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STABILITY ANALYSIS OF BLOCK Θ-METHODS FOR NEUTRAL MULTIDELAY-DIFFERENTIAL-ALGEBRAIC EQUATIONS*

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Abstract This paper is concerned with the stability analysis of Block θ-methods for solving neutral multidelay-differential-algebraic equations. We shown that if the coefficient matrices of neutral multidelay-differential-algebraic equations satisfying some stability conditions and $\theta\in [\frac{1}{2}, 1]$, then the numerical solution of Block θ-methods for solving neutral multidelay-differential-algebraic equations is asymptotically stable.

Keywords Neutral multidelay differential-algebraic system, block θ-methods, asymptotic stability.

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1. Introduction

In this paper, we are concerned with neutral multidelay- differential-algebraic systems(NMDAEs) with the following :

$$\begin{cases} \sum\limits_{i=0}^{l} L_i y'(t-\tau_i) + \sum\limits_{i=0}^{l} M_i y(t-\tau_i) = 0, & t \geq 0, \\ y(t) = \varphi(t), & t \in [-\tau, 0), \end{cases} \tag{1.1}$$

where $L_i, M_i \in C^{d\times d}$ are given matrixes, and L_0 is singular. $\varphi(t) \in C^d$ is a given differentiable vector-valued function. $y(t) = (y_1(t), y_2(t), \ldots, y_d(t))^T, \tau_l > \tau_{l-1} > \ldots \tau_1 > \tau_0 = 0, \tau_i, (i = 1, \cdots, l)$ are constant delay terms.

This type of equation, or special cases of these system arise in a wide range of applications, including multi-body mechanics, prescribed path control, electrical designs, chemically reacting systems and so on (see [1]).

Due to the constraint of delay and algebraic conditions, it is very difficult to get theoretical solution of NMDAEs. So the numerical treatments of NMDAEs becomes very necessary. Furthermore, numerical stability is an important part in numerical analysis. Some paper considered the stability for differential-algebraic equations(DAEs) and delay differential-algebraic systems(DDAEs), these paper can refer [3, 7, 9, 10] and so on. As we know, there are only a few papers discuss the asymptotic stability of (1.1).

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Actually, delay differential algebraic equations can be simplified as a delay differential equations, we can discuss its numerical method of stability through the characteristic equations. Stephen L. Campell [3] give asymptotic stability conditions for equation (1.1) and (1.2). In addition, using $BDF-$methods and θ-methods, numerical stability have been discussed, and give the corresponding numerical asymptotic stability conditions.

$$\sum_{i=0}^{l} L_i y'(t-i\tau) + \sum_{i=0}^{l} M_i y(t-i\tau) = 0. \tag{1.2}$$

In [2], Chengjian Zhang discussed the asymptotic stability of linear multistep methods and Runge-Kutta methods for neutral multidelay-differential-algebraic systems with $\tau_i = i\tau$, as (1.2). However, in this paper, we aim to study asymptotic stability of Block θ-methods for solving neutral multidelay-differential-algebraic systems of (1.1).

The paper is organized as follows. In section 2, we review basic notations and asymptotic stability criteria for analytical solutions of system (1.1). In section 3, we introduce Block θ-methods and the corresponding results. The main results of the paper will be introduced in section 4, it is shown that if the coefficient matrices of neutral multidelay-differential-algebraic equations satisfying some stability conditions and $\theta \in [\frac{1}{2}, 1]$, then the numerical solution of Block θ-methods for solving neutral multidelay-differential-algebraic equations is asymptotically stable.

2. A Review of Asymptotic Stability Results

First of all, we need to introduce some notations : the sets C^+, C^-, C^0 denote the sets of complex numbers with strictly positive real part, strictly negative real part, and zero real part, respectively. The matrix pencil $\{A, B\}$ is said to be regular if there exists $\lambda \in C$ such that matrix $\lambda A + B$ is nonsingular. λ is an eigenvalue of regular matrix pencil $\{A, B\}$ if $det\{\lambda A + B\} = 0$. $\sigma(A, B)$ and $\rho(A, B)$ denote the set of the spectrum and the spectral radius of the pencil $\{A, B\}$, respectively. The concept is also extended to the case of a tuple of matrices $\{A_i\}_{i=0}^n$ by defining $\sigma(\{A_i\}_{i=0}^n) = \{\lambda \in C : det(\sum_{i=0}^n \lambda^{n-i} A_i) = 0\}$, and $\rho(\{A_i\}_{i=0}^n) = max\{|\lambda| : \lambda \in C : det(\sum_{i=0}^n \lambda^{n-i} A_i) = 0\}$.

