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ON THE EQUIVALENCE OF DIFFERENTIAL SYSTEMS*

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Abstract In this article, firstly, we construct some nonlinear differential systems which are equivalent to some known systems. Secondly, we discuss the equivalence between some linear differential systems in a different method. And then we apply the obtained results to the study of the qualitative properties of these systems simultaneously.

Keywords Reflecting function, equivalent systems, periodic solutions.

MSC(2000) 34A12.

1. Introduction

In this paper, we deal with differential system

$$x' = X(t,x), \quad t \in R, \quad x \in D \subset R^n. \tag{1.1}$$

We assume that system (1.1) has continuously differentiable right-hand sides and which has a general solution $\phi(t;t_0,x_0)$. To study the property of the solutions of differential system (1.1) is not only important for the theory of ordinary differential equation but is also for practical significance. As a rule we cannot integrate system (1.1) by quadrature and study properties of the solution directly. In this case we have to look for other methods of studying system (1.1). In [4] there was an elaborated method of the reflecting function which give us an opportunity to do this. The reflecting function for system (1.1) is defined in some region near the hyperplane $t = 0$ by the formula $F(t,x) := \phi(-t;t,x)$. If system (1.1) is 2ω-periodic with respect to t, then $F(-\omega,x)$ is its Poincaré mapping [4, 1]. Therefore, the solution $\phi(t;-\omega,x_0)$ which can be extended to $[-\omega,\omega]$ is 2ω-periodic if and only if $F(-\omega,x_0) = x_0$.

The reflective function $F(t,x)$ of system (1.1) can be found sometimes even for the case where the system (1.1) cannot be integrated by quadrature. For example, every system (1.1) for which $X(-t,x) = -X(t,x)$ has an reflective function given by the formula $F(t,x) \equiv x$. We know this due to the following property. A differentiable function $F(t,x)$ is the reflective function of system (1.1) if and only if the following basic relation

$$F'_t + F'_x X(t,x) + X(-t,F) = 0, \quad F(0,x) = x \tag{1.2}$$

holds.

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So if we can find the solution of the basic relation (1.2), then we can find the initial data for periodic solutions of (1.1) and investigate the character of the stability for those solutions.

If the system

$$x' = Y(t, x) \tag{1.3}$$

has the same reflective function $F(t, x)$ as the system (1.1), then $Y(0, x) = X(0, x)$ and systems

$$\begin{aligned} &F'_t + F'_x X(t, x) + X(-t, F) = 0, \\ &F'_t + F'_x Y(t, x) + Y(-t, F) = 0, \\ &F(0, x) = x \end{aligned}$$

are compatible. At this moment, we call system (1.3) is equivalent to system (1.1).

To check whether the above systems are compatible we can use the Frobenius theorem [2]. Doing this in practice, however, is a very hard task.

If we can neither solve the system (1.1) nor the problem (1.2), then it is good enough to construct any system (1.3) which is equivalent to (1.1). To do this, sometimes we can use:

Lemma 1.1. *[5] Let the vector functions $\triangle_k$ $(k = 1, 2, ..., m)$ be solutions of the equation*

$$\triangle'_t + \triangle'_x X(t, x) = X'_x \triangle \tag{1.4}$$

and $\alpha_k(t)(k = 1, 2, ..., m)$ be any scalar continuous odd functions. Then every system of the form

$$x' = X(t, x) + \sum_{k=1}^{m} \alpha_k(t) \triangle_k(t, x) \tag{1.5}$$

is equivalent to system (1.1) (here m is any natural number or even $m = \infty$). So if we find some solutions of equation (1.4), we can construct system (1.5), which has the same reflective function as system (1.1).

Usually, however, we cannot find out the general equation (1.4). In this paper, first, we will find out some solutions of (1.4), and construct some nonlinear differential systems which are equivalent to some known systems. Secondly, we will discuss in a different method the equivalence between the simplest systems and some linear differential systems for which the identity (1.4) does not necessarily hold.

