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# AFFINELY ADJUSTABLE ROBUST OPTIMIZATION MODEL FOR MULTI-PERIOD PRODUCTION AND INVENTORY SYSTEM UNDER RISK PREFERENCE*

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**Abstract** A problem of optimizing multi-period centralized production and inventory system with waste disposal subjected to uncertain demands is investigated in this paper. Assuming limited information of distributions, that is, only the mean, support and some measures of deviations are available for the demand, a joint ellipsoid uncertainty set is constructed to control the degree of conservatism of the production policies associated with the integrated managers risk preference. Using Affinely Adjustable Robust Counterpart method, we develop an uncertain optimization model in pursuit of maximizing the overall revenue through adaptively controlling multi-period production policies, and relax it to one deterministic robust counterpart which is in fact a tractable second order cone problem.

**Keywords** Multi-period production and inventory, affinely adjustable robust counterpart, risk preference, joint ellipsoid uncertainty set.

**MSC(2000)** 90B05, 90C25, 90C30.

## 1. Introduction

In an extremely complicated environment, the supply chains are inevitably facing several kinds of uncertainty, which coursing the prediction of market demands often deviated from the actual situation, bringing about lots of risk to overall supply chain performance. On the one hand, in pursuit of meeting the customer needs punctually and avoiding out of stock, the supply chain should consider of a certain inventory level in the process of production. On the other hand, for the purpose of the cost reduction and enhancing the competitiveness, it should control the stock to reduce inventory level as far as possible. Therefore, how to arrange production inventory policy subject to uncertain market demand will seriously affect the whole enterprise profit.

The focus of robust optimization is to protect the system against the worst instances of the uncertainty in a certain given set. Theoretically, it was originally developed by Ben-Tal and Nemirovski [1, 2] and independently by El-Ghaoui et al.[8, 9] to address the imperfect knowledge of parameters in mathematical programming problems with an ellipsoidal uncertainty structure. Furthermore, Bertsimas and Sim [4, 5] and Bertsimas et al.[6] developed another robust technique to model

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trade-offs between performance and risk using polyhedral uncertainty sets or risk measures. Practically, based on the framework of robust optimization, Bertsimas and Thiele [7] established a robust inventory control model to address the problem of optimally controlling a supply chain subject to uncertain demand in discrete time, and proved that the model can be effectively handled by solving a linear programming problem. Ben-Tal [3] took a re-parametrization technique for discrete time linear control systems named affinely adjustable robust counterpart method based on robust optimization, which lead to tractable deterministic optimization problems. Under the assumption that the market demand were not exactly known with only limited information such as mean, support, and some measures of deviations, See and Sim [10] proposed a robust optimization approach to address a multi-period inventory control problem, and obtained the parameters of the replenishment policies by solving a tractable deterministic optimization problem in the form of second order cone optimization problem.

In this paper, we address the problem of optimizing multi-period production and inventory system with waste disposal, where the demands can be partially forecasted statistically based on historical data. We make the same assumption about uncertain demands as See and Sim [10] did that the demands may be correlated and ambiguous, which have limited information of distributions, that is, only the mean, support, and some measures of deviations are available. Using Affinely Adjustable Robust Counterpart method proposed by Ben-Tal [3] , our goal is to develop a tractable methodology that uses past demand history to adaptively control multi-period production policies, at the same time to control the degree of conservatism of the solutions associated with decision makers risk tolerance as its done by Bertsimas and Sim[4, 5].

## 2. Problem description and Model construction

In order to concentrate on the production policy, we construct a simplified supply chain framework with $I$ producers, one warehouse and one waste treatment centre, which centralized by an integrated manager. Assuming that all producers manufacture a homogeneous single item, and that the materials required for production can be satisfied from market any time, we consider the framework over a finite discrete horizon of $T$ periods from $t$=1 to $t$=$T$. The timeline of events is as follows:

1. At the beginning of period $t$, the integrated manager arranges production policies for each producer before observing the demand. We assume that the $i-th(i = 1, \cdots, I)$ producer can produce plants $x_i(t)$ according to the plan at per-unit cost $c_i(t)$, and each plant can be sold by a unified price $p_t$. In the limit of production capability, producer $i$ has the maximal production capacity of $P_i(t)$, and the maximal cumulative production capacity of $Q_i$ throughout the planning horizon.