Definition 2.1. Assume the pencil $\{L_0, M_0\}$ is regular, then the solution $y(t)$ of the system (1.1) is called asymptotically stable if there exist a constant ε and a norm $||\cdot||$ such that

$$\max_{-\tau_l \le t \le 0} ||\varphi(t) - \widehat{\varphi(t)}|| \le \varepsilon \quad \text{and} \quad \lim_{t\to\infty} ||y(t) - \widehat{y(t)}|| = 0$$

for any other solution $\widehat{y(t)}$ corresponding to a consistent initial function $\widehat{\varphi(t)}$.

Lemma 2.1. *([3]) Assume that system* (1.1) *is regular, then for every delay* $\tau_i \ge 0$, *where* $\tau_l > \tau_{l-1} > \ldots > \tau_1 > \tau_0 = 0$. *If it satisfies the following two conditions:*

(A1) $\sigma(L_0, M_0) \in c^-$,

(A2) $\sup_{\Re(s)\ge 0} \rho\Big[\sum_{i=1}^{l} |(sL_0 + M_0)^{-1}(sL_i + M_i)|\Big] < 1.$

then the solution of system (1.1) *is asymptotically stable.*

3. Block θ-methods

The Block θ-methods have the form of equation (3.1):

$$Y_{n+1} = Y_n + \theta hBF(Y_{n+1}) + (1-\theta)hBF(Y_n) \quad (n = 0, 1, 2, \ldots), \tag{3.1}$$

where

$$\begin{aligned}
&B = (b_{ij})_{k\times k} \text{ and } Y_{n+1} = (y_{n,1}, y_{n,2} \ldots, y_{n,k})^T,\\
&F(Y_{n+1}) = (f_{n,1}, f_{n,2}, \ldots, f_{n,k})^T \text{ and } y_{n,i} \approx y(t_{n,i}),\\
&f_{n,i} = f(t_{n,i}, y_{n,i}) \text{ and } t_{n,i} = t_0 + nk + ih,\\
&i = 1, 2, \cdots, k,\ j = 1, 2, \cdots, k \text{ and } n = 0, 1, 2, \cdots .
\end{aligned}$$

Lemma 3.1. *([6]) If B is the coefficient matrix of k-dimensional θ-methods of order k, then the spectrum of B is k(note as $\sigma[B] = k$).*

Lemma 3.2. *([6]) If the order of k-dimensional Block θ-methods $p \geq 1$, then $k \in \sigma[B]$.*

Lemma 3.3. *([6]) If $\frac{1}{2} \leq \theta \leq 1$, then a Block θ-method is A-stable if and only if*

$$\sigma[B] = \{k\} \subseteq \{x \in R : x > 0\}.$$

Lemma 3.4. *([6]) The highest order p of k-dimensional Block θ-methods* (3.1) *is*

$$p = \begin{cases} k & k = 1, \theta \neq \frac{1}{2},\\ k+1 & k = 1, \theta = \frac{1}{2},\\ k & k \geq 2, 0 \leq \theta \leq 1.\end{cases}$$

4. Asymptotic stability of Block θ-methods

Applying Block θ-methods

$$Y_{n+1} = Y_n + \theta hBF(Y_{n+1}) + (1-\theta)hBF(Y_n) \quad (n = 0, 1, 2, \ldots)$$

to system (1.1), then lead to the difference equation (4.1):

$$\begin{aligned}
&\sum_{i=0}^{l}[(I\otimes L_i)Y_{n+1-l_i+\delta_i} - (I\otimes L_i)Y_{n-l_i+\delta_i} + \theta h(B\otimes M_i)Y_{n+1-l_i+\delta_i}\\
&+ (1-\theta)h(B\otimes M_i)Y_{n-l_i+\delta_i}] = 0,
\end{aligned} \tag{4.1}$$

where

$$\begin{aligned}
&\tau_i = (l_i - \delta_i)h, \quad 1 \leq i \leq l, \quad \delta_i \in [0, 1),\\
&Y_{n+1} = (y_{n,1}, y_{n,2}, \cdots, y_{n,k})^T,\\
&y_{n,i} \approx (y_1(t_{n,i}), y_2(t_{n,i}), \ldots, y_d(t_{n,i}))^T,\\
&Y_{n+j-l_i+\delta_i} = \sum_{p=-q}^{r} L_p(\delta_i) Y_{n+1-l_i+p},\\
&L_p(\delta) = \prod_{k=-q,k\neq p}^{r} \frac{\delta - k}{p - k}.
\end{aligned}$$

