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A CUBIC DIFFERENTIAL SYSTEM WITH NINE LIMIT CYCLES

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Abstract Advances in Computer Algebra software have made calculations possible that were previously intractable. Our particular interest is in the investigation of limit cycles of nonlinear differential equations. We describe some recent developments in handling very large computations involving resultants and present an example of a nonlinear differential system of degree three with nine small amplitude limit cycles surrounding a focus. We know of no examples of cubic systems with more than this number bifurcating from a fine focus, as opposed to a centre. Our example appears to be the first to have been obtained without recourse to some numerical calculation.

Keywords Nonlinear differential equations, limit cycle, bifurcation, polynomial system.

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1. Introduction

There continues to be much interest in polynomial differential equations in the plane and, in particular, their closed orbits: this is the general area of Hilbert's 16^{th} Problem. Systems

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1.1}$$

are considered, where P, Q are polynomials. There are two particular questions: given a class of systems, how many limit cycles can exist? and how can they be configured? The latter can be thought of as asking how many 'nests' of limit cycles can there be and how are these arranged, and the former how many limit cycles can there be in a nest.

In order to ensure that the presentation is reasonably self-contained, we summarise the mathematical background.

A *limit cycle* is an isolated closed orbit and a *critical point* is a point where both P and Q are zero. It is known that a given system of the form (1.1) has finitely many limit cycles, but it is not known whether there is a uniform bound for systems of a given degree. Even for systems in which P, Q are quadratics the maximum possible number of limit cycles remains unknown.

One way to approach a problem is to start with a known structure and to introduce perturbations. Since a closed orbit must encircle a critical point, much effort has been devoted to estimating how many limit cycles can bifurcate from a critical point of focus type, a centre or from the orbits forming the period annulus.

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A *centre* is a critical point all orbits in a neighbourhood of which are closed. The *period annulus* is the set of closed orbits around it. A *focus* is a critical point in a neighbourhood of which the angular variable θ tends to infinity on orbits. We restrict our attention to non-degenerate critical points, that is critical points for which the eigenvalues of the linearisation of the system are non-zero.

Suppose that the origin is a non-degenerate critical point of focus type. In a neighbourhood of the origin the system can be written in the canonical form

$$\dot{x} = \lambda x + y + p(x,y), \quad \dot{y} = -x + \lambda y + q(x,y), \tag{1.2}$$

where p, q are polynomials without linear terms or constants.

The critical point at the origin is a *fine focus* if $\lambda = 0$. Limit cycles that bifurcate from a fine focus are said to be of *small amplitude.* The number of small amplitude limit cycles that can bifurcate from a fine focus is bounded by the *order* of the fine focus. The order can be defined as follows. It is known (see Nemytskii & Stepanov[9]) that there is an analytic function $V(x,y)$, defined in a neighbourhood of a fine focus, such that $\dot{V}$, the rate of change of V along orbits, is of the form

$$\dot{V} = \sum_{k=1}^{\infty} \eta_{2k} r^{2k},$$

where $r^2 = x^2 + y^2$ and the η_{2k} are the *focal values.* The order of the fine focus is k if $\eta_{2l} = 0$ for $l \leq k$, but $\eta_{2k+2} \neq 0$. At most k limit cycles can bifurcate from a fine focus of order k, but the maximum is often not attained.

Quadratic systems have been studied extensively. It was proved early in the development of the subject that no more than three limit cycles can bifurcate from a focus or centre for quadratic systems, see Bautin [2]. In a recent paper, Zhao [14] proves that no more than five limit cycles can emerge from the orbits forming a period annulus and conjectures that the maximum number is three.

Much less is known about cubic systems. In Zoladek [15] it is shown that no more than eleven limit cycles can bifurcate from a centre in a cubic system. This is a difficult paper to understand. Using a different approach, Yu & Han [13] start with the integrable system of Zoladek's example and introduce perturbations under which nine limit cycles bifurcate from a centre. Using their method they are unable to obtain more than nine. Christopher [3] gives an example of another cubic system in which eleven limit cycles bifurcate from a centre. His approach is simpler, though it does not necessarily lead to the maximum number. The idea is to start with values of the coefficients in p and q for which the origin is a centre of (1.2). All coefficients are perturbed and the linear parts of successive focal values set to zero until they cease to be independent. These approaches obviously require information about the conditions for a centre; a complete set of centre conditions for the general cubic system is not known.

In James & Lloyd [5] an example of a system with eight limit cycles bifurcating from a focus was given. The approach adopted there, as elsewhere, was to calculate focal values, obtain a fine focus of high order and to introduce a sequence of perturbations each of which reverses the stability of the critical point. In this exchange of stability a limit cycle is produced (Hopf bifurcation). This approach is explained in more detail later in this section. Conceptually it is straightforward, however determining the order of a fine focus in this way involves some very complicated calculations; it is possible only by using computer algebra and is computationally demanding. Our implementation of this approach involves the computation of determinants of matrices whose elements are large multivariate polynomials. Some specialised and bespoke software is required for the more complicated calculations.

