$$f(u, v) = \sigma v - eu - cu(u + v)$$

and

$$g(u, v) = bu - fv - dv(u + v).$$

Then the divergence of the vector field (f, g) is

$$div(f, g) = \frac{\partial f}{\partial u} + \frac{\partial g}{\partial v} < 0.$$

It follows from Dulac's criterion (see Strogatz [7], p202) that system (2.1) has no periodic solutions. The last proposition and Poincaré-Bendixon Theorem imply that the positive solution of system (2.1) tends either to the origin or to (u^*, v^*).

<div style="text-align:right">□</div>

3. Spatially inhomogeneous model

Ordinary differential equations such as described above, assume an homogenous environment. However, the search of food by the fish population often happen by random movement. This Fickian dispersion would be modeled by the laplacian operator. Adding diffusion terms to the system (2.1), we obtain a reaction-diffusion system like (1.1).

3.1. local existence of solutions.

In this section, we are concerned with local existence of solutions in the Banach space $X = C(\overline{\Omega})$.

Proposition 3.1. *There exists $t^* > 0$ such that the problem (1.1) has a unique smooth solution in $[0, t^*)$. Furthermore, if t^* is maximal and $t^* < \infty$, then*

$$\lim_{t\to t^*} (|u|_\infty + |v|_\infty) = +\infty.$$

This result is well known and we give only a sketch of the proof. The first step is to convert the system using variation of constants formula, to an integral equation:

$$U(t) = T(t)U_0 + \int_0^t T(t - s)F(U(s))ds,$$

where
$$U = (u, v), \quad U_0 = (u_0, v_0)$$
and
$$F(U) = (\sigma v - eu - cu(u + v), bu - fv - dv(u + v)).$$

Here $\{T(t)\}_{t \geq 0}$ is an analytic semigroup on X, generated by the closure of the operator $A = (d_i \Delta)$ on a convenient domain for which Neumann conditions hold. Since F is locally Lipschitzian, one can prove existence of a local solution defined on maximal interval $[0, t^*)$. Using assumptions on F and that $T(t)$ is analytic, we deduce the smoothness properties of the solutions, see for instance proposition (2.1) in Dung & Smith [5].

3.2. Positivity of the solution

We will prove that problem (1.1) preserves positiveness.

Proposition 3.2. *If u_0, $v_0 \geq 0$, then $u(t, .)$ and $v(t, .) \geq 0$ for $t \geq 0$.*

Proof. Note that $u(t, x) = u^+(t, x) - u^-(t, x)$. The function u satisfies:

$$\begin{cases} \frac{\partial}{\partial t} u - d_1 \Delta u = \sigma v - eu - cu(u + v) & \text{in} \quad \Omega \times [0, t^*], \\ u(0, x) = u_0 & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on} \quad \partial\Omega \times [0, t^*]. \end{cases} \tag{3.1}$$

Let $\|u\| = \sup\limits_{[0, t^*]} \|u(t)\|_{C(\overline{\Omega})}$.

Multiply the equation in (3.1) by u^- and integrate over Ω, this implies that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u^-|^2 \, dx + d_1 \int_\Omega |\nabla u^-|^2 \, dx = -\sigma \int_\Omega vu^- + e \int_\Omega uu^- + c \int_\Omega u(u + v)u^- \, dx. \tag{3.2}$$

We now estimate each of the terms of the right hand side of (3.2) separately. By Young inequality we will have

$$\begin{cases} \sigma \int_\Omega vu^- \leq \frac{\sigma}{\varepsilon} \int_\Omega |u^-|^2 \, dx + \sigma\varepsilon \int_\Omega v^2 \, dx, \\ e \int_\Omega uu^- \leq \frac{e}{\varepsilon} \int_\Omega |u^-|^2 \, dx + e\varepsilon \int_\Omega u^2 \, dx, \\ c \int_\Omega u^2 u^- \, dx \leq c \int_\Omega u(\frac{1}{\varepsilon} |u^-|^2 + \varepsilon u^2) \, dx, \\ c \int_\Omega uu^- v \leq c \int_\Omega u(\frac{1}{\varepsilon} |u^-|^2 + \varepsilon v^2) \, dx. \end{cases}$$