Let

$$Y_n = z^n X,$$

where

$$X = (\xi_{11}, \xi_{21}, \cdots, \xi_{d1}, \xi_{12}, \xi_{22}, \cdots, \xi_{d2}, \cdots, \xi_{1k}, \cdots, \xi_{dk})^T \in C^{dk},$$
$$Y_{n+j-l_i+\delta_i} = \sum_{p=-q}^{r} L_p(\delta_i) Y_{n+1-l_i+p} = \sum_{p=-q}^{r} L_p(\delta_i) z^{n+1-l_i+p} X.$$

Let

$$T(z, \delta_i) = \sum_{p=-q}^{r} L_p(\delta_i) z^p,$$
$$Y_{n+j-l_i+\delta_i} = T(z, \delta_i) z^{n+1-l_i} X.$$

Then equation (4.1) is equivalent to

$$\sum_{i=1}^{l} \Big[(I \otimes L_i) z^{n+1-l_i} - (I \otimes L_i) z^{n-l_i} + h\theta(B \otimes M_i) z^{n-l_i} + h(1-\theta)(B \otimes M_i) z^{n+1-l_i}\Big] T(z, \delta_i) X + \Big[(I \otimes L_0) z^{n+1} - (I \otimes L_0) z^n + h\theta(B \otimes M_0) z^n + h(1-\theta)(B \otimes M_0) z^{n+1}\Big] X = 0,$$

so the characteristic equation of difference equation (4.1) is:

$$P_m(z) = det\Big\{(I \otimes L_0)(z-1) + h(\theta z + (1-\theta))(B \otimes M_0) + \sum_{i=1}^{l} [(I \otimes L_i)(z-1) + h(1-\theta))(B \otimes M_0)] T(z, \delta_i)\Big\} = 0.$$

Using the properties of Kronecker product (cf.[4]), we have

$$P_m(z) = \prod_{\mu \in \sigma(B)} det\Big\{L_0(z-1) + h\mu(\theta z + (1-\theta)) M_0 + \sum_{i=1}^{l} [L_i(z-1) + h\mu(\theta z + (1-\theta)) M_i] T(z, \delta_i) z^{-L_i}\Big\} = 0.$$

Let

$$q_m(z) = det\Big\{L_0(z-1) + h\mu(\theta z + (1-\theta)) M_0 + \sum_{i=1}^{l} [L_i(z-1) + h\mu(\theta z + (1-\theta)) M_i] T(z, \delta_i) z^{-L_i}\Big\}.$$

Then

$$P_m(z) = \prod_{\mu \in \sigma(B)} q_m(z).$$

Let's introduce polynomial

$$r(z,\delta)=\sum_{p=-q}^{r}L_p(\delta)z^{p+q},$$

where

$$z\in C,\ \ \delta\in[0,1),$$
$$L_p(\delta)=\prod_{k=-q,k\neq p}^{r}\frac{\delta-k}{p-k}.$$

Definition 4.1. A numerical methods is called asymptotic stable, if and only if the numerical solutions of system (1.1) satisfy $\lim\limits_{n\to\infty}\|y_{n,i}\|=0$.

Lemma 4.1. *([5, 8])*

(1) $|r(z,\delta_i)|\leq 1$ *if and only if* $q\leq r\leq q+2, |z|=1$ *and* $0\leq\delta<1$ *holds;*

(2) if $q+r>0, q\leq r\leq q+2, |z|=1, 0<\delta<1$, *then the sufficient and necessary condition of* $r(z,\delta)=1$ *is that* $z=1$.

Then

$$T(z,\delta_i)=\sum_{p=-q}^{r}L_p\delta_i z^p=r(z,\delta_i)z^{-p}\qquad(i=1,2,\cdots,l).$$

If z belongs to a unit circle,then according to Lemma 4.1: $|T(z,\delta_i)|\leq 1$. And if $|z|=\infty$, then $|T(\infty,\delta_i)|=0$. Then using Maximum modulus principle can leads to the following result:

When $|z|\geq 1,\delta_i\in[0,1)$holds, then $|T(z,\delta_i)|\leq 1$. That is

$$|T(z,\delta_i)|=|\sum_{p=-q}^{r}L_p\delta_i z^p|\leq 1\quad(i=1,2,\cdots,l).$$

Definition 4.2. Let $W\in C^{n\times n}$ with elements w_{ij} and $|W|$ denote the nonnegative matrix in $R^{n\times n}$ with element $|w_{ij}|$. For two matrices $U,V\in C^{n\times n}$, we write $U\leq V$ if and only if $u_{ij}\leq v_{ij}$ for each $i,j\in\{1,2,\cdots,n\}$.