Yu & Corless [12] gave an example in which nine limit cycles can be bifurcated from a focus, namely

$$\begin{aligned}\dot{x} &= \lambda x + y + x^2 + cy^2 + fx^3 + gx^2y - 3pxy^2 + ky^3,\\ \dot{y} &= -x + \lambda y - 2xy + lx^3 + (m-2f)x^2y + (n-g)xy^2 + py^3.\end{aligned}$$

In contrast to our approach they use numerical techniques to complete the calculations.

We revisited the question of limit cycles bifurcating from a fine focus in the light of the significant advances in Computer Algebra, some of which were described in Pearson & Lloyd [10]. To go beyond the bifurcation of eight limit cycles using this approach requires a surprisingly large increase in computational sophistication. We perform all our computerised calculations on a Dual 3.2 G Hz Xeon processor, with 2 GB of memory, running a Linux Gentoo operating system.

We consider cubic systems and show that

$$\begin{aligned}\dot{x} &= \lambda x + y + x^2 - 2b_1xy + (a_3-1)y^2\\ &\quad + a_4x^3 + a_5x^2y - (2a_4+b_7)xy^2 - b_4y^3,\\ \dot{y} &= -x + \lambda y + b_1x^2 + 2xy - b_1y^2\\ &\quad + b_4x^3 + (b_5 - a_4 - 2b_7)x^2y + (b_6 - a_5)xy^2 + b_7y^3,\end{aligned}$$

with certain relationships between the coefficients a_i, b_i, can have nine limit cycles bifurcating from a fine focus at the origin. Our computer calculations are algebraic and as such involve exact arithmetic.

The number of focal values required to give a fine focus of maximum order is not known in advance. The approach we use is to calculate focal values for a given system until we find the first one in the sequence that is necessarily non-zero when all preceding focal values are zero. This gives us the order of the fine focus and its stability is determined by the sign of the first non-zero focal value. We use the computer algebra procedure FINDETA (Lloyd & Pearson [8]) to calculate the first few focal values. Each of these is then expressed modulo the ideal generated by the previous ones; that is the relations $\eta_2 = \eta_4 = \ldots = \eta_{2k} = 0$ are used to eliminate some of the variables in η_{2k+2}.

The substitutions are in general rational. Since our requirement is knowledge of the stability of the critical point at the origin we only need the signs of the reduced focal values. The convention is that strictly positive factors are removed and, where a substitution results in a rational expression for the reduced focal value, any non-square factors in the denominator are moved to the numerator. Thus the signs of the expressions are maintained. The polynomial obtained in this way from η_{2k+2} is the *Liapunov quantity* $L(k)$. We note that for system (1.2), $L(0) = \lambda$. We determine Liapunov quantities from the focal values we have calculated.

The reduction of the focal values to find the Liapunov quantities is equivalent to finding a basis for the focal values. We do not use Gröbner basis software to find the basis for two main reasons. Firstly, we do not know *a priori* the number of polynomials that make up the basis. Secondly, it is feasible to perform the calculations only in the simplest examples.

In order to bifurcate limit cycles, we start with a fine focus of order k, that is $L(0) = \ldots = L(k-1) = 0$, $L(k) \neq 0$. We perturb one of the parameters in $L(k-1)$, such that the Liapunov quantity becomes non-zero and of opposite sign to $L(k)$. The stability of the fine focus is reversed and a limit cycle bifurcates. Successive perturbation of other parameters, at each stage reducing the order and reversing the stability of the fine focus, produces more limit cycles. Provided the perturbations are small enough existing limit cycles are not destroyed.

As we have already mentioned, the computations we must perform to manipulate the focal values in order to obtain the Liapunov quantities are extremely demanding of computer resources; storage requirements are large and long processing times are common. The calculation of resultants of large, multivariate polynomials is the most onerous task we face (the resultant can be thought of as the elimination of a variable from the polynomials). The polynomials we encounter are of high degree and the terms have large integer coefficients - we shall give an idea of their size in section 2. We employ various techniques to obtain the resultants we require and, for the most difficult examples, use the Computer Algebra system *Fermat*, developed by Robert Lewis [6], in their calculation. Here we use two procedures that we have implemented in Fermat to calculate the resultant; both are based on the Bézout matrix formulation of the resultant, often known as the Cayley or Dixon method, which we now summarise.