Here ε is any positive number. It follows via (3.2) combined with the previous inequalities that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u^-|^2 \, dx$$
$$\leq \frac{(\sigma + e + 2c\|u\|)}{\varepsilon} \int_\Omega |u^-|^2 \, dx + \varepsilon(\sigma + c\|u\|) \int_\Omega v^2 \, dx + \varepsilon(e + c\|u\|) \int_\Omega u^2 \, dx.$$

By choosing ε small enough, we can eliminate the second and the third term of the right hand side of the last inequality. We arrive at the differential inequality:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u^-|^2 \, dx \leq C_0(\sigma + e + 2c\|u\|) \int_\Omega |u^-|^2 \, dx$$

for some constant C_0 independent of u. Therefore

$$\int_\Omega \left|u(t)^-\right|^2 dx \le \left\|u^-(0,x)\right\|^2_{L^2(\Omega)} + 2C_0(\sigma + e + 2c\left\|u\right\|)\int_0^t \int_\Omega \left|u(t)^-\right|^2 dx.$$

Since $u(0,x) = u_0(x) \ge 0$ then $u^-(0,x) = 0$.
Gronwall's inequality implies that

$$\int_\Omega \left|u^-\right|^2 dx = 0.$$

Hence

$$u^-(t,x) = 0$$

and u is positive.

The positiveness of v is obtained in a similar manner. This completes the proof of the proposition. □

3.3. Global existence of the solution

Global existence, that is the solutions are defined on the whole $t \ge 0$ is established for positive solutions.

Proposition 3.3. *The solutions u, v provided by proposition (3.1) are defined on $[0, +\infty)$.*

Proof. Let $u_1(t) = \int_\Omega u(t,x)dx$ and $v_1(t) = \int_\Omega v(t,x)dx$.
Integrating the equations in (1.1) over Ω we obtain

$$\begin{cases} \frac{d}{dt}u_1(t) \le \sigma v_1(t), \\ \frac{d}{dt}v_1(t) \le bu_1(t). \end{cases}$$

Therefore:

$$\frac{d}{dt}(u_1(t) + v_1(t)) \le 2(\sigma + b)(u_1(t) + v_1(t))$$

and

$$u_1(t) + v_1(t) \le u(0) + v(0) + 2(\sigma + b)\int_0^t (u(s) + v(s))ds.$$

By Gronwall's inequality we find that:

$$u_1(t) + v_1(t) \le (u(0) + v(0))\exp 2(\sigma + b)t. \tag{3.3}$$

Since $u(t,x) \le u_1(t)$ and $v(t,x) \le v_1(t)$, we conclude from (3.3) that $(u(t), v(t))$ exists globally on $C(\overline{\Omega})$. □

4. Steady state solution

The corresponding steady state system of (1.1) is

$$\begin{cases} -d_1\Delta u = \sigma v - eu - cu(u + v) = f_1(u,v) & \text{in} \quad \Omega, \\ -d_2\Delta v = bu - fv - dv(u + v) = f_2(u,v) & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{4.1}$$

Observe that the equilibrium given by (2.1) is also a steady state of system (1.1).

Proposition 4.1. *The trivial equilibrium* $(0,0)$ *is unstable iff* $b\sigma > ef$.