2. During the process of production, the $i - th(i = 1, \cdots, I)$ producer will generate some waste at a rate of $\eta_i\,(0 < \eta_i < 1)$ . Supposing of a unified treatment of all waste to the waste treatment centre, one unit waste can be translated into some useful goods with a conversion ratio $\delta\,(0 < \delta < 1)$ at a per-unit procession cost $c_{00}$, the useful goods can be sold at a per-unit price $p_0$.

3. At the beginning of period $t$, the integrated manager faces an initial inventory level $v_t$ and the demand dt for the period $t$ is realized at the end of this period. We assume that the demand must be satisfied any time, and that the holding cost

incurred at a per-unit holding cost $c_0$, when excessive inventory is carried to the next period. Further, we define that inventory has minimal level of $v_{min}$, the maximal storage capacity of $v_{max}$. The initial inventory level of the system is $v_1$.

The inventory level at the $(t{+}1)$-th period can be given by

$$v\left(t+1\right)=v\left(t\right)+\sum_{i=1}^{I}x_i\left(t\right)-d_t. \tag{2.1}$$

Eliminating $v$-variables, the equation (2.1) can be rewritten as follows

$$v\left(t+1\right)=v\left(1\right)+\sum_{s=1}^{t}\left(\sum_{i=1}^{I}x_i\left(s\right)-d_s\right). \tag{2.2}$$

We assume an integrated manager whose objective is to determine the dynamic production policies $x_i\left(t\right)$ for each producer $i$ from period $t{=}1$ to period $t{=}T$ so as to maximize the total profit generated from production process and waste conversion process over all producers and the entire planning period. The problem can be modeled by the following liner programming

$$\begin{aligned}
&\max_{x_i(t)}\ \sum_{t=1}^{T}p_t d_t-\sum_{t=1}^{T}\sum_{i=1}^{I}c_i\left(t\right)x_i\left(t\right)-c_0\sum_{t=1}^{T}\left(v\left(1\right)+\sum_{s=1}^{t-1}\left(\sum_{i=1}^{I}x_i\left(s\right)-d_s\right)\right)\\
&+\sum_{t=1}^{T}\left(\sum_{i=1}^{I}x_i\eta_i\right)\left(\delta p_0-c_{00}\right)\\
&s.t.\quad 0\le x_i\left(t\right)\le P_i\left(t\right),0\le\sum_{t=1}^{T}x_i\left(t\right)\le Q_i, i=1,...,I;\ t=1,...,T,\\
&\qquad V_{min}\le v\left(1\right)+\sum_{s=1}^{t}\left(\sum_{i=1}^{I}x_i\left(s\right)-d_s\right)\le V_{max}t=1,...,T.
\end{aligned} \tag{2.3}$$

# 3. Robust Multi-period Model with Risk Preference

## 3.1. Uncertain demand set under risk preferences

In the heat of global competition, the market demand changes so randomly that are not necessarily identically distributed. Although the associated exact distribution information of demand is difficult to obtain, we have reasonable estimates for the mean value and its range according to experience and historical data. Then, the uncertainty of demand $d_t$ can be expressed as a bounded random variable which fluctuates in a symmetric interval with a nominal value $d_t^*$ and a deviation $\Delta_t$, that is, $d_t\in\left[d_t^*-\Delta_t,d_t^*+\Delta_t\right]\left(t=1,\cdots,T\right)$ . Then, the uncertainty set for all periods can be described as follows

$$\Omega_1=\left\{d=\left(d_1,d_2,...,d_t,...d_T,\right)\in R^T\mid d_t\in\left[d_t^*-\Delta_t,d_t^*+\Delta_t\right],t=1,...,T\right\}. \tag{3.1}$$

When facing with continuously changing market, different decision makers are always likely to have different risk preferences. As a collaborative consortium, supply chain can set a parameter to wholly control or represent the risk preference of decision maker. Hence, the uncertainty set of demand with risk preference can be described as follows

$$\Omega_2 = \left\{ d = (d_1, d_2, ..., d_t, ...d_T,) \in R^T \mid \sum_{t=1}^{T} \left( \frac{d_t - d_t^*}{\Delta_t} \right)^2 \leq \Gamma \right\}, \tag{3.2}$$

where the parameter $\Gamma$ represents the risk preferences of decision makers. When $\Gamma = 0$ , the uncertainty set is singleton, which is equivalent to the traditional case with deterministic demand, so that the decisions in this case are the most vulnerable to the market fluctuations. When $\Gamma = T$ , the uncertainty set of demand is equivalent to formula (3.1) so that the decisions in this case are the most conservative and are completely immune to the market fluctuations. When $0 < \Gamma < T$ , the conservative level of decision makers is between entirely open to entirely conservative.

