Journal of Applied Analysis and Computation
Volume 2, Number 3, August 2012

Website:http://jaac-online.com/
pp. 251–272

ATTRACTORS FOR A CAGINALP PHASE-FIELD MODEL TYPE ON THE WHOLE SPACE $\mathbb{R}^3$

Brice Doumbé Bangola

Abstract We consider in this paper a generalization of Caginalp phase-field system derived from a generalization of the Maxwell-Cattaneo law in an unbounded domain namely $\mathbb{R}^3$ in our case ; which make the analysis challenging. We prove the well-posedness of the problem and the dissipativity of the associated semigroup. Finally, we study the long time behavior of solutions in terms of attractors.

Keywords Caginalp system, Maxwell-Cattaneo law, well-posedness, dissipativity, long time behavior of solutions, global attractors.

MSC(2000) 35B41, 35B45.

1. Introduction

This article deals with a generalization of a system well known in the phase transition theory as the Caginalp system (see Caginalp [10]). This system was proposed by Caginalp in order to model melting-solidification phenomena in certain classes of materials. Since, it has been extensively studied as well in bounded domains as unbounded domains (see, for example, Miranville & Quinatanilla [16], Cherfils & Miranville [11], Miranville & Quintanilla [17], [18], Bates & Zheng [1], Brochet etc. [2], Brochet & Hilhorst [3] and Brochet etc. [4]).

We are concerned in this paper with the following initial-boundary value problem

$$\frac{\partial u}{\partial t} - \Delta u + f(x,u) = \frac{\partial \alpha}{\partial t}, \text{ in } [0,T]\times\mathbb{R}^3, \tag{1.1}$$

$$\frac{\partial^2 \alpha}{\partial t^2} + \frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t} - \Delta\alpha + \lambda\alpha = -u - \frac{\partial u}{\partial t}, \ \lambda > 0, \text{ in } [0,T]\times\mathbb{R}^3, \tag{1.2}$$

$$u|_{t=0} = u_0, \alpha|_{t=0} = \alpha_0, \frac{\partial \alpha}{\partial t}|_{t=0} = \alpha_1, \ x \in \mathbb{R}^3, \tag{1.3}$$

$$\lim_{|x|\to\infty} |u(t,x)| = \lim_{|x|\to\infty} |\alpha(t,x)| = 0, \ \forall\ t \in [0,T], \tag{1.4}$$

where $u = u(t,x)$ is the phase field or order parameter and $\alpha = \alpha(t,x)$ the thermal displacement variable that is to say $\alpha(t) = \int_0^t \theta(\tau)d\tau + \alpha_0$, where θ denotes the relative temperature appearing in the original Caginalp phase-field model. For simplicity, we take all physical constants equal to 1. This kind of problems in bounded domain has been studied in the papers by Miranville & Quintanilla [16], [19] (cf,

Email address :bdoumbe@math.univ-poitiers.fr(B. D. Bangola)
Laboratoire de Mathéatiques et Applications, Université de Poitiers, SP2MI, Boulevard Marie et Pierre Curie - Téléport 2, 86962 Chasseneuil Futuroscope Cedex, Poitiers, France

also, [17] with a nonlinear coupling term). Concerning the study of the asymptotic behavior in unbounded domains, the literature is so vast, one can mention for example, Babin & Vishik [5], Zelik [25] and the references therein, where the existence of global attractor of finite dimensions has been proved using Sobolev weighted spaces. The main difficulty here, is the loss of compactness of the embedding

$$H^1(\mathbb{R}^3) \subset L^p(\mathbb{R}^3), \forall\, p \in [2, 6].$$

In that case, it is impossible to use the classical compactness argument to prove for example the existence of the global attractor. To overcome this drawback, we are going to use a method applied successfully in papers by Conti & Mola [12], Morillas & Valero [20], Belleri & Pata [6], Conti etc. [13] and Pata [23]. This method is based on a decomposition of solutions and a use of suitable cut-off functions. We first establish the well-posedness of the system, then we discuss the existence of absorbing sets. Finally, we study long time behavior of solutions in terms of attractors.

2. Mathematical setting

2.1. Notation

We introduce following Hilbert spaces :

$$H = L^2(\mathbb{R}^3), V = H^1(\mathbb{R}^3), W = H^2(\mathbb{R}^3).$$

We denote by $(.,.)$ and $\|.\|$ the scalar product and the norm in H respectively. The symbol $\langle .,. \rangle$ stands for the duality product. Identifying H and its dual space H', one has the continuous and dense (but not compact) embeddings

$$W \subset V \subset H \subset V'.$$

The scalar product on V' is defined by

$$\langle v, w \rangle = \langle v, (I - \Delta)^{-1} w \rangle, \forall v, w \in V',$$

where the operator $(I - \Delta)^{-1}$ is a bijection from V' to V. Finally, we consider the phase space

$$\mathcal{H} = V \times V \times H,$$

endowed with the norm

$$\|(u, v, w)\|^2_{\mathcal{H}} = \|u\|^2_V + \|v\|^2_V + \|w\|^2.$$

2.2. Assumptions.

We make the following assumptions on the non linearity $f : \mathbb{R}^4 \to \mathbb{R}$, $f(x, .) \in C^2(\mathbb{R})$, $\forall\, x \in \mathbb{R}^3$ and let

$$F(x, s) = \int_0^s f(x, \tau) d\tau.$$

We now assume as in the paper by Belleri & Pata [6], that there exist $r_0 > 0$, and positive constants $c_i,\ i = 0, \cdots, 5$, and $\gamma \in [1, 3]$ such that

$$f(x, 0) \in V, \tag{2.1}$$

$$|f'(x, 0)| \le c_0,\ \forall\ x \in \mathbb{R}^3, \tag{2.2}$$

$$|f''(x, s) \le c_1(1 + |s|^\gamma),\ \forall\ x \in \mathbb{R}^3,\ \forall\ s \in \mathbb{R}, \tag{2.3}$$

$$\lim_{|s|\to+\infty} \inf \frac{f(x, s)}{s} \ge 0, \quad \text{uniformly as } |x| \le r_0, \tag{2.4}$$

$$(f(x, s) - f(x, 0))s \ge c_2 s^2,\ \forall\ s \in \mathbb{R}, |x| > r_0, \tag{2.5}$$

$$\lim_{|s|\to+\infty} \inf \frac{f(x, s)s - c_3 F(x, s)}{s^2} \ge 0,\ \text{uniformly as } |x| \le r_0, \tag{2.6}$$

$$f'(x, s) \ge -c_4,\ \forall\ s \in \mathbb{R},\ |x| > r_0, \tag{2.7}$$

$$|\nabla_x f'(x, s)| \le c_5(1 + |s|^2),\ \forall\ s \in \mathbb{R}, \tag{2.8}$$

where the prime denotes derivation with respect to the second variable of f and ∇_x stands for the partial derivative in x.

For $u \in V$, let

$$\mathcal{F}(u) = \int_{\mathbb{R}^3} F(x, u(x))dx.$$

The quantity $\mathcal{F}(u)$ is well defined owing to (2.1)-(2.3) and the inclusion $V \subset L^6(\mathbb{R}^3)$.

We state preliminary results that will be useful in the course of our investigation.

Lemma 2.1. *For every $r_0 > 0$ fixed, there exists a strictly positive constant c which depends on r_0 such that*

$$\int_{|x|>r_0} v^2(x)dx \ge c\|v\|^2 - \|\nabla v\|^2,\ \forall\ v \in V. \tag{2.9}$$

Proof. We define $B_{r_0} := \{x \in \mathbb{R}^3 : |x| \le r_0\}$ the closed ball of $\mathbb{R}^3$ centered at zero of radius r_0. We set

$$< v >:= \frac{1}{|B_{r_0}|} \int_{|x|\le r_0} v(x)dx,$$

where $|B_{r_0}|$ denotes the three dimensional Lebesgue's measure of B_{r_0}.

$$\int_{|x|\le r_0} |v(x) - < v > |^2 dx = \int_{|x|\le r_0} |v(x)|^2 dx - |B_{r_0}| < v >^2 . \tag{2.10}$$

Applying Poincare's inequality for functions with null mean, we then have

$$\int_{|x|\le r_0} |v(x)|^2 dx \le c\|\nabla v\|^2 + |B_{r_0}| < v >^2 . \tag{2.11}$$

Since,

$$\int_{|x|\le r_0} |v(x) - < v > |^2 dx \le c \int_{|x|\le r_0} |\nabla v(x)|dx,\ \forall\ v \in H^1(B_{r_0}),\ c \ge 0.$$

Adding $\int_{|x|>r_0} |v(x)|^2 dx$ at each member of (2.11), one gets owing to Hölder's inequality

$$\|v\|^2 \le c\|\nabla v\|^2 + |B_{r_0}|^{2/3} \|v\|^2_{L^6(\mathbb{R}^3)} + \int_{|x|>r_0} |v(x)|^2 dx. \tag{2.12}$$

Recalling an inequality due to Gagliardo, Nirenberg and Sobolev :

$$\|v\|_{L^6(\mathbb{R}^3)} \leq c'\|\nabla v\|,$$

hence

$$\|v\|^2 \leq (c + c'|B_{r_0}|^{2/3})\|\nabla v\|^2 + \int_{|x|>r_0} |v(x)|^2 dx, \tag{2.13}$$

which completes the proof. □

Lemma 2.2. *(see [23]) We assume that conditions* (2.1)-(2.7) *are satisfied. Therefore for all $\nu > 0$ small, there exist $c(\nu) \geq 0$ and $\rho(\nu) > 0$ such that for all $u \in V$,*

$$\langle f(x,u), u\rangle - c_6\mathcal{F}(u) \geq -\nu\|u\|^2 - c(\nu), \tag{2.14}$$

$$\mathcal{F}(u) \geq -\nu\|u\|^2 - c(\nu), \tag{2.15}$$

$$\langle f(x,u), u\rangle - \rho(\nu)\|u\|^2 \geq -\frac{c_2}{2}\|\nabla u\|^2 - c(\nu), \tag{2.16}$$

for some $c_6 > 0$ independent of ν.