Proof.   The linearized problem around $(0,0)$ is

$$
\begin{cases}
\frac{\partial}{\partial t}u - d_1\Delta u = \sigma v - eu & \text{in } \Omega \times \mathbb{R}^+, \\
\frac{\partial}{\partial t}v - d_2\Delta v = bu - fv & \text{in } \Omega \times \mathbb{R}^+, \\
u(0,x) = u_0, v(0,x) = v_0 & \text{in } \Omega, \\
\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega \times \mathbb{R}.^+
\end{cases}
$$

Let $X = C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$ and $Y = C(\overline{\Omega})$. Define $L : X \to Y$ by

$$
L\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -d_1\Delta u \\ -d_2\Delta v \end{pmatrix} - \begin{pmatrix} \sigma v - eu \\ bu - fv \end{pmatrix}.
$$

It is proved in Brown & Zhang [2] that L has a principal eigenvalue, i.e; there exists $\lambda_1 \in \mathbb{R}$ and strictly positive functions $u, v \gg 0$ such that

$$
\begin{cases}
-d_1\Delta u - \sigma v + eu = \lambda_1 u, \\
-d_2\Delta v - bu + fv = \lambda_1 v.
\end{cases}
\tag{4.2}
$$

By integrating on Ω both sides of of system (4.2), we obtain

$$
\begin{cases}
-\sigma \int_\Omega v + e \int_\Omega u = \lambda_1 \int_\Omega u, \\
-b \int_\Omega u + f \int_\Omega v = \lambda_1 v \int_\Omega.
\end{cases}
$$

Let

$$
\xi = \frac{\int_\Omega u}{\int_\Omega v} > 0,
$$

so we have

$$
\begin{cases}
\frac{-\sigma}{\xi} + e = \lambda_1, \\
-b\xi + f = \lambda_1.
\end{cases}
$$

To determine λ_1, we need to solve the algebraic equation for ξ

$$
b\xi^2 + (e - f)\xi - \sigma = 0.
$$

By a simple calculation, we find that

$$
\lambda_1 = \frac{2(ef - b\sigma)}{\sqrt{(f-e)^2 + 4b\sigma} + (f+e)}.
$$

Hence $\lambda_1 > 0$ iff $ef - b\sigma > 0$. $\square$

Remark 4.1. We have shown that addition of diffusion does not have a destabilizing effect on the trivial equilibrium of the ODE equations.

5. Bifurcation of the steady state solution

We will show that the non trivial equilibrium can emerge from the trivial one. We introduce a positive parameter λ.

Let

$$X = \left\{ u \in W^{2,p}(\Omega), \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \right\}, Y = L^p(\Omega) \text{ with } p > n,$$

and consider the bifurcation problem

$$L \begin{pmatrix} u \\ v \end{pmatrix} + \lambda A \begin{pmatrix} u \\ v \end{pmatrix} + \lambda G \begin{pmatrix} u \\ v \end{pmatrix} = 0, \tag{5.1}$$

where the operators L, A and G are defined from $X \times X$ into $Y \times Y$ by

$$L \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} d_1 \Delta u \\ d_2 \Delta v \end{pmatrix},$$

$$A \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \sigma v - eu \\ bu - fv \end{pmatrix}$$

and

$$G \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -cu(u+v) \\ -dv(u+v) \end{pmatrix}.$$

We analyze the local structure of the set of positive solutions of (5.1) near $\lambda = 0$. To make this analysis, we will find out the bifurcation equations at this value by Lyapunov-Schmidt Method.

The null space of L is given by

$$\ker L = \{(u,v) \in X \times X, (u,v) = (\alpha, \beta) \in \mathbb{R}^2\}.$$

The spaces $X \times X$ and $Y \times Y$ can be decomposed as follow:

$$X \times X = \ker L + X_1$$

and

$$Y \times Y = \ker L + Y_1,$$

where X_1 and Y_1 are the L^2 orthogonal complements of $X \times X$ and $Y \times Y$ respectively.

Let P and Q be the orthogonal projections on X_1 and Y_1 respectively.