## 3.2. Robust model and affinely adjustable robust counterpart

Assuming that the production policies $x_i(t)$ is made based on the demands $d_r$ observed at period $r$, which satisfies $r \in I_t$ ,where $I_t$ is a given subset of $\{1, \cdots, t\}$. Further, assuming we should specify our production policies when the demand is uncertain but can be restricted into the uncertainty set $\Omega_2$ . Considering the AARC methodology proposed by Ben-Tal [3], we restrict our production policy with affine decision rules

$$x_i(t) = y_i^0(t) + \sum_{r \in I_t} y_i^r(t) d_r, \tag{3.3}$$

where the coefficients $y_i^r(t)$ are new non-adjustable variables. With this approach, problem (2.3) after few changes becomes the following uncertain Linear Programming problem with variables $y_i^r(t)$ , F

$$\begin{aligned}
&\max_{y_i^0(t), y_i^r(t)} \sum_{t=1}^{T} \sum_{i=1}^{I} \left( y_i^0(t) + \sum_{r \in I_t} y_i^r(t) d_r \right) [\eta_i (\delta p_0 - c_{00}) - c_i(t)] \\
&- \sum_{t=1}^{T} \left( \left( v(1) + \sum_{s=1}^{t-1} \left( \sum_{i=1}^{I} \left( y_i^0(s) + \sum_{r \in I_s} y_i^r(s) d_r \right) - d_s \right) \right) c_0 - p_t d_t \right) \\
&s.t.\ 0 \leq y_i^0(t) + \sum_{r \in I_t} y_i^r(t) d_r \leq P_i^t, i = 1, ..., I, t = 1, ..., T, \\
&\quad 0 \leq \sum_{t=1}^{T} y_i^0(t) + \sum_{r \in I_t} y_i^r(t) d_r \leq Q_i, i = 1, ..., I, \\
&\quad V_{min} \leq v(1) + \sum_{s=1}^{t} \left( \sum_{i=1}^{I} \left( y_i^0(s) + \sum_{r \in I_s} y_i^r(s) d_r \right) - d_s \right) \leq V_{max}, t = 1, ..., T, \\
&\quad \forall d \in \Omega_2.
\end{aligned} \tag{3.4}$$

In order to obtain the deterministic robust counterpart of the robust model (3.4), we can equivalently make some changes on it. Obviously, the following equation holds

$$\sum_{s=1}^{t}\sum_{r\in I_s} y_i^r(s)\,d_r = \sum_{r=1}^{t}\left(\sum_{s\in S=\{s|s\leq t, r\in I_s\}} y_i^r(s)\right) d_r, t = 1,\cdots,T. \tag{3.5}$$

Based on the above equation (3.5), we have the following changes in the objective function and inequality constraints. When introducing the following notations

$$c_{i,s}^t = -c_0, 1 \leq s \leq t-1, c_{i,s}^t = \eta_i\left(\delta p_0 - c_i(t)\right), 1 \leq t \leq T;$$
$$q_s^t = c_0, 1 \leq s \leq t-1, q_t^t = p_t, 1 \leq t \leq T. S = \{s \mid s \leq t, r \in I_s\}.$$

The objective function can be reformulated as follows

$$\begin{aligned}
&\sum_{t=1}^{T}\sum_{i=1}^{I}\left(\left(y_i^0(t) + \sum_{r\in I_t} y_i^r(t)\,d_r\right)\left[\eta_i\left(\delta p_0 - c_{00}\right) - c_i(t)\right]\right.\\
&-\sum_{s=1}^{t-1}\left(y_i^0(s) + \sum_{r\in I_s} y_i^r(s)\,d_r\right)c_0\Bigg) + \sum_{t=1}^{T}\left(p_t d_t + c_0\sum_{s=1}^{t-1}\right) - Tv(1)\,c_0 \\
&= \sum_{t=1}^{T}\sum_{i=1}^{I}\sum_{s=1}^{t} c_{i,s}^t y_i^0(s) + \sum_{t=1}^{T}\sum_{r=1}^{t}\left(\sum_{i=1}^{I}\sum_{s\in S} c_{i,s}^t y_{i,s}^r(s) + q_r^t\right) d_r - Tv(1)\,c_0.
\end{aligned} \tag{3.6}$$