Proof. Let prove (2.14) :
It follows from (2.6) that for every $\nu > 0$ fixed, there exists a strictly positive constant L depending on ν such that

$$f(x,u)u - c_3F(x,u) \geq -\nu u^2, \ |u| > L. \tag{2.17}$$

The function $f - c_3F$ being locally bounded in $\mathbb{R}^4$, then there exists $c_6 > 0$ depends on ν such that

$$f(x,u)u - c_3F(x,u) \geq -c_6, \ |u| \leq L, \ |x| \leq r_0. \tag{2.18}$$

Summing (2.17) and (2.18), one has

$$f(x,u)u - c_3F(x,u) \geq -\nu u^2 - c_6, \ \forall\ u \in \mathbb{R}, \ |x| \leq r_0. \tag{2.19}$$

Owing to (2.7), we write

$$(f(x,u) - f(x,s))(u-s) \geq -c_4(u-s)^2, \ |x| > r_0, \ u > s. \tag{2.20}$$

We then have owing to (2.5), (2.20) and Young's inequality

$$\begin{aligned}
&f(x,u)u - c_3F(x,u)\\
=&c_3(f(x,u) - F(x,u)) + c_6 f(x,u)u\\
=&c_3\int_0^u (f(x,u) - f(x,s))ds + c_6(f(x,u) - f(x,0))u + c_6 f(x,0)u\\
\geq& - c_3c_4\int_0^u (u-s)ds + c_6u^2 + c_6 f(x,0)u\\
\geq&\frac{-c_3c_4}{2}u^2 + \frac{c_2c_6}{2}u^2 - \frac{c_6}{2c_2}|f(x,0)|^2\\
\geq& - \frac{c_6}{2c_2}|f(x,0)|^2, \ |x| > r_0,
\end{aligned} \tag{2.21}$$

where $c_6 = 1 - c_3 > 0$. Integrating (2.19) over $|x| \leq r_0$ and (2.21) over $|x| > r_0$, and summing up the resulting inequalities, we get (2.14).

To obtain (2.15), we start by considering (2.4), we then have that for all $\nu > 0$ there exists $M > 0$ depending on ν such that

$$\frac{f(x,u)}{u} \geq -2\nu, \ |u| > M.$$

Without loss of generality, we consider the case $u > M$, noting that we proceed analogously for the case $u < -M$, and $|x| \leq r_0$, then

$$\begin{aligned} F(x,u) &= \int_0^M f(x,s)ds + \int_M^u \frac{f(x,s)s}{s}ds \\ &\geq \int_0^M f(x,s)ds - 2\nu \int_M^u sds \\ &\geq -c_8 - \nu u^2, \end{aligned} \tag{2.22}$$

where $c_8 > 0$ depends on ν. Now from (2.5) and Young's inequality, we write

$$\begin{aligned} F(x,u) &= \int_0^u \frac{(f(x,s) - f(x,0))s}{s}ds + \int_0^u f(x,0)ds \\ &\geq \frac{c_2}{2}u^2 + |f(x,0)u| \\ &\geq \frac{3c_2}{8}u^2 - \frac{2}{c_2}|f(x,0)|^2 \\ &\geq -\frac{2}{c_2}|f(x,0)|^2, \ |x| > r_0. \end{aligned} \tag{2.23}$$

Integrating (2.22) over $|x| \leq r_0$ and (2.22) over $|x| > r_0$ we get adding the inequalities obtained the estimate (2.15).

Finally, using (2.4) and arguing as for (2.19), we then have that for all $\nu > 0$ small, there exists a constant c_9 depending on ν such that

$$f(x,u)u \geq -\nu u^2 - c_9, \ |x| \leq r_0. \tag{2.24}$$

The assumption (2.5), the Young inequality and the Lemma 2.1 yield

$$f(x,u)u \geq \frac{c_2}{2}u^2 - \frac{1}{2c_2}|f(x,0)|^2, \ |x| > r_0. \tag{2.25}$$

Hence the Lemma 2.1 implies

$$\begin{aligned} \int_{|x|>r_0} f(x,u)udx &\geq \int_{|x|>r_0} u^2dx - \frac{1}{2c_2}\|f(x,.)\|^2 \\ &\geq \frac{cc_2}{2}\|u\|^2 - \frac{c_2}{2}\|\nabla u\|^2 - \frac{1}{2c_2}\|f(x,.)\|^2. \end{aligned} \tag{2.26}$$

Integrating (2.24) over $|x| \leq r_0$ and summing the resulting estimate and (2.26), one obtains

$$\langle f(x,u), u\rangle \geq \rho(\nu)\|u\|^2 - \frac{c_2}{2}\|\nabla u\|^2 - c(\nu), \tag{2.27}$$

with $\rho(\nu) = \frac{cc_2-\nu}{2}$ and $c(\nu) = c_9|B_{r_0}| + \frac{1}{2c_2}\|f(x,.)\|^2$, which finish the proof.

□

Lemma 2.3. *(see Pata [23, Lemma 2.7]) Let* $\Phi : \mathcal{H} \longrightarrow \mathbb{R}$ *be a continuous function which satisfies (in the sense of distributions)*

$$\frac{d}{dt}\Phi(z(t)) + \delta\|z(t)\|^2_{\mathcal{H}} \leq k, \tag{2.28}$$

for some $\delta, k > 0$, *and* $z \in \mathcal{C}(\mathbb{R}^+; \mathcal{H})$. *In addition, assume that*

$$\inf_{t\in\mathbb{R}^+} \Phi(z(t)) \geq -m,\ \Phi(z(0)) \leq M, \tag{2.29}$$

for some $m, M \geq 0$. *Then*

$$\Phi(z(t)) \leq \sup_{v\in\mathcal{H}} \{\Phi(v); \delta\|v\|^2_{\mathcal{H}} \leq 2k\},\ \forall t \geq t_0, \tag{2.30}$$

where $t_0 = \frac{m+M}{k}$.

3. Existence and uniqueness of solutions

The aim of this section is to establish the well-posedness of the system (1.1)-(1.4).

Our first existence result is the following

Theorem 3.1. *Under the assumptions listed above, namely,* (2.1)- (2.7) *and for every initial data* $(u_0, \alpha_0, \alpha_1) \in \mathcal{H}$, *the problem* (1.1)-(1.4) *possesses at least one solution* $(u, \alpha, \frac{\partial\alpha}{\partial t})$ *such that* $u \in L^2(0, T; V)$, $\frac{\partial u}{\partial t} \in L^2(0, T; H)$, $\alpha \in L^2(0, T; V)$ *and* $\frac{\partial\alpha}{\partial t} \in L^2(0, T; H)$.

Proof. The proof is carried out via the classical method of Faedo-Galerkin (see [7]). □

Theorem 3.2. *Under the hypothesis of the Theorem 3.1, with* $\gamma = 1$ *in* (2.3), *the solution to the problem* (1.1)-(1.4) *is unique with the above regularity.*

Proof. Let $(u^{(1)}, \alpha^{(1)}, \frac{\partial\alpha^{(1)}}{\partial t})$ and $(u^{(2)}, \alpha^{(2)}, \frac{\partial\alpha^{(2)}}{\partial t})$ be two solutions to (1.1)-(1.2) with initial data $(u_0^{(1)}, \alpha_0^{(1)}, \alpha_1^{(1)})$ and $(u_0^{(2)}, \alpha_0^{(2)}, \alpha_1^{(2)})$ respectively. We set

$$u = u^{(1)} - u^{(2)} \text{ and } \alpha = \alpha^{(1)} - \alpha^{(2)}.$$

Therefore (u, α) satisfies :

$$\frac{\partial u}{\partial t} - \Delta u + f(x, u^{(1)}) - f(x, u^{(2)}) = \frac{\partial\alpha}{\partial t}, \tag{3.1}$$

$$\frac{\partial^2\alpha}{\partial t^2} + \frac{\partial\alpha}{\partial t} - \Delta\frac{\partial\alpha}{\partial t} - \Delta\alpha + \lambda\alpha = -u - \frac{\partial u}{\partial t}. \tag{3.2}$$

Multiplying (3.1) by u and $\frac{\partial u}{\partial t}$, and then (3.2) by $\frac{\partial\alpha}{\partial t}$ summing up the resulting equations, one obtains

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|u\|^2 + \|\nabla u\|^2 + \lambda\|\alpha\|^2 + \|\nabla\alpha\|^2 + \|\frac{\partial\alpha}{\partial t}\|^2\Big) + \|\nabla u\|^2 \\
&+ \|\frac{\partial u}{\partial t}\|^2 + \left\|\frac{\partial\alpha}{\partial t}\right\|^2 + \left\|\nabla\frac{\partial\alpha}{\partial t}\right\|^2 \\
= &- \int_{\mathbb{R}^3}(f(x, u^{(1)}) - f(x, u^{(2)}))u dx - \int_{\mathbb{R}^3}(f(x, u^{(1)}) - f(x, u^{(2)}))\frac{\partial u}{\partial t} dx.
\end{aligned} \tag{3.3}$$