Each element $(u, v) \in X \times X$ admits a unique decomposition of the form:

$$(u, v) = (\alpha, \beta) + U, \quad (\alpha, \beta) \in \mathbb{R}^2, \quad U = P(u, v) \in X_1.$$

It is clear that (5.1) is equivalent to

$$\begin{cases} QLU + \lambda QAU + \lambda QG((\alpha, \beta) + U) = 0, \\ A(\alpha, \beta) + (I - Q)G((\alpha, \beta) + U) = 0. \end{cases} \tag{5.2}$$

Lemma 5.1. *The operator* $QL : X_1 \to X_1$ *is invertible.*

Proof.   The operator QL is injective and surjective. Indeed, let $x = \begin{pmatrix} u \\ v \end{pmatrix} \in X_1$

such that $QL \begin{pmatrix} u \\ v \end{pmatrix} = 0$, then

$$L \begin{pmatrix} u \\ v \end{pmatrix} \in \ker L.$$

This implies that there exists $(\alpha, \beta) \in \mathbb{R}^2$ such that

$$L\left(\begin{array}{c} u \\ v \end{array}\right) = (\alpha, \beta),$$

then

$$\begin{cases} d_1 \Delta u = \alpha, \\ d_2 \Delta v = \beta, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, \quad on \quad \partial\Omega. \end{cases}$$

Integrating over Ω, we obtain that

$$(\alpha, \beta) = (0, 0)$$

and

$$L\left(\begin{array}{c} u \\ v \end{array}\right) = (0, 0).$$

This implies that

$$(u, v) \in X_1 \cap \ker L = \{(0, 0)\}.$$

Let $(x_1, x_2) \in X_1$ be fixed. From classical results of elliptic equations. The system

$$\begin{cases} d_1 \Delta u = x_1, \\ d_2 \Delta v = x_2, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, \quad on \quad \partial\Omega. \end{cases}$$

has a solution $(u, v) \in X \times X$. Hence $(u, v) = (\alpha, \beta) + U \in \ker L + X_1$. It follows that U is a solution of the previous system. □

Consider the equation

$$T(\lambda, U, (\alpha, \beta)) = QLU + \lambda QAU + \lambda QG((\alpha, \beta) + U) = 0.$$

Lemma 5.2. *There exits a neighborhood O_λ of $\lambda = 0$, a neighborhood $O_U \subset X_1$ of $U = 0$ and a function $\varphi : O_\lambda \to O_U$ such that $T(\lambda, \varphi(\lambda, (\alpha, \beta)), (\alpha, \beta)) = 0$.*

Proof. We have $T(0, 0, (\alpha, \beta)) = 0$. The derivative of T at the point $\lambda = 0$ and $U = 0$, gives $D_U T(0, 0, (\alpha, \beta))V = QLV$ which is invertible. Applying the Implicit Function Theorem, we obtain the desired result. □

Using the previous lemma, the bifurcation problem (5.1) is equivalent to:

$$F(\lambda, (\alpha, \beta)) = A(\alpha, \beta) + (I - Q)G((\alpha, \beta) + \varphi(\lambda, (\alpha, \beta)) = 0.$$

Theorem 5.1. *If $b\sigma > ef$, then (5.1) has a positive solution $(u(\lambda), v(\lambda))$ such that $(u(0), v(0)) = (0, 0)$.*

Proof. Let

$$\varphi(\lambda, (\alpha, \beta)) = (\varphi_1(\lambda, (\alpha, \beta), \varphi_2(\lambda, (\alpha, \beta)))$$

and

$$G((\alpha, \beta) + \varphi(\lambda, (\alpha, \beta))) = \left(\begin{array}{c} -c\left[\, \alpha(\alpha + \beta) + (2\alpha + \beta)\varphi_1 + \varphi_1^2 + \varphi_1\varphi_2 + \alpha\varphi_2\right] \\ -d\left[\, \beta(\alpha + \beta) + (2\beta + \alpha)\varphi_2 + \varphi_2^2 + \varphi_1\varphi_2 + \beta\varphi_1\right] \end{array}\right).$$