Other results concerning the reflective function and its applications can be found in works of Mironenko [4, 5], Musafirov [6] and others [3, 7, 8, 9].

2. Main results

Now, we consider linear system

$$x' = A(t)x, \quad t \in R, \quad x \in R^n, \tag{2.1}$$

where $A(t)$ is a continuously matrix function for $t \in R$.

Let

$$u(t, x) = (u_1(t, x), u_2(t, x), ..., u_n(t, x)),$$

where $u_i(t, x) = c_i$ are independent first integrals of (2.1), $c_i(i = 1, 2, ..., n)$ are constants, $\Phi(t)$ is a fundamental matrix of (2.1).

Lemma 2.1. *Let*

$$\Delta(t,x) = \Phi(t)S(u(t,x)),$$

where $S(u)$ is an arbitrary continuously differentiable vector function. Then it satisfies the relation

$$\Delta'_t + \Delta'_x A(t)x - A(t)\Delta = 0. \tag{2.2}$$

Proof. For $u'_t + u'_x A(t)x = 0$, $\Phi'(t) = A(t)\Phi(t)$,

$$\begin{aligned}&\Delta'_t + \Delta'_x A(t)x - A(t)\Delta\\ =&\Phi'(t)S(u) + \Phi(t)S'(u)u'_t + \Phi(t)S'(u)u'_x A(t)x - A(t)\Phi(t)S(u) \equiv 0.\end{aligned}$$

□

Theorem 2.1. *Let $\alpha_k(t)$ be any scalar continuous odd functions, and*

$$\Delta_k = \Phi(t)S_k(u),\ \ S_k(u)(k = 1, 2, ..., m)$$

be any continuously differentiable vector functions. Then the reflective function of system

$$x' = A(t)x + \sum_{k=1}^{m} \alpha_k(t)\Phi(t)S_k(u) \tag{2.3}$$

is the same as system (2.1).

Besides this, if system (2.3) *is a 2ω-periodic system. Then its solution $x(t)$ defined on interval $[-\omega, \omega]$ is 2ω-periodic, if and only if $y(t)$ $(y(-\omega) = x(-\omega))$ is 2ω-periodic solution of* (2.1).

Proof. By lemma (2.1), $\Delta_k(k = 1, 2, .., m)$ are the solutions of equation (2.2), and according to Lemma (1.1), the reflective function of (2.3) is the same as system (2.1). By [4], the system (2.3) is equivalent to system (2.1), then their reflective functions coincide, which yields that their Poincaré mappings coincide. It implies that the present theorem is true.

□

Similarly, we could obtain the following results:

Corollary 2.1. *Let $\Delta_k = \Phi(t)D_k(u)\Phi^{-1}(t)x$, where $D_k(u)$ is an arbitrary continuously differentiable matrix function. Then the reflective function of system*

$$x' = A(t)x + \sum_{k=1}^{m} \alpha_k(t)\Phi(t)D_k(u)\Phi^{-1}(t)x$$

is the same as system (2.1).

Corollary 2.2. *The reflective function of system*

$$x' = Ax + \sum_{k=1}^{m} \alpha_k(t)e^{At}D_k(u)e^{-At}x$$

is the same as system $x' = Ax$, where A is a constant matrix.

Remark 2.1. This result is same as the theorem (2.1) of paper [3].

Corollary 2.3. *Let $\alpha_k(t)$ be any scalar continuous odd functions, and*

$$\Delta_k = S_k(u)X(x),\ \ S_k(u)(k=1,2,...,m)$$

be arbitrary continuously differentiable scalar functions. Then the reflective function of system

$$x' = X(x) + \sum_{k=1}^{m} \alpha_k(t) S_k(u) X(x) \tag{2.4}$$

is the same as system $X' = X(x)$. Where $u(t,x) = (u_1(t,x), u_2(t,x), ..., u_n(t,x))$, $u_i(t,x) = c_i(i=1,2,...,n)$ are the independent first integrals of $x' = X(x)$.