Let p_1, p_2 be irreducible, multivariate polynomials of degrees n_1, n_2 respectively in the variable v. Let $R(p_1, p_2, v)$ represent the resultant of the polynomials p_1, p_2 with respect to the variable v. Assume $n_2 \geq n_1$, and write p_1 and p_2 as polynomials in the single variable v, with polynomial coefficients, b_i, c_i, in the remaining variables. We have

$$p_1(v) = \sum_{i=0}^{n_1} b_i v^i \quad \text{and} \quad p_2(v) = \sum_{i=0}^{n_2} c_i v^i.$$

Let

$$\beta(x, y) = \frac{p_1(x)p_2(y) - p_1(y)p_2(x)}{x - y}.$$

Clearly $x - y$ is a factor of the numerator, so β is a polynomial of degree $n_2 - 1$ in x and y. The elements of the Bézout matrix, B, are given by

$$B(i, j) = \text{ coefficient of } \; x^{i-1} y^{j-1} \; \text{ in } \; \beta,$$

for $i, j = 1, 2, \ldots, n_2$. The resultant is

$$R(p_1, p_2, v) = \pm c_{n_2}^{n_1 - n_2} \det(B),$$

where $c_{n_2} \neq 0$ and $\det(B)$ represents the determinant of B. The elements of the Bézout matrix are functions of the coefficients of v in p_1, p_2 and the order of the matrix is equal to the highest degree of v in p_1 or p_2, namely n_2. The matrix elements are often extremely large multivariate polynomials whose terms have large integer coefficients. We shall give an idea of just how large these polynomials and the integer coefficients become later in this paper. The resultant is obtained by calculating $det(B)$ and dividing this by the factor $c_{n_2}^{n_2 - n_1}$. We note that, unless $n_1 = n_2$, the resultant is a polynomial of lower degree and generally fewer terms than the determinant.

The calculation of the determinant is computationally very demanding. Fermat has an efficient built-in function to calculate the determinant of a matrix which uses heuristics to determine the best method, depending on the type of matrix elements. We use this determinant function in one of our routines to calculate resultants. Sometimes the size of the matrices or the complexity of their elements means that this is not feasible and we must calculate the determinant by other means.

Resultants often contain many repeated factors and we know that the factor $c_{n_2}^{n_2-n_1}$ must be present in the determinant. It is expeditious to remove any factors as they arise in the calculation of the determinant. We can remove common factors from the rows and columns of the matrix B before the calculation of the determinant begins. In Pearson & Lloyd [11] we described a procedure in which the determinant is evaluated using expansion by minors, with factors removed as the process proceeds. We used this software in Pearson & Lloyd [10]. A slightly different approach, the "Early Discovery of Factors" (EDF) method is described by Lewis [7]. In summary, row reductions are performed, with common factors of the rows of the reduced matrices removed during the process.

We are extremely grateful to Robert Lewis for making his EDF software available to us. This method is particularly effective when the matrix entries are univariate polynomials with very large integer coefficients.

Since very large integers arise in the computations, it would be useful if modular arithmetic could be used in some of the calculations. We make the following general observation:-

$$R(p_1, p_2, v) \bmod p = R(p_1 \bmod p, p_2 \bmod p, v) \bmod p,$$

where p is a prime number and only integer coefficients are computed modulo p. Within Fermat, computations modulo p are equivalent to considering integers in the range $[-(\frac{p-1}{2}), (\frac{p-1}{2})]$. In some cases, where we only need to show that a certain polynomial, G say, is not a factor of a resultant, we are able to calculate the resultant modulo some prime. We calculate the modular resultant and determine its factors. We then aim to show that G is not a factor of the resultant by comparing the modular factors of G with those of the resultant.

2. The example with nine limit cycles

Consider the general cubic differential system

$$\begin{aligned}
\dot{x} &= \lambda x + y + a_1x^2 + (a_2 + 2b_1)xy + (a_3 - a_1)y^2 \\
&\quad + a_4x^3 + a_5x^2y + (a_6 - 3b_7)xy^2 + a_7y^3, \\
\dot{y} &= -x + \lambda y + b_1x^2 + (b_2 - 2a_1)xy + b_3y^2 \\
&\quad + b_4x^3 + (b_5 - a_6 - 3a_4)x^2y + (b_6 - a_5)xy^2 + b_7y^3,
\end{aligned} \tag{2.1}$$

where the coefficients a_i, b_i are real. The specific form of the coefficients is chosen to simplify the computations. James & Lloyd [5] show that when $a_2 = -4b_1$, $b_2 = 4a_1$, $b_3 = -b_1$, $a_4 = a_6 = b_7 = 0$, there are systems of the form (2.1) with eight small amplitude limit cycles surrounding the origin. Here we seek a system for which there are nine small amplitude limit cycles.