The equation

$$F(0, (\alpha, \beta)) = 0$$

implies that

$$A(\alpha, \beta) + G(\alpha, \beta) = 0,$$

so

$$(\alpha, \beta) = (0, 0) \text{ or } (\alpha, \beta) = (\alpha^*, \beta^*) \text{ with } 0 < \alpha^* < \frac{\sigma}{c}, 0 < \beta^* < \frac{b}{d}.$$

The linearized operator

$$D_{(\alpha, \beta)} F(0, (0, 0)) V = AV$$

is invertible. If $b\sigma - ef > 0$, then the Implicit Function Theorem implies the existence of a neighborhood $O_{(0,0)}$ of $(0, 0)$, a neighborhood O_0 of 0 and a function $\Phi : O_0 \to O_{(0,0)}$ such that

$$F(\lambda, \Phi(\lambda)) = 0.$$

Similarly

$$D_{(\alpha, \beta)} F(0, (\alpha^*, \beta^*)) = \begin{pmatrix} -e - 2c\alpha^* - c\beta^* & \sigma - c\alpha^* \\ b - d\beta^* & -f - d\alpha^* - 2d\beta^* \end{pmatrix}.$$

Let

$$M = D_{(\alpha, \beta)} F(0, (\alpha^*, \beta^*)) = \begin{pmatrix} -e - 2c\alpha^* - c\beta^* & \sigma - c\alpha^* \\ b - d\beta^* & -f - d\alpha^* - 2d\beta^* \end{pmatrix}.$$

Since (α^*, β^*) is a solution of

$$\begin{cases} \sigma\beta^* - e\alpha^* - c\alpha^*(\alpha^* + \beta^*) = 0, \\ b\alpha^* - f\beta^* - d\beta^*(\alpha^* + \beta^*) = 0, \end{cases}$$

then it satisfies

$$\begin{aligned} (-e - c\alpha^*)\alpha^* + (\sigma - c\alpha^*)\beta^* &= 0, \\ (b - d\beta^*)\alpha^* + (-f - d\alpha^*)\beta^* &= 0, \end{aligned}$$

so (α^*, β^*) is an eigenvector of the matrix

$$B = \begin{pmatrix} (-e - c\alpha^*) & (\sigma - c\alpha^*) \\ (b - d\beta^*) & (-f - d\alpha^*) \end{pmatrix}$$

associated to the eigenvalue 0. Since the trace $trB < 0$, we conclude that B has two eigenvalues: $\lambda_1 = 0$ and $\lambda_2 < 0$.

We have

$$\begin{aligned} \sigma - c\alpha^* &> 0, \\ b - d\beta^* &> 0. \end{aligned}$$

This implies that M is irreducible and $M < B$. Applying Perron-Frobenius Theorem (Dung & Smith [5], p60), we find that

$$\lambda_1(M) < \lambda_1(B) = 0.$$

Hence M is invertible. The implicit function Theorem implies the existence of a neighborhood $O_{(\alpha^*, \beta^*)}$ of (α^*, β^*) and a neighborhood O_0 of 0 and a function $\Psi : O_0 \to O_{(\alpha^*, \beta^*)}$ such that

$$F(\lambda, \Psi(\lambda)) = 0.$$

The two nontrivial solutions defined on a neighborhood of $\lambda = 0$ are given by

$$(u_1(\lambda), v_1(\lambda)) = \Phi(\lambda) + \varphi(\lambda, \Phi(\lambda))$$

and

$$(u_2(\lambda), v_2(\lambda)) = \Psi(\lambda) + \varphi(\lambda, \Psi(\lambda)).$$

The principal eigenvalue of the operator $L + \lambda A$ is

$$Z(\lambda) = \frac{2\lambda(b\sigma - ef)}{\sqrt{(f-e)^2 + 4b\sigma} + (f+e)}.$$