Accordingly, we can also do some changes in inequality constraints separately

$$0 \leq \sum_{t=1}^{T} y_i^0(t) + \sum_{r\in I_t} y_i^r(t)\,d_r \leq Q_i$$

$$\Downarrow$$

$$0 \leq \sum_{t=1}^{T} y_i^0(t) + \sum_{t=1}^{T}\left(\sum_{s\in S} y_i^r(s)\right) d_r \leq Q_i \tag{3.7}$$

$$V_{min} \leq v(1) + \sum_{s=1}^{t}\left(\sum_{i=1}^{I}\left(y_i^0(s) + \sum_{r\in I_s} y_i^r(s)\,d_r\right) - d_s\right) \leq V_{max}$$

$$\Downarrow$$

$$V_{min} \leq v(1) + \sum_{s=1}^{t}\sum_{i=1}^{I} y_i^0(s) + \sum_{r=1}^{t}\left(\sum_{i=1}^{I}\sum_{s\in S} y_i^r(s)\right) d_r - \sum_{s=1}^{t} d_s \leq v_{max}. \tag{3.8}$$

In summary, the uncertain optimization problem has an equivalently version as follow

$$\begin{aligned}
&\max_{y_i^0(t),y_i^r(t)} F\\
&s.t.\ \sum_{t=1}^{T}\sum_{i=1}^{I}\sum_{s=1}^{t} c_{i,s}^t y_i^0(s)+\sum_{t=1}^{T}\sum_{r=1}^{t}\left(\sum_{i=1}^{I}\sum_{s\in S} c_{i,s}^t y_{i,s}^r(s)+q_r^t\right)d_r - Tv(1)c_0 \le F,\\
&\quad 0\le y_i^0(t)+\sum_{r\in I_t} y_i^r(t)\,d_r \le P_i^t, i=1,...,I, t=1,...,T,\\
&\quad 0\le \sum_{t=1}^{T} y_i^0(t)+\sum_{t=1}^{T}\left(\sum_{s\in S} y_i^r(s)\right)d_r \le Q_i, i=1,...,I,\\
&\quad V_{min}\le v(1)+\sum_{s=1}^{t}\sum_{i=1}^{I} y_i^0(s)+\sum_{r=1}^{t}\left(\sum_{i=1}^{I}\sum_{s\in S} y_i^r(s)\right)d_r-\sum_{s=1}^{t} d_s \le v_{max},\\
&\quad t=1,...,T.
\end{aligned} \tag{3.9}$$

Where $i, s^t = -c_0, 1 \le s \le t-1, c_{i,s}^t = \eta_i(\delta p_0 - c_i(t)), 1\le t\le T; q_s^t = c_0, 1 \le s \le t-1, q_t^t = p_t, 1 \le t \le T$.

On the basis of the above model (3.9), then we discuss the robust counterpart.

**Lemma 3.1.** *Considering the following optimization problem*

$$Z^* = \min_y \left\{a^T y \mid y^T y \le 1\right\}. \tag{3.10}$$

*The optimal value can be given as* $Z^* = -\|a\|_2$.