We first scale x, y by a_1; let $x \to \frac{x}{a_1}$, $y \to \frac{y}{a_1}$, to give a system of the form (2.1) with $a_1 = 1$. In the following the scaled variables, where $a_i \to \frac{a_i}{a_1}$, $b_i \to \frac{b_i}{a_1}$, $i = 1, 2, 3$ and $a_i \to \frac{a_i}{a_1^2}$, $b_i \to \frac{b_i}{a_1^2}$, $i = 4, 5, 6, 7$, are used.

In order to obtain nine limit cycles by successive perturbation of the coefficients the system must have at least nine coefficients (λ and eight of the a_i, b_i). We are free to choose relationships between up to six of the coefficients a_i, b_i to simplify the calculations. We note that not every selection will lead to nine limit cycles. We make the choices following a process of trial and error.

Let $\lambda = 0$, $a_1 = 1$. We choose the relationships between the quadratic coefficients that were used by James & Lloyd [5], that is $b_3 = -b_1$, $a_2 = -4b_1$, $b_2 = 4$, and calculate the focal values η_4 and η_6 for (2.1). From η_4, we find that $L(1) = 4a_3b_1 + b_5$. Let $b_5 = -4a_3b_1$, then from η_6, we find

$$L(2) = a_3\Phi - b_6(2a_4 - 2b_7 + a_6) - 3a_6(a_7 + b_4),$$

where Φ is a polynomial in $a_3, a_4, a_5, a_6, a_7, b_1, b_4, b_6, b_7$. We can choose two more relationships between the coefficients and we do this such that

$$b_6(2a_4 - 2b_7 + a_6) + 3a_6(a_7 + b_4) = 0,$$

and hence a_3 is a factor of $L(2)$. We consider various options and find that η_8 has the fewest terms when $a_7 = -b_4$ and $a_6 = 2(b_7 - a_4)$.

So, with all six choices made, we consider the differential system with nine parameters

$$\begin{aligned}\dot{x} &= \lambda x + y + x^2 - 2b_1xy + (a_3 - 1)y^2 \\ &\quad + a_4x^3 + a_5x^2y - (2a_4 + b_7)xy^2 - b_4y^3, \\ \dot{y} &= -x + \lambda y + b_1x^2 + 2xy - b_1y^2 \\ &\quad + b_4x^3 + (b_5 - a_4 - 2b_7)x^2y + (b_6 - a_5)xy^2 + b_7y^3.\end{aligned} \tag{2.2}$$

We proceed to calculate focal values for (2.2) and from them determine the corresponding Liapunov quantities. We use the convention described in the Introduction that strictly positive factors are removed and that expressions in the denominator, which are not strictly positive, are moved to the numerator. We have

$$\begin{aligned}L(0) &= \lambda, \\ L(1) &= 4a_3b_1 + b_5.\end{aligned}$$

Let $b_5 = -4a_3b_1$, then

$$L(2) = a_3(10a_3^2b_1 + 10b_1b_4 - 2a_5b_1 - 2b_1b_6 - 4a_3b_1 - (a_4 - b_7)(2 + 5a_3)).$$

Let $a_3 = 0$. Then

$$L(3) = 2a_4^3 - 2a_4^2b_7 + a_4b_4b_6 - a_4b_6^2 - 2a_4b_7^2 - b_4b_6b_7 + 2b_7^3 + a_5b_6(a_4 - b_7).$$

Assume that $b_6(a_4 - b_7) \neq 0$ and let

$$a_5 = \frac{-(2a_4^3 - 2a_4^2b_7 + a_4b_4b_6 - a_4b_6^2 - 2a_4b_7^2 - b_4b_6b_7 + 2b_7^3)}{b_6(a_4 - b_7)}.$$

Then $L(3) = 0$ and

$$L(4) = (a_4 + b_7)\Omega((a_4 + b_7)\Psi_0 + 40b_4b_6(a_4 - b_7)\Psi_1),$$

where $\Omega = a_4 - b_7 - b_1 b_6 - a_4 b_1^2 + b_1^2 b_7$ and

$$\begin{aligned}\Psi_0 =& 80a_4^2b_6 - 160a_4b_6b_7 - 20b_6^3 + 80b_6b_7^2 + 320a_4^3b_1 - 960a_4^2b_1b_7\\ &- 80a_4b_1b_6^2 + 960a_4b_1b_7^2 + 80b_1b_6^2b_7 - 320b_1b_7^3 + 272a_4^4 - 1088a_4^3b_7\\ &- 80a_4^2b_1^2b_6 + 16a_4^2b_6^2 + 1632a_4^2b_7^2 + 160a_4b_1^2b_6b_7 - 32a_4b_6^2b_7\\ &- 1088a_4b_7^3 + 20b_1^2b_6^3 - 80b_1^2b_6b_7^2 - 13b_6^4 + 16b_6^2b_7^2 + 272b_7^4,\\ \Psi_1 =& 4b_6 + 16a_4b_1 - 16b_1b_7 + 12a_4^2 - 24a_4b_7 - 4b_1^2b_6 + 3b_6^2 + 12b_7^2.\end{aligned}$$