Corollary 2.4. *Suppose that $u_i(t,x_i) = c_i(i=1,2,...,n)$ are the first integrals of equations $x_i' = f_i(x_i)g_i(t)(i=1,2,...,n)$. Then the reflective function*

$$x' = X(t,x) + \sum_{k=1}^{m} \alpha_k(t)\Delta_k(t,x)$$

is the same as equation $x' = X(t,x)$. Where

$$\begin{aligned}
&\Delta_k = (\Delta_{k1}, \Delta_{k2}, ..., \Delta_{kn})^T, \\
&X(t,x) = (f_1(x_1)g_1(t), f_2(x_2)g_2(t), ..., f_n(x_n)g_n(t))^T, \\
&\Delta_{ki} = S_{ki}(u_i(t,x_i))f_i(x_i),\ \ \text{and } \alpha_k(t)
\end{aligned}$$

are arbitrary scalar odd functions, $g_k(t)$ and $f_k(x)$ and $S_{ki}(u)$ are arbitrary scalar continuously differentiable functions$(i=1,2,..,n,\ \ k=1,2,...,m)$.

Remark 2.2. If all the equivalent systems in Corollary (2.1)-(2.4) are 2ω-periodic with respect to t, then the qualitative properties of their 2ω-periodic solutions coincide.

Example 2.1. The reflective function of system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{2.5}$$

is the same as system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \sum_{k=1}^{m} \alpha_k(t) e^{\sin t} \begin{pmatrix} S_1 \cos\beta + S_2 \sin\beta \\ -S_1 \sin\beta + S_2 \cos\beta \end{pmatrix}, \tag{2.6}$$

where $\beta = 1 - \cos t$, $\alpha_k(t)(k=1,2,...,m)$ are arbitrarily continuous scalar odd functions, $S_1 = S_1(u_, u_2)$, $S_2 = S_2(u_1, u_2)$ are arbitrarily continuously differentiable functions, $u_1 = e^{-\sin t}(x\cos\beta - y\sin\beta)$, $u_2 = e^{-\sin t}(x\sin\beta + y\cos\beta)$.

Furthermore, if $\alpha_k(t)(k=1,2,..,m)$ are 2π-periodic odd functions, then all the solutions (2.6) defined on interval $[-\pi,\pi]$ are 2π-periodic, which is obtained by the fact that the reflective function of system (2.5) is $F(t,x,y) = (e^{-2\sin t}x, e^{-2\sin t}y)^T$ and $F(-\pi,x,y) \equiv (x,y)^T$.

This example satisfies all the conditions of Theorem (2.1).

The identity (1.4) only is a sufficient condition for system (1.5) to be equivalent to system (1.1). There are some systems in the form (1.5) which are equivalent to (1.1), but for which the identity (1.4) is not valid. For example: $x' = \frac{1}{2}x^2$ is equivalent to equation $x' = \frac{x^2}{2+2tx+x^2} = \frac{1}{2}x^2 - \frac{t}{2}\Delta$ with reflective function $F = \frac{x}{1+tx}$, but $\Delta = \frac{x^3(2+tx)}{2+2tx+x^2}$ does not satisfy the relation (1.4).

In the following part, we will study in a different method the equivalence between the simple systems (or the simplest systems)and some linear systems.

We denote $A := A(t)$, $\bar{A} := A(-t)$, $F := F(t)$. The notation " $detA(t) \neq 0$ " means that in some deleted neighborhood of $t = 0$ and $\mid t \mid$ being small enough $detA(t)$ is different from zero.

Definition 2.1. [4] For a continuous differentiable function $F(t,x)$ which satisfies that $F(-t, F(t,x)) = x$, $F(0,x) = x$, if $X(t,x) = -\frac{1}{2}{F'_x}^{-1}(t,x)F'_t(t,x)$, then the system (1) is called **the simple system**.