**Proof.** The objective function, obvious a linear one, is convex, while the constraint function can be written as $g(y) = y^T y - 1$, which has a definite Hessian matrix $\bigtriangledown^2 g(y) = 2I \succ 0$ , is also convex. As a convex optimization, its optimal solution is equivalent to its KKT point. Assume the optimal solution vector is $y^*$, then $y^*$ must satisfy the following equation system

$$\{\bigtriangledown f(y^*) + \lambda \bigtriangledown g(y^*) = 0; \lambda \bigtriangledown g(y^*) = 0; \lambda \ge 0\}. \tag{3.11}$$

By solving system (14), we have $\lambda = \frac{1}{2}\|a\|_2, y^* = -\frac{a}{\|a\|_2}$ , then the optimal value $Z^* = -\|a\|_2$ . □

**Lemma 3.2.** *There exists a relaxed relationship between the following constraints*

$$\sum_{s=1}^{t} d_s x_s \le y, \forall d \in \Omega_2 \underset{relaxation}{\Rightarrow} \sum_{s=1}^{t} d_s^* x_s + \sqrt{\Gamma}\left(\sum_{s=1}^{t}\Delta_s^2 x_s^2\right)^{\frac{1}{2}}, t=1,\cdots,T. \tag{3.12}$$

**Proof.** Firstly, we prove the conclusion when $t{=}T$. We have

$$\sum_{t=1}^{T} d_t x_t \le y, \forall d\in\Omega_2 \Leftrightarrow \max_{d\in\Omega_2}\sum_{t=1}^{T} d_t x_t \le y. \tag{3.13}$$

According to the above reformulation, we have following sub-optimization problem

$$Z_T^* = \max_{d_t}\left\{\sum_{s=1}^{t} d_s x_s \mid \sum_{t=1}^{T}\left(\frac{d_t - d_t^*}{\Delta_t}\right)^2 \le \Gamma\right\}. \tag{3.14}$$

Let $\overline{d_t} = d_t - d_t^*$ , we arrive at the following equivalent problem

$$Z_T^* = \sum_{t=1}^{T} d_t^* x_t - \min_{\frac{\overline{d_t}}{\Delta_t\sqrt{\Gamma}}} \left\{ \sqrt{\Gamma} \sum_{t=1}^{T} \left( -\frac{\overline{d_t}}{\Delta_t\sqrt{\Gamma}} \right) \Delta_t x_t \mid \sum_{t=1}^{T} \left( -\frac{\overline{d_t}}{\Delta_t\sqrt{\Gamma}} \right)^2 \leq 1 \right\}. \quad (3.15)$$

According to the lemma 3.1, we have the optimal value of the sub-optimization problem as

$$Z_T^* \leq \sum_{t=1}^{T} d_t^* x_t + \sqrt{\Gamma} \left( \sum_{t=1}^{T} \Delta_t^2 x_t^2 \right)^{\frac{1}{2}}. \quad (3.16)$$

Then, we have the conclusion for $t{=}T$

$$\sum_{t=1}^{T} d_t x_t \leq y, \forall d \in \Omega_2 \underset{relaxation}{\Rightarrow} \sum_{t=1}^{T} d_t^* x_t + \sqrt{\Gamma} \left( \sum_{t=1}^{T} \Delta_t^2 x_t^2 \right)^{\frac{1}{2}}. \quad (3.17)$$

Next, we prove the conclusion when $t = 1, \cdots, T-1$. As the inequality for $t = 1, \cdots, T-1$ can be convert to the case of $t{=}T$ as follows

$$\sum_{s=1}^{t} d_s x_s \leq y, \forall d \in \Omega_2 \Leftrightarrow \sum_{s=1}^{t} d_s x_s + \sum_{s=t+1}^{T} d_s \times 0, \forall d \in \Omega_2, t = 1, ..., T-1. \quad (3.18)$$

Then we have the conclusion according to above proof

$$\sum_{s=1}^{t} d_s x_s \leq y, \forall d \in \Omega_2 \underset{relaxation}{\Rightarrow} \sum_{s=1}^{t} d_s^* x_s + \sqrt{\Gamma} \left( \sum_{s=1}^{t} \Delta_s^2 x_s^2 \right)^{\frac{1}{2}}, t = 1, \cdots, T. \quad (3.19)$$