Assume that $b_6(a_4 - b_7)(a_4 + b_7)\Omega\Psi_1 \neq 0$. Let

$$b_4 = -\frac{1}{40}\frac{(a_4 + b_7)\Psi_0}{b_6(a_4 - b_7)\Psi_1},$$

then $L(4) = 0$. To simplify the calculations we let $a_4 = m + b_7$. We calculate

$$L(5) = b_6 m\Omega X^2(m\Upsilon_0 + 2b_7\Upsilon_1),$$

where $X = 4m^2 + b_6^2 \neq 0$, under current assumptions, and

$$\begin{aligned}\Upsilon_0 =& 12096b_6^3 + 123648b_1b_6^2m - 25536b_1^2b_6^3 + 408576b_1^2b_6m^2 + 22832b_6^4\\ &+ 91328b_6^2m^2 - 161280b_1^3b_6^2m + 430080b_1^3m^3 + 155008b_1b_6^3m\\ &+ 620032b_1b_6m^3 + 14784b_1^4b_6^3 - 236544b_1^4b_6m^2 - 31840b_1^2b_6^4\\ &+ 127360b_1^2b_6^2m^2 + 1018880b_1^2m^4 + 14316b_6^5 + 114528b_6^3m^2\\ &+ 229056b_6m^4 + 37632b_1^5b_6^2m - 99712b_1^3b_6^3m - 398848b_1^3b_6m^3\\ &+ 48624b_1b_6^4m + 388992b_1b_6^2m^3 + 777984b_1m^5 - 1344b_1^6b_6^3\\ &+ 9008b_1^4b_6^4 + 36032b_1^4b_6^2m^2 - 9996b_1^2b_6^5 - 79968b_1^2b_6^3m^2\\ &- 159936b_1^2b_6m^4 + 3213b_6^6 + 38556b_6^4m^2 + 154224b_6^2m^4 + 205632m^6,\\ \Upsilon_1 =& 6720b_6^3 + 80640b_1b_6^2m - 20160b_1^2b_6^3 + 322560b_1^2b_6m^2 + 15920b_6^4\\ &+ 63680b_6^2m^2 - 161280b_1^3b_6^2m + 430080b_1^3m^3 + 127360b_1b_6^3m\\ &+ 509440b_1b_6m^3 + 20160b_1^4b_6^3 - 322560b_1^4b_6m^2 - 31840b_1^2b_6^4\\ &+ 127360b_1^2b_6^2m^2 + 1018880b_1^2m^4 + 12156b_6^5 + 97248b_6^3m^2\\ &+ 194496b_6m^4 + 80640b_1^5b_6^2m - 127360b_1^3b_6^3m - 509440b_1^3b_6m^3\\ &+ 48624b_1b_6^4m + 388992b_1b_6^2m^3 + 777984b_1m^5 - 6720b_1^6b_6^3\\ &+ 15920b_1^4b_6^4 + 63680b_1^4b_6^2m^2 - 12156b_1^2b_6^5 - 97248b_1^2b_6^3m^2\\ &- 194496b_1^2b_6m^4 + 3213b_6^6 + 38556b_6^4m^2 + 154224b_6^2m^4 + 205632m^6.\end{aligned}$$

Assume that $\Upsilon_1 \neq 0$ and let $b_7 = -m\Upsilon_0/2\Upsilon_1$. Then

$$\begin{aligned}L(6) &= \Omega\Psi_1\Phi_1,\\ L(7) &= \Omega\Psi_1\Upsilon_1(b_1^2+1)\Phi_2,\\ L(8) &= \Omega\Psi_1\Phi_3,\\ L(9) &= \Omega\Psi_1\Upsilon_1(b_1^2+1)\Phi_4,\end{aligned}$$

where the Φ_i, $i = 1, \cdots, 4$ are irreducible polynomials in m, b_1, b_6. Our requirement that all coefficients are real means that $b_1^2 + 1 > 0$ and, under current assumptions, $\Omega\Psi_1\Upsilon_1 \neq 0$. It remains to consider the Φ_i, $i = 1, 2, 3, 4$.

The degrees of the variables and the number of terms in the Φ_i are shown in the table below.

	Φ_1	Φ_2	Φ_3	Φ_4	Φ_3 mod Φ_1	Φ_4 mod Φ_1
m	16	22	30	36	22	26
b_1	16	22	32	38	44	56
b_6	16	22	30	36	15	15
Terms	425	980	2598	4224	3104	5144

All the remaining variables occur to high degree in the Φ_i and the polynomials Φ_3, Φ_4 in particular have a large number of terms. We eliminate b_6 using our resultant procedure that incorporates the built-in determinant function of Fermat.