By literature [4], if $F(t,x)$ is the reflective function of linear system (2.1), then $F(t,x) = F(t)x$, and we call $F(t)$ **reflective matrix** of (2.1). Thus, for an arbitrary differentiable matrix $F(t)$ which is a reflective matrix of linear system (2.1) if and only if

$$F' + FA + \bar{A}F = 0, \quad F(0) = E.$$

Theorem 2.2. *If system* (2.1) *is a simple system, then system* (2.1) *is equivalent to system*

$$x' = Ax + \sum_{k=1}^{m} \alpha_k(t)A^k x, \tag{2.7}$$

where $\alpha_k(t)(k = 1, 2, ..., m)$ *are arbitrary continuous scalar odd functions.*

Proof. Suppose that F is the reflective matrix of the simple system (2.1). Then $A = -\frac{1}{2}F^{-1}F'$ and $FA = \bar{A}F$ and $F' = -FA - \bar{A}F$. Therefor,

$$\begin{aligned} &F' + F(A + \sum_{k=1}^{m} \alpha_k(t)A^k) + (\bar{A} + \sum_{k=1}^{m} \alpha_k(-t)\bar{A}^k)F \\ =&\sum_{k=1}^{m} \alpha_k(t)(FA^k - \bar{A}^k F) = 0, \end{aligned}$$

i.e., F is a reflective matrix of (2.7) too. □

Theorem 2.3. *Suppose that system* (2.1) *is a simple system, matrix functions* $D_j(j = 1, 2, ..., l)$ *satisfy one of the following relations*

$$D' = AD - DA + \sum_{k=1}^{m} \alpha_k(t)A^k,$$

$$D' = AD - DA + \sum_{k=1}^{m} \alpha_k(t)D^k.$$

Then the system

$$x' = Ax + \sum_{j=1}^{l} \beta_j(t)D_j^k x \tag{2.8}$$

is equivalent to system (2.1), *where* $\alpha_k(t)(k = 1, 2, ..., m)$ *and* $\beta_j(t)(j = 1, 2, ..., l)$ *are arbitrary continuous scalar odd functions.*

Proof. To prove the equivalence between systems (2.1) and (2.8), only need to check that

$$FD_j = \bar{D}_j F, \quad j = 1, 2, ..., l.$$

Let denote $U = FD_j - \bar{D}_j F$. Then $U(0) = 0$,

$$U' = F(D_j' - AD_j + D_j A) + (\bar{D}_j' - \bar{A}\bar{D}_j + \bar{D}_j \bar{A}) - UA - \bar{A}U.$$

If $D_j' = AD_j - D_j A + \sum_{k=1}^m \alpha_k(t) A^k$, then

$$U' = -UA - \bar{A}U, U(0) = 0.$$

If $D_j' = AD_j - D_j A + \sum_{k=1}^m \alpha_k(t) D_j^k$, then

$$U' = -UA - \bar{A}U + \sum_{k=1}^{m} \alpha_k(t) \sum_{i+j=k-1} \bar{A}^i U A^j, \quad U(0) = 0.$$

By the uniqueness of solution of the initial problem of the linear differential system, it follows $U \equiv 0$. Thus F is a reflective matrix of (2.8) too. □

Example 2.2. System

$$x' = -\frac{1}{2}\begin{pmatrix} c + s^3 & (c - s^2 + s^3)e^c \\ -(c + s^2 + s^3)e^{-c} & -c - s^3 \end{pmatrix} x$$

is a simple system with reflective matrix $F = \begin{pmatrix} 1+s & se^c \\ -se^{-c} & 1-s \end{pmatrix}$, where $s = \sin t$, $c = \cos t$. By theorem (2.2), it is equivalent to system