Owing to (2.1)-(2.3) with $\gamma = 1$, one has

$$|f'(x,u)| \leq c(1+|u|^2), c > 0, \ \forall\ x \in \mathbb{R},$$

and applying Hölder's inequality with exponents 1/3, 1/6, 1/2 and owing to the injection $V \subset L^6(\mathbb{R}^3)$, we write

$$\begin{aligned} & -\int_{\mathbb{R}^3} (f(x,u^{(1)}) - f(x,u^{(2)})) \frac{\partial u}{\partial t} dx \\ \leq & c \int_{\mathbb{R}^3} (1+|u^{(1)}|^2 + |u^{(2)}|^2)|u| \left|\frac{\partial u}{\partial t}\right| dx \\ \leq & c(1+\|u^{(1)}\|^2_{L^6(\mathbb{R}^3)} + \|u^{(2)}\|^2_{L^6(\mathbb{R}^3)})\|u\|_{L^6(\mathbb{R}^3)} \left\|\frac{\partial u}{\partial t}\right\| \\ \leq & c(1+\|u^{(1)}\|^2_V + \|u^{(2)}\|^2_V)\|u\|_V \left\|\frac{\partial u}{\partial t}\right\|. \end{aligned} \tag{3.4}$$

By (2.7), we have

$$-\int_{\mathbb{R}^3} (f(x,u^{(1)}) - f(x,u^{(2)}))u dx \leq c_4 \|u\|^2. \tag{3.5}$$

Hence (3.3)-(3.5) and Hölder's inequality imply

$$\frac{\partial \phi}{\partial t} + \left\|\frac{\partial u}{\partial t}\right\|^2 + \left\|\nabla \frac{\partial \alpha}{\partial t}\right\|^2 \leq c\phi, \tag{3.6}$$

where $\phi(t) = \|u(t)\|^2_V + \|\alpha(t)\|^2_V + \left\|\frac{\partial \alpha}{\partial t}(t)\right\|^2$.

We find by the Gronwall Lemma that

$$\begin{aligned} & \|u(t)\|^2_V + \|\alpha(t)\|^2_V + \left\|\frac{\partial \alpha}{\partial t}(t)\right\|^2 \\ \leq & c(\|u_0^{(1)} - u_0^{(2)}\|^2_V + \|\alpha_0^{(1)} - \alpha_0^{(2)}\|^2_V + \|\alpha_1^{(1)} - \alpha_1^{(2)}\|^2). \end{aligned} \tag{3.7}$$

Hence the uniqueness (for $u_0^{(1)} = u_0^{(2)}$, $\alpha_0^{(1)} = \alpha_0^{(2)}$ and $\alpha_1^{(1)} = \alpha_1^{(2)}$) and the continuous dependence on the initial data. □

4. Dissipativity

From what proceeds, we can define a continuous semigroup as follows

$$\begin{aligned} S(t) : \mathcal{H} & \longrightarrow \mathcal{H} \\ z_0 & \mapsto S(t)z_0 = (u(t), \alpha(t), \frac{\partial \alpha}{\partial t}(t)), \ \forall\ t \geq 0, \end{aligned}$$

where $(u, \alpha, \frac{\partial \alpha}{\partial t})$ is the unique solution to our system and $z_0 = (u_0, \alpha_0, \alpha_1)$.

This section is devoted to the existence of absorbing sets for the semigroup $S(t)$, $t \geq 0$, defined on the phase space $\mathcal{H}$. We have the

Theorem 4.1. *Assume that* (2.1)- (2.7) *are satisfied, with* $\gamma = 1$ *in* (2.3). *Then the semigroup* $S(t)$ *possesses a bounded absorbing set in* $\mathcal{H}$.

Proof. Let $R_0 > 0$ be fixed and let the initial data be taken in a ball of $\mathcal{H}$ of radius R_0 and of center 0, $B(0, R_0)$.

We multiply (1.1) by u and $\frac{\partial u}{\partial t}$, one obtains by adding up the two resulting equations

$$\frac{1}{2}\frac{d}{dt}(\|u\|_V^2 + 2\mathcal{F}(u)) + \|\nabla u\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2 + \langle f(x,u), u\rangle = \langle u + \frac{\partial u}{\partial t}, \frac{\partial \alpha}{\partial t}\rangle. \tag{4.1}$$

Let $\epsilon > 0$ be small to precise. We multiply now (1.2) by $\epsilon\alpha + \frac{\partial \alpha}{\partial t}$. We have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\lambda+\epsilon)\|\alpha\|^2 + (1+\epsilon)\|\nabla\alpha\|^2 + \left\|\frac{\partial \alpha}{\partial t}\right\|^2 + 2\epsilon(\alpha, \frac{\partial \alpha}{\partial t})\Big) \\ &\lambda\epsilon\|\alpha\|^2 + \epsilon\|\nabla\alpha\|^2 + (1-\epsilon)\left\|\frac{\partial \alpha}{\partial t}\right\|^2 + \left\|\nabla\frac{\partial \alpha}{\partial t}\right\|^2 \\ =& -\epsilon(u + \frac{\partial u}{\partial t}, \alpha) - (u + \frac{\partial u}{\partial t}, \frac{\partial \alpha}{\partial t}). \end{aligned} \tag{4.2}$$

We sum now (4.1) and (4.2) to have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\varPhi + \|\nabla u\|^2 + \lambda\epsilon\|\alpha\|^2 + \epsilon\|\nabla\alpha\|^2 + (1-\epsilon)\left\|\frac{\partial \alpha}{\partial t}\right\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2 + \left\|\nabla\frac{\partial \alpha}{\partial t}\right\|^2 \\ =& -\epsilon(u + \frac{\partial u}{\partial t}, \alpha) - \langle f(x,u), u\rangle, \end{aligned} \tag{4.3}$$

where

$$\varPhi = \|u\|_V^2 + (\lambda+\epsilon)\|\alpha\|^2 + (1+\epsilon)\|\nabla\alpha\|^2 + \left\|\frac{\partial \alpha}{\partial t}\right\|^2 + 2\epsilon(\alpha, \frac{\partial \alpha}{\partial t}) + 2\mathcal{F}(u), \tag{4.4}$$

satisfies, owing to Lemma 2.2 and for $\nu = \frac{1}{4}$

$$\begin{aligned} \varPhi(t) \geq& \frac{1}{2}\|u(t)\|^2 + \|\nabla u(t)\|^2 + (\lambda+\epsilon)\|\alpha(t)\|^2 + (1+\epsilon)\|\nabla\alpha(t)\|^2 \\ &+ \left\|\frac{\partial \alpha}{\partial t}\right\|^2 + 2\epsilon(\alpha, \frac{\partial \alpha}{\partial t}) - 2c(\frac{1}{4}) \\ \geq& \frac{1}{2}\|u(t)\|^2 + \|\nabla u(t)\|^2 + \lambda\|\alpha(t)\|^2 + (1+\epsilon)\|\nabla\alpha(t)\|^2 \\ &+ (1-\epsilon)\left\|\frac{\partial \alpha}{\partial t}(t)\right\|^2 - 2c(\frac{1}{4}). \end{aligned} \tag{4.5}$$

Hence for $\epsilon < 1$, we then have that there exist two constants $K_1 \geq 0$ and $K_2 > 0$ such that

$$\varPhi(t) + K_1 \geq K_2\Big(\|u(t)\|_V^2 + \|\alpha(t)\|_V^2 + \left\|\frac{\partial \alpha}{\partial t}(t)\right\|^2\Big), \tag{4.6}$$

where the constant K_2 is dependent of ϵ.

Writing (2.16) with $\nu = 1$, and setting $\beta = \frac{\rho(1)}{4}$, we have

$$-\frac{1}{2}\langle f(x,u), u\rangle \leq -2\beta\|u\|^2 + \frac{c_2}{4}\|\nabla u\|^2 + \frac{1}{2}c(1), \tag{4.7}$$

and considering the estimate (2.14) for $\nu = 2\beta$, we then get

$$-\frac{1}{2}\langle f(x,u), u\rangle \leq -\frac{c_3}{2}\mathcal{F}(u) + \beta\|u\|^2 + \frac{1}{2}c(2\beta). \tag{4.8}$$

Summing up (4.7) and (4.8), one gets

$$-\langle f(x,u), u\rangle \leq -\frac{c_3}{2}\mathcal{F}(u) - \beta\|u\|^2 + \frac{c_2}{4}\|\nabla u\|^2 + K_3, \tag{4.9}$$

with $K_3 = \frac{1}{2}(c(1) + c(2\beta))$.

Thus (4.3) becomes

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Phi + \beta\|u\|^2 + (1-\frac{c_2}{4})\|\nabla u\|^2 + \lambda\epsilon\|\alpha\|^2 + \epsilon\|\nabla\alpha\|^2 + (1-\epsilon)\left\|\frac{\partial\alpha}{\partial t}\right\|^2\\ &\frac{c_3}{2}\mathcal{F}(u) + \left\|\frac{\partial u}{\partial t}\right\|^2 + \left\|\nabla\frac{\partial\alpha}{\partial t}\right\|^2 \leq -\epsilon(u + \frac{\partial u}{\partial t}, \alpha) + K_3.\end{aligned} \tag{4.10}$$

Applying Young's inequality we write

$$\epsilon(u,\alpha) \leq \frac{\beta}{2}\|u\|^2 + \frac{\epsilon^2}{2\beta}\|\alpha\|^2 \tag{4.11}$$

and

$$\epsilon(\frac{\partial u}{\partial t}, \alpha) \leq \frac{\lambda\epsilon}{2}\|\alpha\|^2 + \frac{\epsilon}{2\lambda}\left\|\frac{\partial u}{\partial t}\right\|^2, \tag{4.12}$$

on account of (4.10)-(4.12), we get an inequality of the form

$$\frac{d}{dt}\Phi + \delta\Phi + c\left\|\frac{\partial u}{\partial t}\right\|^2 + \left\|\nabla\frac{\partial\alpha}{\partial t}\right\|^2 \leq C, \delta > 0, \tag{4.13}$$

in particular,

$$\frac{d}{dt}\Phi + \delta\Phi \leq C. \tag{4.14}$$

The Gronwall Lemma leads us to

$$\begin{aligned}\Phi(t) &\leq e^{-\delta t}\Phi(0) + C', t \geq 0,\\ &\leq Ke^{-\delta t} + C',\end{aligned} \tag{4.15}$$

where K depends on R_0.