Consider first $\Phi_1 = \Phi_2 = 0$; this is the case if the resultant of Φ_1, Φ_2 with respect to b_6 is zero. The leading coefficients of b_6 in Φ_1, Φ_2 are integers and that of Φ_2, which will occur to the power six in the determinant, has sixteen decimal digits. Let # represent a very large integer. We factorise the determinant and conclude that

$$R(\Phi_1, \Phi_2, b_6) = \# m^{144}(b_1^2 + 1)^{188} R_1,$$

where R_1 is an irreducible polynomial of degrees 20 in m, 40 in b_1 with 221 terms.

In order to calculate the resultant of Φ_1 and Φ_3, with respect to b_6, we reduce the degree of b_6, in Φ_3, by substituting for b_6^{16}, from $\Phi_1 = 0$. The resulting polynomial, Φ_3 mod Φ_1, has more terms, the degree of b_1 is increased and the integer part of the leading coefficient has 38 decimal digits. However the order of the Bézout matrix, from which the resultant is determined, is 16 (the degree of b_6 in Φ_1) instead of 30. This reduction in order of the matrix more than compensates for the increased complexity of its polynomial entries. We calculate

$$R(\Phi_1, \Phi_3 \bmod \Phi_1, b_6) = \# m^{186}(b_1^2 + 1)^{248} R_2,$$

where R_2 is an irreducible polynomial of degrees 42 in m, 92 in b_1, with 1097 terms.

Similarly, we calculate the resultant of Φ_1 and Φ_4 mod Φ_1 with respect to b_6. In this instance m^{72} and integer factors together coming to approximately 10^{131} are removed from the matrix before the determinant is calculated. The determinant, which we find has factors m^{156}, $(b_1^2+1)^{280}$ and an integer factor with 1342 decimal digits, took 79 hours 24 minutes to calculate. We conclude that

$$R(\Phi_1, \Phi_4 \bmod \Phi_1, b_6) = \# m^{228}(b_1^2 + 1)^{280} R_3,$$

where R_3 is an irreducible polynomial of degrees 64 in m, 136 in b_1, with 2373 terms.

In all these resultant calculations the polynomial of highest degree has a leading coefficient of b_6 that is an integer. Hence the non-integer factors of the determinant are true factors of the resultant.

Under current assumptions, $m \neq 0$ and $b_1^2 + 1 > 0$. It remains to consider the possibility that $R_1 = R_2 = R_3 = 0$.

The magnitude of the integer coefficients and the high degrees of the variables, in the polynomials R_1, R_2, R_3 makes the calculation of resultants to eliminate m extremely demanding of computer resources. All the elements of the Bézout matrices are large polynomials in b_1 alone.

For the resultant of R_1 and R_2 the Bézout matrix is of order 42. The leading coefficient of m in R_1 is an integer with 147 decimal digits. Let the leading coefficient of m in R_2 be α. Then $\alpha = \alpha_I \alpha_P$, where α_I is an integer with at least 170 decimal digits and α_P is a polynomial of degree eight in b_1, having a leading coefficient with 33 decimal digits. We note that α^{22}, that is not part of the resultant, will arise in the calculation of the determinant of the Bézout matrix for R_1, R_2. Although α^2 and some large integers, that are common factors of the elements in a row or column, can be removed from this matrix, we were only able to calculate its determinant by using the Early Discovery of Factors technique.

Recall that in the EDF method row reductions are performed and factors common to a row are removed at each stage. After two row reductions an integer of the order of 10^{45} and α_P have been removed from the matrix. When eleven row reductions have been completed a total of 135 factors have been removed from the rows of the matrix. Some of these are as small as the integer 4, but in total they come to at least $10^{469}\alpha_P^{10}$. At the completion of row reduction 21 the spurious factor, α^{22}, has been removed completely. Row reductions 22 to 25 yield a polynomial factor $(b_1^2+1)^{40}$, as well as a large integer. In further row reductions only integer factors are removed. After several days of calculation we found that

$$R(R_1, R_2, m) = \#(b_1^2+1)^{40} Z,$$

where Z is a polynomial of degree 1760 in b_1 with extremely large integer coefficients. We note that b_1 occurs to even powers in Z.