$$\begin{aligned} x' = & -\frac{1}{2}\begin{pmatrix} c + s^3 & (c - s^2 + s^3)e^c \\ -(c + s^2 + s^3)e^{-c} & -c - s^3 \end{pmatrix} x \\ & + \sum_{k=1}^{m} \alpha_k(t)(-\frac{1}{2})^k \begin{pmatrix} c + s^3 & (c - s^2 + s^3)e^c \\ -(c + s^2 + s^3)e^{-c} & -c - s^3 \end{pmatrix}^k x, \end{aligned}$$

where $\alpha_k(t)(k = 1, 2, ..., m)$ are arbitrary continuous differentiable scalar odd functions. Besides, if $\alpha_k(t + 2\pi) = \alpha_k(t)(k = 1, 2, ..., m)$ then all solutions defined on interval $[-\pi, \pi]$ of the above equivalent systems are 2π-periodic.

Definition 2.2. [4] For a continuously differentiable function $F(t, x)$ which satisfies that $F(-t, F(t, x)) = x$, $F(0, x) = x$, if $X(t, x) = -(F_x'(t, x)+E)^{-1} F_t'(t, x)$, then the system (1.1) is called **the simplest system (SS)** with reflective function $F(t, x)$.

Meanwhile, linear system (2.1) is the **SS** with reflective function $F(t, x) = F(t)x$, we also call this linear system is **SS** with reflective matrix $F(t)$.

By literature [4], we know that if the system (1.1) is the **SS** with reflective function $F(t, x)$, then $X(t, x) = X(-t, F(t, x))$. Conversely, if this identity is valid and its solution F satisfies the relation (1.2), then the system (1.1) is the **SS** with reflective function $F(t, x)$.[4]

Remark 2.3. If $A(t) + A(-t) = 0$ and $A(t) \neq 0$, then the system (2.1) is not the **SS**.

Theorem 2.4. *If* $det(A) \neq 0$ *and*

$$A + A'A^{-1} + \bar{A} + \bar{A}'\bar{A}^{-1} = 0, \quad \lim_{t\to 0} \bar{A}^{-1}A = E, \tag{2.9}$$

then the system (2.1) *is the* **SS** *with reflective matrix* $F = \bar{A}^{-1}A$.

Proof. Since $F = \bar{A}^{-1}A$, $A = \bar{A}F$, $A' = \bar{A}'F + \bar{A}F'$. Thus,

$$\bar{A}(F' + FA + \bar{A}F) = (A + A'A^{-1} + \bar{A} + \bar{A}'\bar{A}^{-1})A = 0,$$

i.e., $F = \bar{A}^{-1}A$ is a reflective matrix of (2.1). Then, the system (2.1) is the **SS** with reflective matrix $F = \bar{A}^{-1}A$. □

From this theorem implies the following conclusions easily.

Corollary 2.5. *If* $det(A) \neq 0$, *and*

$$A' + A^2 = \sum_{j=1}^{n} \alpha_j(t)(S_j(A) + S_j(\bar{A}))A, \quad \lim_{t\to 0} \bar{A}^{-1}A = E,$$

then the system (2.1) *is the* **SS** *with reflective matrix* $F = \bar{A}^{-1}A$, *where* $\alpha_j(t)(j = 1, 2, ..., n)$ *are arbitrary scalar odd functions,* $S_j(j = 1, 2, ..., n)$ *are arbitrary differentiable functions.*

Corollary 2.6. *If* $det(A) \neq 0$, *and*

$$A' + A^2 = (S(t, A) - S(-t, \bar{A}))A, \quad \lim_{t\to 0} \bar{A}^{-1}A = E,$$

then the system (2.1) *is the* **SS** *with reflective matrix* $F = \bar{A}^{-1}A$, *where* $S(t, A)$ *is an arbitrary differentiable function.*

By the literature [4], we have

Corollary 2.7. *If the conditions of theorem* (2.2) *are satisfied, then the system* (2.1) *is equivalent to the following systems*

$$x' = Ax + A^{-1}\bar{A}S(t, x) - S(-t, \bar{A}^{-1}Ax)$$

and

$$x' = Ax + A^{-1}(S(t, Ax) - S(-t, Ax)).$$

Now consider linear system

$$x' = Bx, \tag{2.10}$$

where $B = B(t)$ is a continuous differentiable matrix function.