□

**Theorem 3.1.** *Original uncertain optimization problem* (3.9) *can be relaxed to the following deterministic robust counterpart which is in fact a second order cone problem as follows*

$$\begin{aligned}
&\max_{y_i^0(t), y_i^r(t)} F \\
&s.t.\ \alpha_r = \sum_{t=1}^{T-r+1} \left( \sum_{i=1}^{I} \sum_{s \in S'} c_{i,s}^t y_i^r(s) + q_r^{t+r-1} \right), r = 1, \cdots, T, \\
&\quad \sum_{t=1}^{T} \sum_{i=1}^{I} \sum_{s=1}^{t} c_{i,s}^t y_i^0(s) + \sum_{r=1}^{T} d_r^* \alpha_r + \sqrt{\Gamma} \left( \sum_{r=1}^{T} \Delta_r^2 \alpha_r^2 \right)^{\frac{1}{2}} - Tv(1)\, c_0 \leq F, \\
&\quad y_i^0(t) + \sum_{r \in I_t} y_i^r(t)\, d_r^* - \sqrt{\Gamma} \left( \sum_{r \in I_t} y_i^r(t)^2 \Delta_r^2 \right)^{\frac{1}{2}} \geq 0, \\
&\quad y_i^0(t) + \sum_{r \in I_t} y_i^r(t)\, d_r^* + \sqrt{\Gamma} \left( \sum_{r \in I_t} y_i^r(t)^2 \Delta_r^2 \right)^{\frac{1}{2}} \leq P_i(t), \\
&\quad i = 1, \cdots, I; t = 1, \cdots, T, \\
&\quad \delta_i^r = \sum_{s \in S} y_i^r(s), i = 1, \cdots, I; r = 1, \cdots, T,
\end{aligned} \quad (3.20)$$

$$\sum_{t=1}^{T} y_i^0(t) + \sum_{r=1}^{T} \delta_i^r d_r^* - \sqrt{\Gamma}\left(\sum_{r=1}^{T} \delta_i^{r2} \Delta_r^2\right)^{\frac{1}{2}} \geq 0,$$

$$\sum_{t=1}^{T} y_i^0(t) + \sum_{r=1}^{T} \delta_i^r(t) d_r^* + \sqrt{\Gamma}\left(\sum_{r=1}^{T} \delta_i^{r2} \Delta_r^2\right)^{\frac{1}{2}} \leq Q_i, i = 1, \cdots, I,$$

$$\beta_t^r = \sum_{i=1}^{I} \sum_{s \in S} y_i^r(s) - 1, r = 1, \cdots, t, t = 1, \cdots, T,$$

$$v(1) + \sum_{s=1}^{t} \sum_{i=1}^{I} y_i^0(s) + \sum_{r=1}^{t} \beta_t^r d_r^* - \sqrt{\Gamma}\left(\sum_{r=1}^{t} \beta_t^{r2} \Delta_r^2\right)^{\frac{1}{2}} \geq V_{min},$$

$$v(1) + \sum_{s=1}^{t} \sum_{i=1}^{I} y_i^0(s) + \sum_{r=1}^{t} \beta_t^r d_r^* + \sqrt{\Gamma}\left(\sum_{r=1}^{t} \beta_t^{r2} \Delta_r^2\right)^{\frac{1}{2}} \leq v_{max},$$

$$t = 1, \cdots, T,$$

*where* $S' = \{s \mid s \leq t + r - 1, r \in I_r\}$.

**Proof.** According to the model (3.9), considering constraint come from the objective function, we define the following additional variables

$$\alpha_t^r = \sum_{i=1}^{I} \sum_{s \in S} c_{i,s}^t y_i^r(s) + q_r^t. \tag{3.21}$$

Let $\alpha_r = \sum_{t=1}^{T-r+1} \alpha_{t+r-1}^r$ , according to the lemma 3.2, the original constraint can be equivalently converted to the following one:

$$\sum_{t=1}^{T} \sum_{i=1}^{I} \sum_{s=1}^{t} c_{i,s}^t y_i^0(s) + \sum_{t=1}^{T} \sum_{r=1}^{t} \left(\sum_{i=1}^{I} \sum_{s \in S} c_{i,s}^t y_{i,s}^r(s) + q_r^t\right) d_r - Tv(1) c_0 \leq F$$

$$\Downarrow$$

$$\sum_{t=1}^{T} \sum_{i=1}^{I} \sum_{s=1}^{t} c_{i,s}^t y_i^0(s) + \sum_{r=1}^{T} d_r \alpha_r - Tv(1) c_0 \leq F$$

$$\Downarrow relaxation$$

$$\sum_{t=1}^{T} \sum_{i=1}^{I} \sum_{s=1}^{t} c_{i,s}^t y_i^0(s) + \sum_{r=1}^{T} d_r^* \alpha_r + \sqrt{\Gamma}\left(\sum_{r=1}^{T} \Delta_r^2 \alpha_r^2\right)^{\frac{1}{2}} - Tv(1) c_0 \leq F. \tag{3.22}$$