And then

$$\|S(t)(u_0,\alpha_0,\alpha_1)\|^2_{\mathcal{H}} \leq Ke^{-\delta t} + C'. \tag{4.16}$$

Let $R > 0$ large enough. Therefore,

$$\|S(t)(u_0,\alpha_0,\alpha_1)\|^2_{\mathcal{H}} \leq R, \forall t \geq t_R, \tag{4.17}$$

where $t_R = \max\{0, -\frac{1}{\delta}\log\left(\frac{R-C'}{K}\right)\}$, which completes the proof. □

Remark 4.1. By the Theorem 4.1, we note that the ball of $\mathcal{H}$ of radius R centered at zero is a bounded absorbing set for the semigroup $S(t)$, $t \geq 0$.

Corollary 4.1. *The set*

$$\beta_0 = \bigcup_{t\geq 0} S(t)B_R,$$

where B_R stands for the ball of radius R centered at zero, is a connected bounded absorbing and invariant set for the semigroup $S(t)$ in $\mathcal{H}$ (that is, $S(t)\beta_0 \subset \beta_0$ for all $t \geq 0$, and for any bounded set $\mathcal{B} \subset \mathcal{H}$, there exists $t_0 = t_0(\mathcal{B})$ such that $S(t)\mathcal{B} \subset \beta_0$, for all $t \geq t_0$).

Proof. The proof is immediately obtained by construction. □

5. Global attractor

For a good understanding of the asymptotic behavior of solutions to our problem ; someone might be tempted to exploit compactness arguments to prove the existence of the global attractor. But due to the unboundedness of the domain one could not obtain appropriate compact estimates. An additional drawback appears due to the lack of regularizing effects of initial data. To solve these difficulties, we decompose the solution using also suitable cut-off functions.

Now we state the main result of this section.

Theorem 5.1. *Assume that* (2.1)-(2.8) *hold. Then the semigroup $S(t)$, $t \geq 0$, associated to the problem* (1.1)-(1.4) *possesses the (connected) global attractor $\mathcal{A}$ in $\mathcal{H}$.*

The proof is based on the following abstract result.

Theorem 5.2. *(see Temam [24, page 56]) Let $(S(t), \mathcal{X})$ be a dynamical system, with $\mathcal{X}$ a Banach space. Assume that*

(i) there exists an invariant bounded absorbing set $\beta_0 \subset \mathcal{H}$ for the semigroup $S(t)$, $t \geq 0$;

(ii) for every $\eta > 0$, there exist $t_\eta \geq 0$ and a (relative) compact set $\mathcal{K}_\eta \subset \mathcal{X}$ such that

$$\delta_{\mathcal{X}}(S(t_\eta)\beta_0, \mathcal{K}_\eta) \leq \eta, \tag{5.1}$$

where $\delta_{\mathcal{X}}$ denotes the usual Hausdorff semidistance in $\mathcal{X}$. Then the ω-limit set of β_0 is the (connected) global attractor for $S(t)$.

We established that the semigroup $S(t)$, $t \geq 0$, possesses an invariant bounded absorbing set $\beta_0 \subset \mathcal{H}$. Our aim is to prove that β_0 satisfies (ii). To do so, we split the solution of our problem into tree parts, using suitable cut-off functions.

5.1. Cut-off functions and decomposition of solutions

Let us assume that (2.1)-(2.8) hold, with $\gamma = 1$. For every $r > r_0$ fixed, we introduce two positives functions $\varphi_r^1, \varphi_r^2 \in C^\infty(\mathbb{R}^3)$ such that

$$\begin{cases} \varphi_r^1(x) + \varphi_r^2(x) = 1, & \text{for} \quad x \in \mathbb{R}^3, \\ \varphi_r^1(x) = 0, & \text{for} \quad |x| \leq r, \\ \varphi_r^2(x) = 0, & \text{for} \quad |x| \geq r+1. \end{cases}$$

We deduce from (2.2)-(2.3) the existence of $\nu > 0$ such that

$$\frac{f(x,s) - f(x,0)}{s} \geq -2c_1, \tag{5.2}$$

for $|s| < \nu$. Furthermore, from (2.4), there exists $L > 0$ such that (5.2) still holds for $|s| > L$ and $|x| \leq r_0 + 1$. Finally, due to the local boundedness of the non linearity f, we write for $\nu \leq |s| < L$

$$\frac{f(x,s) - f(x,0)}{s} \geq -M, \tag{5.3}$$

for some $M > 0$ and $|x| \leq r_0 + 1$. So, up to redefining M, estimate (5.3) is satisfied for all $s \in \mathbb{R}$ and $|x| \leq r_0 + 1$. We then decompose the non linearity f as follows

$$f = f_r^1 + f_r^2,$$

where

$$\begin{aligned} f_r^1(x,s) &= [f(x,s) - f(x,0)]\varphi_r^1(x) + [f(x,s) - f(x,0) + c_2 s + Ms]\varphi_r^2(x), \\ f_r^2(x,s) &= f(x,0)\varphi_r^1(x) + [f(x,0) - c_2 s - Ms]\varphi_r^2(x), \end{aligned}$$

for every $s \in \mathbb{R}$ and almost every $x \in \mathbb{R}^3$. Notice that f_r^1 fulfills (2.2)-(2.3) (replacing c_1 by $c_1 + c_2 + M$), and $f_r^1(x,0) \equiv 0$. In that case, (2.5) and (5.3) imply

$$f_r^1(x,s)s \geq c_2 s^2, \ \forall\ s \in \mathbb{R}, \ \text{a.e.} x \in \mathbb{R}^3. \tag{5.4}$$

Therefore, setting

$$F_r^1(x,s) = \int_0^s f_r^1(x,\tau)d\tau,$$

we then have

$$F_r^1 \geq 0 \text{ and } |F_r^1(x,s)| \leq c(s^2 + s^4), \forall\ x \in \mathbb{R}^3, \ \forall\ s \in \mathbb{R}. \tag{5.5}$$

From now, as in the paper by [23] (see also [12] and [6]), we decompose the solution to the problem (1.1)-(1.4) with initial data $z_0 \in \beta_0$ (where β_0 is the invariant bounded absorbing set of the Corollary 4.1) as follows

$$S(t)z_0 = z_1(t) + z_2(t),$$

where

$$z_1(t) = (u^d(t), \alpha^d(t), \frac{\partial \alpha^d}{\partial t}(t))$$

and

$$z_2(t) = (u^c(t), \alpha^c(t), \frac{\partial \alpha^c}{\partial t}(t)),$$

are solutions to

$$\frac{\partial u^d}{\partial t} - \Delta u^d + f_r^1(x,u^d) = \frac{\partial \alpha^d}{\partial t}, \tag{5.6}$$

$$\frac{\partial^2 \alpha^d}{\partial t^2} + \frac{\partial \alpha^d}{\partial t} - \Delta \frac{\partial \alpha^d}{\partial t} - \Delta \alpha^d + \lambda \alpha^d = -u^d - \frac{\partial u^d}{\partial t}, \tag{5.7}$$

$$\lim_{x \to +\infty} |u^d(x,t)| = \lim_{x \to +\infty} |\alpha^d(x,t)| = 0, \tag{5.8}$$

$$u^d(0) = u_0, \alpha^d(0) = \alpha_0, \frac{\partial \alpha^d}{\partial t}(0) = \alpha_1, \tag{5.9}$$

and

$$\frac{\partial u^c}{\partial t} - \Delta u^c + f_r^1(x,u) - f_r^1(x,u^d) + f_r^2(x,u) = \frac{\partial \alpha^c}{\partial t}, \tag{5.10}$$

$$\frac{\partial^2 \alpha^c}{\partial t^2} + \frac{\partial \alpha^c}{\partial t} - \Delta \frac{\partial \alpha^c}{\partial t} - \Delta \alpha^c + \lambda \alpha^c = -u^c - \frac{\partial u^c}{\partial t}, \tag{5.11}$$

$$\lim_{x\to+\infty} |u^c(x,t)| = \lim_{x\to+\infty} |\alpha^c(x,t)| = 0, \tag{5.12}$$

$$u^c(0) = \alpha^c(0) = \frac{\partial \alpha^c}{\partial t}(0) = 0, \tag{5.13}$$

respectively. Notice that, arguing as in the proofs of Theorem 3.1 and Theorem 4.1 we allow to show the well-posedness and dissipativity results of systems (5.6)-(5.9) and (5.10)-(5.13).

Remark 5.1. For all $r > r_0$ fixed, and for every $T > 0$ fixed, there exists a strictly positive constant C depending on r and T such that solutions z_1 to (5.6)-(5.9) and z_2 to (5.10)-(5.13) fulfill

$$\|z_1(t)\|^2_{\mathcal{H}} \leq C$$

and

$$\|z_2(t)\|^2_{\mathcal{H}} \leq C,$$

for all $t \in [0,T]$ and for $z_0 \in \beta_0$.