If $\lambda > 0$, then $Z(\lambda) > 0$, and the results in Brwn and Zhang [2] and Canada etc. [4] imply that (5.1) has a unique positive solution that is $(u_2(\lambda), v_2(\lambda))$. $\qquad \square$

6. Asymptotic behavior

Sufficient conditions are obtained by Pao [6] to ensure the convergence of the time-dependent solution to a steady state solution between upper and lower solutions. It is crucial to remark that since all the coefficients are constant, the system (4.1) has a unique positive solution (u, v) in the region $\left[0, \frac{\sigma}{c}\right] \times \left[0, \frac{b}{d}\right]$ when $\lambda_1 < 0$ (see for instance Canada etc. [4]). The dynamic of the system is considered in the following theorem.

Let (φ_1, φ_2) be the principal eigenfunction associated to λ_1.

Proposition 6.1. *Let $(u(t, u_0), v(t, v_0))$ be the solution of (1.1) and (u, v) be the positive solution of (4.1).*

Assume that $b\sigma > ef$. If the initial distribution (u_0, v_0) satisfies

$$\frac{\sigma}{c \max \varphi_1} \varphi_1 \leq u_0 \leq \frac{\sigma}{c}, \quad \frac{b}{d \max \varphi_2} \varphi_2 \leq v_0 \leq \frac{b}{d},$$

then the solution $(u(t, u_0), v(t, v_0))$ converges to (u, v) as $t \to \infty$.

Assume that $b\sigma < ef$. If the initial distribution (u_0, v_0) satisfies

$$0 \leq u_0 \leq \frac{\sigma}{c}, \quad 0 \leq v_0 \leq \frac{b}{d},$$

then the solution $(u(t, u_0), v(t, v_0))$ converges to $(0,0)$ as $t \to \infty$.

Proof. We distinguish two cases:

Case when $b\sigma > ef$ and initial values verify

$$\frac{\sigma}{c \max \varphi_1} \varphi_1 \leq u_0 \leq \frac{\sigma}{c}, \quad \frac{b}{d \max \varphi_2} \varphi_2 \leq v_0 \leq \frac{b}{d}.$$

Let

$$\begin{pmatrix} \underline{u} \\ \underline{v} \end{pmatrix} = \varepsilon \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

with ε such that

$$0 < \varepsilon < \min\left\{\frac{\sigma}{c\max\varphi_1}, \frac{b}{d\max\varphi_2}\right\}$$

and

$$\left(\begin{array}{c}\overline{u}\\\overline{v}\end{array}\right) = \left(\begin{array}{c}\frac{\sigma}{c}\\\frac{b}{d}\end{array}\right)$$

be respectively a system of ordered lower and upper solutions of (1.1). Since the system is quasimonotone in $\langle u_*, u^*\rangle$, with

$$u_* = (\underline{u}, \underline{v}), \quad u^* = (\overline{u}, \overline{v}).$$

Theorem 8.3.1 in Pao [6] implies that (1.1) has a unique solution satisfying

$$\left(\begin{array}{c}\underline{u}\\\underline{v}\end{array}\right) \le \left(\begin{array}{c}u(t, u_0)\\v(t, v_0)\end{array}\right) \le \left(\begin{array}{c}\overline{u}\\\overline{v}\end{array}\right).$$

In particular for fixed $t_1 > 0$, we have :

$$\left(\begin{array}{c}\underline{u}\\\underline{v}\end{array}\right) \le \left(\begin{array}{c}u(t_1, u_0)\\v(t_1, v_0)\end{array}\right) \le \left(\begin{array}{c}\overline{u}\\\overline{v}\end{array}\right). \tag{6.1}$$