Finally we use modular calculations to show that no factor of Z is also a factor of $R(R_1, R_3, m)$. The modulus we use is chosen to ensure that leading terms in the polynomials do not disappear. Often quite small prime numbers satisfy this requirement but then the modular resultant has very many factors which we must subsequently consider. We choose a larger prime number to reduce the number of factors whilst still making the calculations feasible in a reasonable time. We establish that the maximum degrees of the variables in R_1 and R_3 are not changed when the integer coefficients are reduced modulo 44449. So the resultant calculated modulo 44449 will have the same degree as the actual resultant. We calculate

$$R(R_1, R_3, m) \bmod 44449 = \#(b_1^2+1)^{40} W,$$

where W is a polynomial of degree 2640 in b_1. We find that W has 11 modular factors,

$$W = w_{24}\hat{w}_{24}w_{36}w_{56}w_{84}w_{132}w_{308}\hat{w}_{308}w_{476}w_{576}w_{616},$$

where $w_i, \hat{w}_i$ are polynomials of degree i in b_1. Similarly we find the factors of Z modulo 44449:

$$Z \bmod 44449 = z_{28}z_{36}\hat{z}_{36}z_{48}\hat{z}_{48}z_{56}z_{140}z_{360}z_{266}\hat{z}_{266}z_{476},$$

where $z_i, \hat{z}_i$ are polynomials of degree i in b_1. We conclude that $R(R_1, R_2, m)$ and $R(R_1, R_3, m)$ cannot be zero simultaneously. Hence, if $\Phi_1 = \Phi_2 = \Phi_3 = 0$, then $\Phi_4 \neq 0$.

Considering Z as a polynomial in $\omega = b_1^2$ we find that $Z = 0$ has a real positive zero ω^* in the interval $(0.6, 0.7)$. When $b_1^2 = \omega^*$ there is a value of m such that $R_1 = R_2 = 0$, but $R_3 \neq 0$, and hence a value of b_6, such that $\Phi_1 = \Phi_2 = \Phi_3 = 0$, with $\Phi_4 \neq 0$. Under current assumptions there are values of the coefficients such that $L(0) = \cdots = L(8) = 0$, $L(9) \neq 0$, and hence the origin is a fine focus of order 9.

Theorem 2.1. *The origin is a fine focus of order at most nine for system* (2.2) *when* $b_6(a_4 - b_7)(a_4 + b_7)\Psi_1\Upsilon_1\Omega \neq 0$ *and*

$$\lambda = 0, \;\; b_5 = 0, \quad a_3 = 0, \;\; a_5 = \frac{-(2m^2(a_4+b_7) + b_4 b_6 m - a_4 b_6^2)}{b_6 m},$$

$$b_4 = -\frac{1}{40}\frac{(a_4+b_7)\Psi_0}{b_6 m \Psi_1}, \;\; b_7 = -m\Upsilon_0/2\Upsilon_1, \quad \Phi_1 = \Phi_2 = \Phi_3 = 0, \;\; \Phi_4 \neq 0,$$

where $m = a_4 - b_7$ *and* $\Psi_0, \Psi_1, \Upsilon_0, \Upsilon_1, \Phi_1, \Phi_2, \Phi_3, \Phi_4$ *are as defined above.*

Proof. When the conditions of Theorem 2.1 hold the Liapunov quantities $L(0) = \cdots = L(8) = 0$ and $L(9) \neq 0$. The origin is a fine focus of order nine. When $\Phi_1 = \Phi_2 = \Phi_3 = 0$ then Φ_4 cannot be zero; the origin cannot be of order greater than nine. □

Corollary 2.1. *Up to nine limit cycles can be bifurcated from the origin in system* (2.2) *with the conditions given in Theorem 2.1.*

Proof. The origin is a fine focus of order nine when the conditions of Theorem 2.1 hold. Then $L(0) = \cdots = L(8) = 0$ and $L(9) \neq 0$, where

$$\begin{aligned}
&L(0) = \lambda, \;\; L(1) = 4a_3b_1 + b_5,\\
&L(2) = -a_3(5a_3m + 2(2a_3b_1 + a_4 + b_7) + 2b_1(a_5 + b_6) - 10b_1(b_4 + a_3^2)),\\
&L(3) = 2a_4^3 - 2a_4^2b_7 + a_4b_4b_6 - a_4b_6^2 - 2a_4b_7^2 - b_4b_6b_7 + 2b_7^3 + a_5b_6m,\\
&L(4) = (a_4+b_7)\Omega((a_4+b_7)\Psi_0 + 40b_4b_6(a_4-b_7)\Psi_1),\\
&L(5) = b_6m\Omega(m\Upsilon_0 + 2b_7\Upsilon_1), \;\; L(6) = \Omega\Psi_1\Phi_1,\\
&L(7) = \Omega\Psi_1\Upsilon_1\Phi_2, \;\; L(8) = \Omega\Psi_1\Phi_3, \;\; L(9) = \Omega\Psi_1\Upsilon_1\Phi_4,
\end{aligned}$$

and $a_4 = m + b_7$; Ω, $\Psi_0, \Psi_1, \Upsilon_0, \Upsilon_1, \Phi_1, \Phi_2, \Phi_3, \Phi_4$ are as defined above.

Starting with a fine focus of order nine at the origin we bifurcate successive limit cycles by a sequence of perturbations of the parameters. At each perturbation the order of the origin as a fine focus is reduced and its stability is reversed; a limit cycle bifurcates. Provided the perturbations are small enough existing limit cycles are not destroyed and the "far field" is unaffected.