5.2. Existence of the global attractor

Let now state a series of Lemmata that will play an important role in order to establish the main result of this section.

Lemma 5.1. *Under conditions of Theorem 5.1, for every $\eta > 0$, there exist a time $t_\eta > 0$ and $r_\eta > r_0$ such that the solution $z_1(t_\eta)$ to (5.6)-(5.9) corresponding to $r = r_\eta$ at the time t_η satisfies the inequality*

$$\|z_1(t_\eta)\|_{\mathcal{H}} \leq \frac{1}{2}\eta, \tag{5.14}$$

for every $z_0 \in \beta_0$.

Proof. Let consider the system (5.6)-(5.9). Multiply (5.6) by $u^d + \frac{\partial u^d}{\partial t}$ and integrate over $\mathbb{R}^3$. We get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|u^d\|^2 + \|\nabla u^d\|^2 + 2\mathcal{F}^1(u^d)) + \|\nabla u^d\|^2 + \left\|\frac{\partial u^d}{\partial t}\right\|^2 \\ &= -\langle f_r^1(.,u^d), u^d\rangle + (u^d + \frac{\partial u^d}{\partial t}, \frac{\partial \alpha^d}{\partial t}), \end{aligned} \tag{5.15}$$

where $\mathcal{F}^1(u^d) = \int_{\mathbb{R}^3} F^1(x, u^d(x))dx$.

Let us introduce again $\epsilon > 0$ which is small enough. Multiplying (5.7) by $\frac{\partial \alpha^d}{\partial t} +$

$\epsilon\alpha^d$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big((\lambda+\epsilon)\|\alpha^d\|^2+(1+\epsilon)\|\nabla\alpha^d\|^2+\left\|\frac{\partial\alpha^d}{\partial t}\right\|^2+2\epsilon(\alpha^d,\frac{\partial\alpha^d}{\partial t})\Big)\\
&+\lambda\epsilon\|\alpha^d\|^2+\epsilon\|\nabla\alpha^d\|^2+(1-\epsilon)\left\|\frac{\partial\alpha^d}{\partial t}\right\|^2+\left\|\nabla\frac{\partial\alpha^d}{\partial t}\right\|^2\\
=&-(u^d+\frac{\partial u^d}{\partial t},\frac{\partial\alpha^d}{\partial t})-\epsilon(u^d+\frac{\partial u^d}{\partial t},\alpha^d).
\end{aligned} \tag{5.16}$$

Summing (5.15) and (5.16), one gets

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Phi(z_1)+\|\nabla u^d\|^2+\lambda\epsilon\|\alpha^d\|^2+\epsilon\|\nabla\alpha\|^2(1-\epsilon)\left\|\frac{\partial\alpha^d}{\partial t}\right\|^2\\
&+\left\|\frac{\partial u^d}{\partial t}\right\|^2+\left\|\nabla\frac{\partial\alpha^d}{\partial t}\right\|^2=-\langle f_r^1(.,u^d),u^d\rangle-\epsilon(u^d+\frac{\partial u^d}{\partial t},\alpha^d),
\end{aligned} \tag{5.17}$$

where

$$\begin{aligned}
\Phi(z_1(t))=&\|u^d(t)\|^2+\|\nabla u^d(t)\|^2+(\lambda+\epsilon)\|\alpha^d(t)\|^2+(1+\epsilon)\|\nabla\alpha^d(t)\|^2\\
&+\left\|\frac{\partial\alpha^d}{\partial t}(t)\right\|^2+2\epsilon(\alpha^d(t),\frac{\partial\alpha^d}{\partial t}(t))+2\mathcal{F}^1(u^d(t)),
\end{aligned} \tag{5.18}$$

satisfies for $\epsilon<1$

$$\begin{aligned}
\Phi(z_1(t))\geq&(1+c_2)\|u^d(t)\|^2+\|\nabla u^d(t)\|^2+\lambda\|\alpha^d(t)\|^2\\
&+(1+\epsilon)\|\nabla\alpha^d(t)\|^2+(1-\epsilon)\left\|\frac{\partial\alpha^d}{\partial t}(t)\right\|^2\\
\geq&\delta\|z_1(t)\|^2_{\mathcal{H}},\ \forall\ t\geq 0,\ \delta=\delta(\epsilon)>0
\end{aligned} \tag{5.19}$$

and

$$\Phi(z_1(0))\leq\mu, \tag{5.20}$$

where μ is a positive constant independent of initial data.

Noting that

$$-\langle f_r^1(.,u^d),u^d\rangle\leq -c_2\|u^d\|^2, \tag{5.21}$$

we then write

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Phi(z_1)+c_2\|u^d\|^2+\|\nabla u^d\|^2+\alpha\epsilon\|\alpha^d\|^2+\epsilon\|\nabla\alpha^d\|^2\\
&+(1-\epsilon)\left\|\frac{\partial\alpha^d}{\partial t}\right\|^2+\left\|\frac{\partial u^d}{\partial t}\right\|^2+\left\|\nabla\frac{\partial\alpha^d}{\partial t}\right\|^2\\
\leq&\epsilon(u^d,\alpha^d)+\epsilon(\frac{\partial u^d}{\partial t},\alpha^d).
\end{aligned}$$

Applying once again Young's inequality we get

$$\begin{aligned}
\epsilon(u^d,\alpha^d) &\leq \frac{c_2}{2}\|u^d\|^2+\frac{\epsilon^2}{2c_2}\|\alpha^d\|^2,\\
\epsilon(\frac{\partial u^d}{\partial t},\alpha^d) &\leq \frac{\lambda\epsilon}{2}\|\alpha^d\|^2+\frac{\epsilon}{2\lambda}\left\|\frac{\partial u^d}{\partial t}\right\|^2,
\end{aligned}$$

we end up with the differential inequality

$$\frac{d}{dt}\Phi(z_1)+k\|z_1\|_{\mathcal{H}}^2+c\left\|\frac{\partial u^d}{\partial t}\right\|^2+\left\|\nabla\frac{\partial \alpha^d}{\partial t}\right\|^2\leq \omega,\ c>0,$$

for some $k>0$ and $\omega\in(0,1)$.

By virtue of Lemma 2.3, there exists $t_\eta>0$ such that

$$\Phi(z_1(t_\eta))\leq \sup_{x\in\mathcal{H}}\Big\{\Phi(x):k\|x\|\leq 2\omega\Big\},$$

we then conclude that

$$\|z_1(t_\eta)\|_{\mathcal{H}}^2\leq \frac{1}{2}\eta,\ \forall\ t_\eta\geq \frac{\mu-\frac{k}{2}\eta}{\omega},$$

as claimed. □

In the sequel, we fix $\eta>0$. We choose $t_\eta>0$ as in Lemma 5.1 and we then have, due to Remark 5.1 that

$$\begin{aligned}\sup_{t\in[0,t_\eta]}\sup_{z_0\in\beta_0}\Big\{&\|\nabla u(t)\|+\left\|\frac{\partial\alpha}{\partial t}(t)\right\|,\|\nabla\alpha(t)\|,\|\nabla u^d(t)\|,\\ &\|\nabla\alpha^d(t)\|,\left\|\frac{\partial\alpha^d}{\partial t}(t)\right\|,\|\nabla u^c(t)\|,\|\nabla\alpha^c(t)\|,\left\|\frac{\partial\alpha^c}{\partial t}(t)\right\|\Big\}\leq C.\end{aligned}\tag{5.22}$$

Let $\rho>0$ be given. We now introduce family of smooth functions $\psi_\rho:\mathbb{R}^3\to[0,1]$ defined by :

$$\psi_\rho(x)=\begin{cases}0, & \text{if}\quad |x|\leq\rho+1,\\ 1, & \text{if}\quad |x|\geq 2(\rho+1),\end{cases}$$

such that :

$$|\nabla\psi_\rho(x)|\leq\frac{C}{\rho+1},\tag{5.23}$$

$$|\nabla\psi_\rho^2(x)|\leq\frac{C}{\rho+1}\psi_\rho(x),\tag{5.24}$$

$$|\Delta\psi_\rho(x)|\leq\frac{C}{\rho+1},\tag{5.25}$$

with $C>0$.

Then, for every fixed $\rho>0$, following the paper by Conti & Mola [12], we decompose the solution $z_2(t)$ to (5.10)-(5.13) into the sum :

$$z_2(t)=\check{z}_\rho(t)+\hat{z}_\rho(t),\tag{5.26}$$

where

$$\check{z}_\rho(t)=\psi_\rho(x)z_2(t)$$

and

$$\hat{z}_\rho(t)=(1-\psi_\rho(x))z_2(t).$$

The result below says that $\check{z}_\rho(t)$ can be as small as possible for ρ large enough.

Lemma 5.2. *Let $z_2(t_\eta)$ be the solution to* (5.10)-(5.13) *corresponding to $r = r_\eta$ and $t = t_\eta$. Then, there exists $\rho_\eta \geq r_\eta$ such that*

$$\|\check{z}_\rho(t)\|^2_{\mathcal{H}} \leq \frac{1}{2}\eta, \tag{5.27}$$

for every $\rho \geq \rho_\eta$, and for all $z_0 \in \beta_0$.