Theorem 2.5. *Suppose that* $det(A) \neq 0$ *and the relation* (2.9) *is valid and*

$$A'A^{-1} + ABA^{-1} + \bar{A}'\bar{A}^{-1} + \bar{A}\bar{B}\bar{A}^{-1} = 0. \tag{2.11}$$

Then the system (2.10) *is equivalent to system* (2.1).

Proof. By theorem (2.4), we know $F = \bar{A}^{-1}A$ is a reflective matrix of (1.4), then

$$\bar{A}(F' + FB + \bar{B}) = (A'A^{-1} + ABA^{-1} + \bar{A}'\bar{A}^{-1} + \bar{A}\bar{B}\bar{A}^{-1})A = 0,$$

i.e., $F = \bar{A}^{-1}A$ is also reflective matrix of (2.10). □

Corollary 2.8. *If $F = \bar{A}^{-1}A$ is a reflective matrix of* (2.1) *and*

$$A + \bar{A} = ABA^{-1} + \bar{A}\bar{B}\bar{A}^{-1},$$

then the system (2.10) *is equivalent to system* (2.1).

This conclusion can be yielded by using (2.9) and (2.11).

Corollary 2.9. *Suppose that $F = \bar{A}^{-1}A$ is the reflective matrix of system* (2.1). *Then system* (2.1) *is equivalent to system*

$$x' = Ax + A^{-1}\sum_{j=1}^{m} \alpha_j(t)R_j(t)Ax, \tag{2.12}$$

where $\alpha_j(t), R_j(t)(j = 1, 2, ..., m)$ are arbitrary functions such that the right-hand sides of system (2.10) *are continuous differentiable, $\alpha_j(t)$ are scalar odd functions $R_j(t)(j = 1, 2, ..., m)$ are $n \times n$ even matrix functions.*

Proof. It is easy to check that system (2.12) satisfies all the conditions of Corollary (2.8). □

Remark 2.4. If the conditions of theorem (2.5) or corollary (2.8) or corollary (2.9) are satisfied and the systems (2.1) and (2.10) and (2.12) are 2ω-periodic with respect to t, then all solutions of those defined on interval $[-\omega, \omega]$ are 2ω-periodic. This assertion is implied by $F(t) = \bar{A}^{-1}A$ and $F(-\omega) = E$.

Example 2.3. System

$$x' = -\frac{1}{4}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & -a_{11} \end{pmatrix} x$$

is the **SS** with reflective matrix

$$F = \begin{pmatrix} 1 + \sin t & e^{\cos t}\sin t \\ -e^{-\cos t}\sin t & 1 - \sin t \end{pmatrix},$$

where

$$\begin{aligned} a_{11} &= 2\cos t + \sin^3 t, \\ a_{12} &= e^{\cos t}(2\cos t + \sin^3 t - 2\sin^2 t), \\ a_{21} &= -e^{-\cos t}(2\cos t + \sin^3 t + 2\sin^2 t). \end{aligned}$$

This system is equivalent to system

$$x' = -\frac{1}{4}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & -a_{11} \end{pmatrix}\left[E - \frac{1}{4}\sum_{j=1}^{m} \alpha_j(t)R_j(t)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & -a_{11} \end{pmatrix}\right] x, \tag{2.13}$$

where $\alpha_j(t)(j = 1, 2, ..., m)$ are arbitrary continuous scalar odd functions, $R_j(t)(j = 1, 2, ..., m)$ are continuous differentiable 2×2 matrix functions. Besides, if system (2.13) is a 2π-periodic with respect to t, then all solutions of systems (2.13) defined on interval $[-\pi, \pi]$ are 2π-periodic.

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