We begin with $b_1^2 = \omega^*$, where ω^* is the root of $Z = 0$ which lies in $(0.6, 0.7)$, and the corresponding value of $m = m^*$, such that $R_1 = R_2 = 0$. Then there is a value $b_6 = b_6^*$, such that $\Phi_1 = \Phi_2 = \Phi_3 = 0$. The stability of the origin is given by the sign of $\Omega\Psi_1\Upsilon_1\Phi_4$, when $b_1^2 = \omega^*$, $m = m^*$ and $b_6 = b_6^*$.

We perturb b_1 such that $Z \neq 0$, and hence $\Phi_3 \neq 0$, at the same time adjusting m, b_6 such that Φ_1, Φ_2 remain zero. We require $L(8)L(9) < 0$, that is

$$\Upsilon_1\Phi_3\Phi_4 < 0.$$

So we perturb b_1 such that the sign of $\Upsilon_1\Phi_3$ is opposite to that of Φ_4. The stability of the origin is reversed and a limit cycle bifurcates.

To generate a second limit cycle we perturb m such that $L(7)$ becomes non-zero and of opposite sign to $L(8)$, at the same time adjusting b_6 so that Φ_1 remains zero. We require

$$\Upsilon_1\Phi_2\Phi_3 < 0,$$

or equivalently $\Phi_2\Phi_4 > 0$. The stability of the origin is reversed and a second limit cycle bifurcates. Provided the perturbations are small enough the first limit cycle is not destroyed.

The third limit cycle is generated by perturbing b_6 such that $L(6)$ becomes non-zero and of opposite sign to $L(7)$; that is

$$\Upsilon_1\Phi_1\Phi_2 < 0,$$

or equivalently $\Phi_1\Phi_3 > 0$. A fourth limit cycle bifurcates when b_7 is perturbed so that

$$b_6 m(m\Upsilon_0 + 2b_7\Upsilon_1)\Psi_1\Phi_1 < 0.$$

Similarly perturbation of b_4 such that

$$b_6 m(m\Upsilon_0 + 2b_7\Upsilon_1)(a_4 + b_7)((a_4 + b_7)\Psi_0 + 40b_4b_6(a_4 - b_7)\Psi_1) < 0,$$

reverses the stability of the origin and a fifth limit cycle bifurcates. The sixth limit cycle comes from perturbation of a_5 such that

$$((a_4 + b_7)\Psi_0 + 40b_4b_6(a_4 - b_7)\Psi_1)$$
$$\times(2m^2(a_4 + b_7) - a_4b_6^2 + b_4b_6m + a_5b_6m) < 0.$$

Limit cycle seven is bifurcated by perturbing a_3 such that

$$a_3(5a_3m + 2(2a_3b_1 + a_4 + b_7) + 2b_1(a_5 + b_6) - 10b_1(b_4 + a_3^2))$$
$$\times(2m^2(a_4 + b_7) - a_4b_6^2 + b_4b_6m + a_5b_6m) > 0.$$

and number eight by perturbing b_5 such that

$$a_3(5a_3m + 2(2a_3b_1 + a_4 + b_7) + 2b_1(a_5 + b_6) - 10b_1(b_4 + a_3^2))(4a_3b_1 + b_5) > 0.$$

At each perturbation the stability of the origin is reversed and another limit cycle is bifurcated. Finally λ is perturbed such that $\lambda(4a_3b_1 + b_5) < 0$. The ninth and final limit cycle is bifurcated. □

3. Concluding remarks

It seems unlikely that the approach based on finding a basis for Liapunov quantities as described above could be used to determine the maximum number of limit cycles bifurcating from a fine focus in a general cubic system, which has fifteen coefficients in its canonical form. We can always rotate and scale the system to reduce the number of variables to thirteen, which would give rise to no more than thirteen small amplitude limit cycles. However even finding an example of a cubic system with eleven such limit cycles is beyond current capabilities. A search for a transformation of the system to one in which there are fewer coefficients seems to be a more promising starting point.

It is noteworthy that in recent years computational methods have come to play an increasing role in mathematical research (see Borwein [1] for a recent survey). Their use in this paper falls within number 7 of the typology suggested by Borwein [1]. In particular, proofs sometimes involve the extensive use of some sophisticated software.

Proofs based on the use of computer algebra are inevitably impossible to verify completely, see Daly [4]. In some cases we can repeat the calculations using different software implemented on other platforms to reinforce our results. However, as in the above example, we have often struggled to find one successful route to a conclusion so we are unable to contemplate this type of double checking. Importantly, the software is used extensively on examples for which results have been obtained by independent means and this leads to a high degree of confidence in the reliability of the software.

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