Proof. Along the proof c will denote any constant independent of ρ that may even be different from line to line. Let us multiply (5.10) by $\psi_\rho^2(u^c + \frac{\partial u^c}{\partial t})$ and (5.11) by $\psi_\rho^2 \frac{\partial \alpha^c}{\partial t}$ and integrate over $\mathbb{R}^3$. Summing the resulting equations, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|\psi_\rho u^c\|^2 + \|\psi_\rho \nabla u^c\|^2 + \lambda\|\psi_\rho \alpha^c\|^2 + \|\psi_\rho \nabla \alpha^c\|^2 + \left\|\psi_\rho \frac{\partial \alpha^c}{\partial t}\right\|^2\Big)\\
&+ \|\psi_\rho \nabla u^c\|^2 + \left\|\psi_\rho \frac{\partial u^c}{\partial t}\right\|^2 + \left\|\psi_\rho \frac{\partial \alpha^c}{\partial t}\right\|^2 + \left\|\psi_\rho \nabla \frac{\partial \alpha^c}{\partial t}\right\|^2\\
=&-\int_{\mathbb{R}^3}(f_r^1(x,u) - f_r^1(x,u^d))\psi_\rho^2(u^c + \frac{\partial u^c}{\partial t})dx - \int_{\mathbb{R}^3} f_r^2(x,u)\psi_\rho^2(u^c + \frac{\partial u^c}{\partial t})dx\\
&-\int_{\mathbb{R}^3} u^c \nabla \psi_\rho^2 \nabla u^c dx - \int_{\mathbb{R}^3}\frac{\partial u^c}{\partial t}\nabla\psi_\rho^2 \nabla u^c dx - \int_{\mathbb{R}^3}\frac{\partial \alpha^c}{\partial t}\nabla\psi_\rho^2\nabla\alpha^c dx\\
&-\int_{\mathbb{R}^3}\frac{\partial \alpha^c}{\partial t}\nabla\psi_\rho^2\nabla\frac{\partial \alpha^c}{\partial t}dx.
\end{aligned}$$

Since, for $|x| \geq \rho + 1$,

$$f_r^1(x,u) - f_r^1(x,u^d) = f(x,u) - f(x,u^d).$$

Then, Hölder's inequality, the embedding $V \hookrightarrow L^6(\mathbb{R}^3)$, (5.22) and (5.23) imply

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3}(f_r^1(.,u) - f_r^1(.,u^d))\psi_\rho^2 u^c dx\right|\\
\leq& c\int_{\mathbb{R}^3}(1 + m(t))\psi_\rho^2|u^c|^2 dx\\
\leq& c\|\psi_\rho u^c\|^2 + c\|m\|^2_{L^4(\mathbb{R}^3)}\|\psi_\rho u^c\|^2_{L^4(\mathbb{R}^3)}\\
\leq& c\|\psi_\rho u^c\|^2 + c\|\psi_\rho u^c\|^2_V\\
\leq& c\|\psi_\rho u^c\|^2 + c\|\psi_\rho \nabla u^c\|^2 + c\|\nabla\psi_\rho u^c\|^2\\
\leq& c\|\psi_\rho u^c\|^2 + c\|\psi_\rho \nabla u^c\|^2 + \frac{c}{\rho+1},
\end{aligned}$$

where $m(t) = |u(t)|^2 + |u^d(t)|^2$, and similarly,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3}(f_r^1(.,u) - f_r^1(.,u^d))\psi_\rho^2\frac{\partial u^c}{\partial t}dx\right|\\
\leq& c\int_{\mathbb{R}^3}(1 + m(t))\psi_\rho^2|u^c|\left|\frac{\partial u^c}{\partial t}\right|dx\\
\leq& c(1 + \|m\|^2_{L^6(\mathbb{R}^3)})\left\|\psi_\rho\frac{\partial u^c}{\partial t}\right\|\|\psi_\rho u^c\|_{L^6(\mathbb{R}^3)}\\
\leq& c\left\|\psi_\rho\frac{\partial u^c}{\partial t}\right\|^2 + c\|\psi_\rho u^c\|^2_V\\
\leq& c\|\psi_\rho u^c\|^2 + c\|\psi_\rho\nabla u^c\|^2 + \frac{1}{6}\left\|\psi_\rho\frac{\partial u^c}{\partial t}\right\|^2 + \frac{c}{\rho+1}.
\end{aligned}$$

On the other hand,

$$\left|\int_{\mathbb{R}^3} f_r^2(x,u)\psi_\rho^2 u^c dx\right| \le \|\psi_\rho f(.,0)\|^2 + \|\psi_\rho u^c\|^2,$$
$$\left|\int_{\mathbb{R}^3} f_r^2(x,u)\psi_\rho^2 \frac{\partial u^c}{\partial t} dx\right| \le \|\psi_\rho f(.,0)\|^2 + \frac{1}{6}\left\|\psi_\rho \frac{\partial u^c}{\partial t}\right\|^2.$$

Finally, on account of (5.22) and (5.24),

$$\begin{aligned}\left|\int_{\mathbb{R}^3} u^c \nabla\psi_\rho^2 \nabla u^c dx\right| &\le \frac{c}{\rho+1}\int_{\mathbb{R}^3} |u^c||\psi_\rho \nabla u^c| dx \\ &\le c\|\psi_\rho \nabla u^c\|^2 + \frac{c}{\rho+1}.\end{aligned}$$

Analogously we have

$$\left|\int_{\mathbb{R}^3} \frac{\partial u^c}{\partial t}\nabla\psi_\rho^2 \nabla u^c dx\right| \le c\|\psi_\rho \nabla u^c\|^2 + \frac{c}{\rho+1},$$
$$\left|\int_{\mathbb{R}^3} \frac{\partial \alpha^c}{\partial t}\nabla\psi_\rho^2 \nabla \alpha^c dx\right| \le c\|\psi_\rho \nabla \alpha^c\|^2 + \frac{c}{\rho+1},$$
$$\left|\int_{\mathbb{R}^3} \frac{\partial \alpha^c}{\partial t}\nabla\psi_\rho^2 \nabla \frac{\partial \alpha^c}{\partial t} dx\right| \le c\left\|\psi_\rho \frac{\partial \alpha^c}{\partial t}\right\|^2 + \frac{c}{\rho+1}.$$

Collecting all the above estimates, we then arrive at

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|\psi_\rho u^c\|^2 + \|\psi_\rho \nabla u^c\|^2 + \lambda\|\psi_\rho \alpha^c\|^2 + \|\psi_\rho \nabla\alpha^c\|^2 + \left\|\psi_\rho \frac{\partial \alpha^c}{\partial t}\right\|^2\Big) \\ &+ \frac{2}{3}\left\|\psi_\rho \frac{\partial u^c}{\partial t}\right\|^2 + \left\|\psi_\rho \nabla\frac{\partial \alpha^c}{\partial t}\right\|^2 \\ \le& c(\|\psi_\rho u^c\|^2 + \|\psi_\rho \nabla u^c\|^2 + \|\psi_\rho \nabla\alpha^c\|^2 + \left\|\psi_\rho \frac{\partial \alpha^c}{\partial t}\right\|^2) \\ &+ c(\|\psi_\rho f(.,0)\|^2 + \frac{c}{\rho+1}).\end{aligned} \tag{5.28}$$

Finally, multiplying (5.11) by $\psi_\rho^2\alpha^c$ and integrating over $\mathbb{R}^3$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|\psi_\rho \alpha^c\|^2 + \|\psi_\rho \nabla\alpha^c\|^2 + 2\int_{\mathbb{R}^3}\psi_\rho^2\alpha^c\frac{\partial \alpha^c}{\partial t}dx\Big) + \lambda\|\psi_\rho\alpha^c\|^2 \\ &+ \|\psi_\rho \nabla\alpha^c\|^2 + \int_{\mathbb{R}^3}\alpha^c\nabla\psi_\rho^2\nabla\frac{\partial \alpha^c}{\partial t}dx + \int_{\mathbb{R}^3}\alpha^c\nabla\psi_\rho^2\nabla\alpha^c dx \\ =& -\int_{\mathbb{R}^3}\psi_\rho^2\alpha^c(u^c + \frac{\partial u^c}{\partial t})dx + \left\|\psi_\rho \frac{\partial \alpha^c}{\partial t}\right\|^2.\end{aligned} \tag{5.29}$$

Analogously, we have

$$\left|\int_{\mathbb{R}^3} \alpha^c \nabla \psi_\rho^2 \nabla \frac{\partial \alpha^c}{\partial t} dx\right| \leq \frac{1}{2}\|\psi_\rho \nabla \frac{\partial \alpha^c}{\partial t}\|^2 + \frac{c}{\rho+1},$$
$$\left|\int_{\mathbb{R}^3} \alpha^c \nabla \psi_\rho^2 \nabla \alpha^c dx\right| \leq c\|\psi_\rho \nabla \alpha^c\|^2 + \frac{c}{\rho+1},$$
$$\left|\int_{\mathbb{R}^3} \psi_\rho^2 \alpha^c \frac{\partial u^c}{\partial t} dx\right| \leq \frac{1}{6}\left\|\psi_\rho \frac{\partial u^c}{\partial t}\right\|^2 + c\|\psi_\rho \alpha^c\|^2,$$
$$\left|\int_{\mathbb{R}^3} \psi_\rho^2 \alpha^c u^c dx\right| \leq \|\psi_\rho \alpha^c\|^2 + \|\psi_\rho u^c\|^2.$$

In light of the above estimates, (5.29) becomes

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|\psi_\rho \alpha^c\|^2 + \|\psi_\rho \nabla \alpha^c\|^2 + 2\int_{\mathbb{R}^3} \psi_\rho^2 \alpha^c \frac{\partial \alpha^c}{\partial t} dx\Big) \\
\leq& c(\|\psi_\rho u^c\|^2 + \|\psi_\rho \alpha^c\|^2 + \|\rho_\rho \nabla \alpha^c\|^2 + \left\|\frac{\partial \alpha^c}{\partial t}\right\|^2) + \frac{1}{6}\left\|\psi_\rho \frac{\partial u^c}{\partial t}\right\|^2 \\
&+ \frac{1}{2}\|\psi_\rho \nabla \frac{\partial \alpha^c}{\partial t}\|^2 + \frac{c}{\rho+1}.
\end{aligned} \tag{5.30}$$