Lemma 4.2. *([4]) Let* $W\in C^{n\times n}$ *and* $V\in R^{n\times n}$. *If* $|W|\leq V$, *then* $\rho(W)\leq\rho(V)$.

Theorem 4.1. *Assume that system* (1.1) *satisfies the conditions of Lemma 2.1 and the Block* θ*-method is A-stable, then the solution of Block* θ*-methods for solving NMDAEs is asymptotically stable.*

Proof. We just have to prove that $p_m(z)$ is a *Schur* polynomial, if and only if every $q_m(z)$ is a *Schur* polynomial. If this was not true, then there must exist $|z_0|\geq 1$, such that

$$\begin{aligned}q_m(z_0)=&det\Big\{L_0(z_0-1)+h\mu(\theta z_0+(1-\theta))M_0+\sum_{i=1}^{l}[L_i(z_0-1)\\&+h\mu(\theta z+(1-\theta))M_i]T(z_0,\delta_i)z_0^{-l_i}\Big\}=0.\end{aligned}$$

Obviously when $\theta \in [\frac{1}{2}, 1]$ and $\theta z_0 + (1-\theta) \neq 0$. We can have that

$$det\Big\{L_0 \frac{z-1}{h\mu(\theta z_0 + (1-\theta))} + M_0 + \sum_{i=1}^{l}[L_i \frac{z_0 - 1}{h\mu(\theta z_0 + (1-\theta))} + M_i]T(z_0, \delta_i) z_0^{-L_i}\Big\} = 0.$$

Let

$$s = \frac{z_0 - 1}{h\mu(\theta z_0 + (1-\theta))}.$$

That is

$$z_0 = \frac{1 + h\mu s(1-\theta)}{1 - h\mu s\theta}.$$

Let $z_0 = a + bi$, then for $\theta \in [\frac{1}{2}, 1]$,

$$\begin{aligned}|z_0|^2 &= 1 + \frac{2h\mu a + h^2\mu^2 a^2 + h^2\mu^2 b^2 - 2(h^2\mu^2 a^2 + h^2\mu^2 b^2)\theta}{(1 - h\mu\theta a)^2 + h^2\mu^2 b^2\theta^2} \\ &\leq 1 + \frac{2h\mu a}{(1 - h\mu\theta a)^2 + h^2\mu^2 b^2}.\end{aligned}$$

If $\mu > 0$, $a < 0$, then $Re(s) < 0 \Rightarrow |z_0|^2 < 1 \Rightarrow |z_0| < 1$. Therefore $\mu > 0$, $|z_0| \geq 1$ can lead to $Re(s) \geq 0$. For

$$det\Big\{L_0 s + M_0 + \sum_{i=1}^{l}(L_i s + M_i)T(z_0, \delta_i) z_0^{-l_i}\Big\} = 0.$$

Then

$$det\Big\{z_0^{l_i} I + \sum_{i=1}^{l}(L_0 s + M_0)^{-1}(L_i s + M_i)T(z_0, \delta_i)\Big\} = 0,$$

$$z_0^{l_i} \in \sigma\Big\{-\sum_{i=1}^{l}(L_0 s + M_0)^{-1}(L_i s + M_i)T(z_0, \delta_i)\Big\}.$$

The above equation, together with Lemma 2.1, 4.1, 4.2 and $|T(z, \delta_i)| \leq 1$, we get

$$\begin{aligned}&\rho\Big[-\sum_{i=1}^{l}(L_0 s + M_0)^{-1}(L_i s + M_i)T(z_0, \delta_i)\Big] \\ \leq\ &\rho\Big[\sum_{i=1}^{l}|(L_0 s + M_0)^{-1}(L_i s + M_i)|\,|T(z_0, \delta_i)|\Big] \\ \leq\ &\rho\Big[\sum_{i=1}^{l}|(L_0 s + M_0)^{-1}(L_i s + M_i)|\Big] \leq 1.\end{aligned}$$

Then it leads to $|z_0|^{l_i} < 1$, which implies that $|z_0| < 1$. All of these contradict with the former assumption that $|z_0| \geq 1$. Now, we complete the proof. □

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