For the second constraint, we directly have by the lemma 3.2:

$$0 \leq y_i^0(t) + \sum_{r \in I_t} y_i^r(t) d_r \leq P_i(t)$$

$$\Downarrow relaxation$$

$$y_i^0(t) + \sum_{r \in I_t} y_i^r(t) d_r^* - \sqrt{\Gamma}\left(\sum_{r \in I_t} y_i^r(t)^2 \Delta_r^2\right)^{\frac{1}{2}} \geq 0, \tag{3.23}$$

$$y_i^0(t) + \sum_{r\in I_t} y_i^r(t)\, d_r^* + \sqrt{\Gamma}\left(\sum_{r\in I_t} y_i^r(t)^2\, \Delta_r^2\right)^{\frac{1}{2}} \le P_i(t)\,. \qquad (3.24)$$

Let $\delta_i^r = \sum_{s\in S} y_i^r(s)$ , according to the lemma 3.2, the third constraint can be equivalently converted to the following one:

$$0 \le \sum_{t=1}^{T} y_i^0(t) + \sum_{t=1}^{T}\left(\sum_{s\in S=\{s|s\le t, r\in I_s\}} y_i^r(s)\right) d_r \le Q_i$$

$$\Downarrow relaxation$$

$$\sum_{t=1}^{T} y_i^0(t) + \sum_{r=1}^{T} \delta_i^r d_r^* - \sqrt{\Gamma}\left(\sum_{r=1}^{T} \delta_i^{r2} \Delta_r^2\right)^{\frac{1}{2}} \ge 0, \qquad (3.25)$$

$$\sum_{t=1}^{T} y_i^0(t) + \sum_{r=1}^{T} \delta_i^r d_r^* + \sqrt{\Gamma}\left(\sum_{r=1}^{T} \delta_i^{r2} \Delta_r^2\right)^{\frac{1}{2}} \le Q_i. \qquad (3.26)$$

Samely, let $\beta_t^r = \sum_{i=1}^{I} \sum_{s\in S} y_i^r(s) - 1$ , according to the lemma 3.2, the forth constraint can be equivalently converted to the following one:

$$V_{min} \le v(1) + \sum_{s=1}^{t}\sum_{i=1}^{I} y_i^0(s) + \sum_{r=1}^{t}\left\{\sum_{i=1}^{I} \sum_{s\in S=\{s|s\le t, r\in I_s\}} y_i^r(s) - 1\right\} d_r \le v_{max}$$

$$\Downarrow relaxation$$

$$v(1) + \sum_{s=1}^{t}\sum_{i=1}^{I} y_i^0(s) + \sum_{r=1}^{t} \beta_t^r d_r^* - \sqrt{\Gamma}\left(\sum_{r=1}^{t} \beta_t^{r2} \Delta_r^2\right)^{\frac{1}{2}} \ge V_{min}, \qquad (3.27)$$

$$v(1) + \sum_{s=1}^{t}\sum_{i=1}^{I} y_i^0(s) + \sum_{r=1}^{t} \beta_t^r d_r^* + \sqrt{\Gamma}\left(\sum_{r=1}^{t} \beta_t^{r2} \Delta_r^2\right)^{\frac{1}{2}} \le v_{max}. \qquad (3.28)$$

In result, the robust counterpart of uncertain optimization problem (3.9) with affine decision rules, can be relaxed to a second order cone problem, which can be effectively solved by some commercial software. □

# 4. Conclusions

In this paper, we address the problem of optimizing overall profit of multi-period production and inventory system with waste disposal subjected to uncertain demands. Our contributions over the related works can be summarized as follows: (a) Assuming only the mean, support, and some measures of deviations are available for each demand, we construct a joint ellipsoid uncertainty set to control the degree of conservatism of the solutions associated with the integrated manager risk preference. (b) Using Affinely Adjustable Robust Counterpart method, we develop an uncertain optimization model to adaptively control multi-period production policies, and relax it to one deterministic robust counterpart which is in fact a tractable second order cone problem. In advance, the computational studies and proofs are required in the future work.

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