Summing up (5.28) and (5.30) and setting

$$\begin{aligned}
\Theta(t) =& \|\psi_\rho u^c\|^2 + \|\psi_\rho \nabla u^c\|^2 + (1+\lambda)\|\psi_\rho \alpha^c\|^2 + 2\|\psi_\rho \nabla \alpha^c\|^2 + \left\|\psi_\rho \frac{\partial \alpha^c}{\partial t}\right\|^2 \\
&+ 2\int_{\mathbb{R}^3} \psi_\rho^2 \alpha^c \frac{\partial \alpha^c}{\partial t} dx,
\end{aligned}$$

we end up with the differential inequality

$$\frac{d}{dt}\Theta(t) + \left\|\psi_\rho \frac{\partial u^c}{\partial t}\right\|^2 + \|\psi_\rho \nabla \frac{\partial \alpha^c}{\partial t}\|^2 \leq c\Theta(t) + c\Gamma(\rho),$$

where

$$\Gamma(\rho) = \|\psi_\rho f(.,0)\|^2 + \frac{c}{\rho+1}.$$

In particular,

$$\frac{d}{dt}\Theta(t) \leq c\Theta(t) + c\Gamma(\rho).$$

Since $\Theta(0) = 0$, Gronwall's Lemma applied on $[0, t_\eta]$ yields

$$\Theta(t_\eta) \leq c t_\eta e^{c t_\eta}\Gamma(\rho).$$

Notice that

$$\|\psi_\rho u^c\|_V^2 + \|\psi_\rho \alpha^c\|_V^2 + \left\|\psi_\rho \frac{\partial \alpha^c}{\partial t}\right\|^2 \leq 2\Theta(t_\eta) + \int_{\mathbb{R}^3} |\psi_\rho|^2 |\alpha^c|^2 dx.$$

From (5.22) and (5.24), we then have

$$\int_{\mathbb{R}^3} |\psi_\rho|^2 |\alpha^c|^2 dx \leq \frac{c}{\rho+1}.$$

Hence, we conclude that

$$\|\tilde{z}(t_\eta)\|_{\mathcal{H}}^2 \leq ct_\eta e^{ct_\eta}\Gamma(\rho) + \frac{c}{\rho+1}.$$

The constant c being independent of ρ, one can choose ρ large enough such that

$$ct_\eta e^{ct_\eta}\Gamma(\rho) + \frac{c}{\rho+1} \leq \frac{1}{2}\eta,$$

which gives the expected result. □

Before stating the next Lemma which provides the compact part in the decomposition of the solution, we are going to introduce suitable Sobolev spaces.

Let $\mathcal{B} \subset \mathbb{R}^3$ be a regular bounded domain.

Let A be the operator defined in $L^2(\mathcal{B})$ by :

$$A = -\Delta, \text{ with domain } \mathcal{D}(A) = H^2(\mathcal{B}) \cap H_0^1(\mathcal{B}).$$

We consider the family of Hilbert spaces $\mathcal{D}(A^{s/2}), s \in \mathbb{R}$, with the scalar products and norms given by :

$$\langle ., . \rangle_{\mathcal{D}(A^{s/2})} = \langle A^{s/2}., A^{s/2}. \rangle$$

and

$$\|.\|_{\mathcal{D}(A^{s/2})} = \|A^{s/2}.\|,$$

respectively.

Remark 5.2. By definition, one has $\mathcal{D}(A^0) = L^2(\mathcal{B})$ and $\mathcal{D}(A^{1/2}) = H^1(\mathcal{B})$.

Now set

$$\mathcal{H}(\mathcal{B}) = \mathcal{D}(A) \times \mathcal{D}(A) \times \mathcal{D}(A^{1/2}).$$

Lemma 5.3. *Let $z_2(t_\eta)$ be the solution to* (5.10)-(5.13) *corresponding to $r = r_\eta$. Let $\rho > \rho_\eta$ be fixed, and one considers the ball $B_\rho = \{x \in \mathbb{R}^3 : |x| \leq 2\rho + 3\}$. Then, there exists a constant $k_{\eta,\rho} > 0$ such that*

$$\|\hat{z}_\rho(t_\eta)\|_{\mathcal{H}(B_\rho)} \leq k_{\eta,\rho},$$

for every $z_0 \in \beta_0$.

Proof. Note that $\hat{z}_\rho$ vanishes for $|x| \geq 2\rho+2$, hence its restriction to B_ρ belongs to $H_0^1(B_\rho)$ for every $t > 0$. Set $A := -\Delta$ the operator with domain $\mathcal{D}(A) = H^2(B_\rho) \cap H_0^1(B_\rho)$. Noting that

$$\begin{aligned}
-A\hat{u^c} =&(1-\psi_\rho)\Delta u^c - 2\nabla\psi_\rho\nabla u^c - \Delta\psi_\rho u^c,\\
-A\hat{\alpha^c} =&(1-\psi_\rho)\Delta \alpha^c - 2\nabla\psi_\rho\nabla \alpha^c - \Delta\psi_\rho \alpha^c,\\
-A\frac{\partial\hat{\alpha^c}}{\partial t} =&(1-\psi_\rho)\Delta \frac{\partial\alpha^c}{\partial t} - 2\nabla\psi_\rho\nabla \frac{\partial\alpha^c}{\partial t} - \Delta\psi_\rho \frac{\partial\alpha^c}{\partial t}.
\end{aligned}$$

It suffices to show that

$$\|u^c(t_\eta)\|_W^2 + \|\alpha^c(t_\eta)\|_W^2 + \|\frac{\partial\alpha^c}{\partial t}(t_\eta)\|_V^2 \leq k_{\eta,\rho},$$

for some $k_{\eta,\rho} > 0$ independent of initial data. Hence, considering (5.22) it just remains to prove that

$$\|\Delta u^c(t_\eta)\|^2 + \|\Delta\alpha^c(t_\eta)\|^2 + \left\|\nabla\frac{\partial\alpha^c}{\partial t}(t_\eta)\right\|^2 \leq k_{\eta,\rho}. \tag{5.31}$$

To do so, we multiply (5.10) by $-\Delta\frac{\partial u^c}{\partial t}$ and integrate over $\mathbb{R}^3$. We get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\Delta u^c\|^2 + \left\|\nabla\frac{\partial u^c}{\partial t}\right\|^2 + \int_{\mathbb{R}^3}(\nabla f_r^1(.,u) - \nabla f_r^1(.,u^d) + \nabla f_r^2(.,u))\nabla\frac{\partial u^c}{\partial t}dx\\ &= -\int_{\mathbb{R}^3}\frac{\partial\alpha^c}{\partial t}\Delta\frac{\partial u^c}{\partial t}dx.\end{aligned} \tag{5.32}$$

Now multiplying (5.11) by $-\Delta\frac{\partial\alpha^c}{\partial t}$ and integrating over $\mathbb{R}^3$. We obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\lambda\|\nabla\alpha^c\|^2 + \|\Delta\alpha^c\|^2 + \left\|\nabla\frac{\partial\alpha^c}{\partial t}\right\|^2) + \left\|\nabla\frac{\partial\alpha^c}{\partial t}\right\|^2 + \left\|\Delta\frac{\partial\alpha^c}{\partial t}\right\|^2\\ &= \int_{\mathbb{R}^3}(u^c + \frac{\partial u^c}{\partial t})\Delta\frac{\partial\alpha^c}{\partial t}dx.\end{aligned} \tag{5.33}$$

The sum of (5.32) and (5.33) yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\Delta u^c\|^2 + \lambda\|\nabla\alpha^c\|^2 + \|\Delta\alpha^c\|^2 + \left\|\nabla\frac{\partial\alpha^c}{\partial t}\right\|^2) + \left\|\nabla\frac{\partial u^c}{\partial t}\right\|^2\\ &+ \left\|\nabla\frac{\partial\alpha^c}{\partial t}\right\|^2 + \left\|\Delta\frac{\partial\alpha^c}{\partial t}\right\|^2\\ &= -\int_{\mathbb{R}^3}(\nabla f_r^1(.,u) - \nabla f_r^1(.,u^d) + \nabla f_r^2(.,u))\nabla\frac{\partial u^c}{\partial t}dx + \int_{\mathbb{R}^3}u^c\Delta\frac{\partial\alpha^c}{\partial t}dx.\end{aligned} \tag{5.34}$$

Hölder's inequality implies

$$\left|\int_{\mathbb{R}^3}u^c\Delta\frac{\partial\alpha^c}{\partial t}dx\right| \leq \frac{1}{2}\|u^c\|^2 + \frac{1}{2}\left\|\Delta\frac{\partial\alpha^c}{\partial t}\right\|^2.$$