Let $(\underline{U}, \underline{V})$ and $(\overline{U}, \overline{V})$ be the solutions of (1.1) corresponding respectively to initials values $(\underline{u}, \underline{v}), (\overline{u}, \overline{v})$. In this case, Lemma 10.4.1 in Pao [6] implies that:

$$\left(\begin{array}{c}\underline{U}\\\underline{V}\end{array}\right) \le \left(\begin{array}{c}u(t + t_1, u_0)\\v(t + t_1, v_0)\end{array}\right) \le \left(\begin{array}{c}\overline{U}\\\overline{V}\end{array}\right).$$

Furthermore, Theorem 8.3.1 in Pao [6] shows that the solutions $\left(\begin{array}{c}\underline{U}\\\underline{V}\end{array}\right)$ and $\left(\begin{array}{c}\overline{U}\\\overline{V}\end{array}\right)$ remain in $\langle u_*, u^*\rangle$, it follows from Theorem 10.4.1 in Pao [6] that

$$(u, v) \le \lim_{t\to\infty} (u(t + t_1, u_0), v(t + t_1, v_0)) \le (u, v).$$

We conclude that

$$\lim_{t\to\infty} (u(t, u_0), v(t, v_0)) = (u, v).$$

Case when $b\sigma < ef$ and initial values verify

$$0 \le u_0 \le \frac{\sigma}{c}, \quad 0 \le v_0 \le \frac{b}{d}.$$

In this case , system (4.1) has only the trivial solution $(0, 0)$, and

$$\left(\begin{array}{c}\underline{u}\\\underline{v}\end{array}\right) = \left(\begin{array}{c}0\\0\end{array}\right)$$

is a lower solution of (1.1). Let $t_1 > 0$ be fixed. Similarly, we have:

$$\left(\begin{array}{c}0\\0\end{array}\right) \le \left(\begin{array}{c}u(t_1, u_0)\\v(t_1, v_0)\end{array}\right) \le \left(\begin{array}{c}\overline{u}\\\overline{v}\end{array}\right). \tag{6.2}$$

By Lemma 10.4.1, Theorem 10.4.1 in Pao [6] and the fact that for $\lambda_1 > 0$, the unique solution of (4.1) is $(0, 0)$, we obtain that

$$\lim_{t\to\infty} (u(t, u_0), v(t, v_0)) = (0, 0).$$

$\square$

Concluding remarks

Besides of the mathematical interest of the results, there is an ecological motivation. In this paper, we propose a simple model with age structure which is more realistic than the well known logistic equation. Both adults and juveniles can become extinct. Such extinction occurs when $b\sigma$ is low or ef is high. On the other hand if $b\sigma$ is high or ef is low, the system has a stable positive equilibrium. When the diffusion is fast, a branch of positive equilibrium for the stationary problem bifurcates from the extinction state.

Acknowledgements

References

[1] H. Amann, *Ordinary differential equations, an introduction to nonlinear analysis*, de Gruter Studies in Mathematics 13, New York, 1990.

[2] K.J. Brown and Y. Zhang, *On a system of reaction-diffusion equations describing a population with two age groups*, Math. Anal. Appl., 282 (2003), 444-452.

[3] S.M. Bouguima, S. Fekih and W. Hennaoui, *Spatial structure in a juvenile-adult model*, Nonlinear Analysis: Real World Applications, 9 (2008), 1184-1201.

[4] A. Canada, P. Magal and J.A. Montero,*Optimal control of harvesting in a nonlinear elliptic system arising from population dynamics*, Math. Anal. Appl., 254 (2001), 571-581.

[5] L. Dung and H. Smith, *A parabolic system modeling microbial competition in an unmixed bio-reactor*, J. Differential Equations, 130 (1996), 59-91.

[6] C.V. Pao, *Nonlinear parabolic and elliptic equations*, Plenum, New York, 1992.

[7] S.H. Strogatz, *Nonlinear dynamics and chaos*, Addison Wesley publishing company, 1994.

[8] F. Verhulst, *Nonlinear differential equations and dynamical systems*, Springer-Verlag, 1990.