From the definition of $f^i,\ i = 1,2$, we write

$$f_r^1(.,u) - f_r^1(.,u^d) + f_r^2(.,u) = f(.,u) - f(.,u^d) - (c_2 + M)u^d\varphi_r^2 - f(.,0)\varphi_r^2,$$

and then

$$\begin{aligned}&\nabla f_r^1(.,u) - \nabla f_r^1(.,u^d) + \nabla f_r^2(.,u)\\ =&\nabla f(.,u) - \nabla f(.,u^d) - (c_2 + M)\nabla u^d\varphi_r^2 - \nabla f(.,0)\varphi_r^2\\ =&f'(.,u)\nabla u - f'(.,u^d)\nabla u^d - (c_2 + M)\nabla u^d\varphi_r^2 + \nabla_x f(x,u) - \nabla_x f(x,u^d)\\ &- \nabla_x f(x,0)\varphi_r^2\\ =&(f'(.,u) - f'(.,u^d))\nabla u^d + f'(.,u)\nabla u^c + \nabla_x f(x,u) - \nabla_x f(x,u^d)\\ &- (c_2 + M)\nabla u^d\varphi_r^2 - \nabla_x f(x,0)\varphi_r^2.\end{aligned}$$

We then deduce from (2.1)-(2.3) and (2.8), with $\gamma = 1$, Hölder's inequality and the estimate (5.22) that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3}(\nabla f_r^1(.,u) - \nabla f_r^1(.,u^d) + \nabla f_r^2(.,u))\nabla\frac{\partial u^c}{\partial t}dx\right| \\
\leq& c\int_{\mathbb{R}^3}(1+|u|+|u^d|)|u^c|\left|\nabla\frac{\partial u^c}{\partial t}\right|dx + c\int_{\mathbb{R}^3}(1+|u|^2)\left|\nabla\frac{\partial u^c}{\partial t}\right|dx \\
\leq& c + \frac{1}{2}\left\|\nabla\frac{\partial u^c}{\partial t}\right\|^2.
\end{aligned}$$

Collecting the above estimates, we are led to an inequality of the form

$$\begin{aligned}
&\frac{d}{dt}(\|\Delta u^c\|^2 + \|\Delta\alpha^c\|^2 + \left\|\nabla\frac{\partial\alpha^c}{\partial t}\right\|^2) + \left\|\nabla\frac{\partial u^c}{\partial t}\right\|^2 + \left\|\nabla\frac{\partial\alpha^c}{\partial t}\right\|^2 \\
&+ \left\|\Delta\frac{\partial\alpha^c}{\partial t}\right\|^2 \leq c.
\end{aligned}$$

In particular,

$$\frac{d}{dt}(\|\Delta u^c\|^2 + \lambda\|\nabla\alpha^c\|^2 + \|\Delta\alpha^c\|^2 + \left\|\nabla\frac{\partial\alpha^c}{\partial t}\right\|^2) \leq c.$$

Since, $u^c(0) = \alpha^c(0) = \frac{\partial\alpha^c}{\partial t}(0) = 0$, we conclude owing to Gronwall's Lemma that

$$\|\Delta u^c(t_\eta)\|^2 + \|\Delta\alpha^c(t_\eta)\|^2 + \left\|\nabla\frac{\partial\alpha^c}{\partial t}(t_\eta)\right\|^2 \leq c,$$

where c depends on η, which completes the proof. $\square$

Proof of Theorem 5.1.

The proof is carried out via a straightforward application of Theorem 5.2. The existence of a bounded absorbing set β_0 for $S(t)$ is guaranteed by Corollary 4.1. It remains to check the second point of the Theorem 5.2. For this, we choose, for all $\eta > 0$, r_η and t_η as above. Let z_2 be the solution of (5.10)-(5.13) corresponding to $r = r_\eta$. From Lemma 5.3, we construct the set

$$\mathcal{K}_\eta = \bigcup_{z_0\in\beta_0}(\hat{z}_{\rho\eta}(t_\eta)),$$

which turns out to be compact in $\mathcal{H}$. We conclude owing to Lemma 5.1 and 5.2 that

$$\delta_{\mathcal{H}}(S(t_\eta)\beta_0, \mathcal{K}_\eta) \leq \eta.$$

Hence, the ω-limit set of β_0, namely

$$\omega(\beta_0) = \bigcap_{s\geq 0}\overline{\bigcup_{t\geq s}S(t)\beta_0},$$

is the global attractor for $S(t)$ (see also, Miranville & Zelik [21] and Hale [15]). Since β_0 is connected, then $\mathcal{A}$ is also. $\square$

Remark 5.3. (a) We emphasize that the study of the dimension of the global attractor seems very complicated or even impossible in that case. Thus, it is not possible to describe the dynamics of the system in terms of a finite number of physical parameters (see for example, Babin & Vishik [8], Eden etc. [14] and Miranville & Zelik [22] for more discussions on this subject).

(b) We can consider the same problem with the non linearity f independent of x, namely, $f = f(s)$. We then are led to a problem which is somewhat simpler to solve. In that case, assumption (2.5) holds on the whole $\mathbb{R}^3$ and reasoning as in Lemma 5.1, one gets that the attractor is the set $\{0\}$ (see [9, Chapitre 4]).

Acknowledgements

This article was initiated by Monica Conti while the author was visiting Politecnico di Milano in Italy in the framework of the project Galilée between Milan (Italy) and Poitiers (France). He wishes to thank Politecnico di Milano for its warm welcome. The author wish specially to thank Alain Miranville for suggesting the problem. The author is indebted to Monica Conti, who read and constructively criticized the original manuscript.

References

[1] P. W. Bates and S. M. Zheng, *Inertial manifolds and inetial sets for the phase-field equations*, J. Dynam. Differential equations, 4 (1992), 375-398.

[2] D. Brochet, D. Hilhorst and A. Novick-Cohen, *Finite-dimensional exponential attractor for a model for order-desorder and phase separation*, Appl. Math. Lett., 7 (1992), 83-87.

[3] D. Brochet and D. Hilhorst, *Universal attractor and inertial sets for the phase-field model*, Appl. Math. Lett., 4 (1991), 59-63.

[4] D. Brochet, D. Hilhorst and X. Chen, *Finite-dimensional exponential attractor for the phase field model*, Appl. Anal., 49 (1993),197-212.

[5] A. V. Babin and M. I. Vishik, *Attractors of partial differential evolution equations in unbouded domain*, Proc. Roy. Soc. Edinburgh Sect. A, 116 (1990), 221-243.

[6] V. Belleri and V. Pata, *Attractors for semilinear strogly damped wave equations on* $\mathbb{R}^3$, Discrete Contin. Dynam. Systems., 7 (2001), 719-735.

[7] V. Belleri, *Analisi asintoticadi equazioni delle onde fortamente smortzate in uno dominio illimitato*, Master thesis, Politecnico di Milano, Italia. April 2005.

[8] A. V. Babin and M. I. Vishik, *Attractors for evolution equations*, volume 25 of studies in Mathematics ans its Applications, North-Holland, Publishing Co., Amsterdam, 1992. Translated ans revised from 1989 Russian original by Babin.

[9] B. D. Bangola, *Etude de modèles de Champs de phase de type Caginalp*, Ph.D. thesis, Université de Poitiers, France. to appear.

[10] G. Caginalp, *An analysis of a phase-field model of a free boundary*, Arch. Rational Mech. Anal., 92 (1986), 205-245.

[11] L. Cherfils and A. Miranville, *On the Caginalp system with dynamics boundary conditions and singular potentials*, Appl. Math., 54 (2009), 89-115.

[12] M. Conti and G. Mola, *Attractors for a phase field model on* $\mathbb{R}^3$, Adv. Math. Sci. Appl., 15 (2005), 527-543.

[13] M. Conti, V. Pata and M. Squassima, *Strongly damped wave equations on* $\mathbb{R}^3$ *with critical nonlinearities*, Commun. Appl. Anal., 9 (2005), 161-176.

[14] A. Eden, C. Foias, B. Nicolenko, and R. Temam, *Exponential attractors for dissipative evolution equations*, volume 37 of RAM : Research in Applied Mathematics. Msson, Paris, 1994.

[15] J. K. Hale, *Asyptotic behavior of dissipative systems*, volume 25 of Mathematical surveys and Monographs, Mathematical American Society, Providence, RI, 1988.

[16] A. Miranville and R. Quintanilla, *A generalization of the Caginalp phase-field system based on the Maxwell-Cattaneo law*, Nonlinear Anal., 71 (2009), 2278-2298.

[17] A. Miranville and R. Quintanilla, *A Caginalp phase-field system with nonlinear coupling*, Nonlinear Anal. Real World Appl., 11 (2010), 2849-2861.

[18] A. Miranville and R. Quintanilla, *Some generaizations of the Caginalp phase-field system*, Appl. Anal., 88 (2009), 897-894.

[19] A. Miranville and R. Quintanilla, *A phase-field model based on the three-phase-lag heat conduction*, Appl. Math. Optim., 63 (2011), 133-150.

[20] F. Morillas and J. Valero, *Asymptotic compactness and attractors for phase-field equation in* $\mathbb{R}^3$, Set-Valued Anal., 16 (2008), 861-897.

[21] A. Miranville and S. Zelik, *Doubly nonlinear Cahn-Hilliard-Gurtin equations*, Hokkaido Math. J., 38 (2009), 315-360.

[22] A. Miranville and S. Zelik, *Attractors for dissipative partial differential equations in bounded and unbounded domains*, In Handbook of differential equations : evolutionary equations. Vol. IV, Handb. Diff. Equ., 4 (2008), 103-200.

[23] V. Pata, *Attractors for a damped wave equation on* $\mathbb{R}^3$ *with linear memory*, Math. Methods Appl. Sci., 23 (2000), 633-653.

[24] R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics*, volume 68 of Applied Mathematical Sciences, Springer-Verlag, New York, second edition, 1997.

[25] S. V. Zelik, *The attractor for a nonlinear hyperbolic equation in the unbounded domain. Discrete Contin*, Dynam. Systemes, 7 (2001), 593-641.