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EVANS FUNCTIONS AND BIFURCATIONS OF STANDING WAVE SOLUTIONS IN DELAYED SYNAPTICALLY COUPLED NEURONAL NETWORKS

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Abstract Consider the following nonlinear singularly perturbed system of integral differential equations

$$\begin{aligned}&\frac{\partial u}{\partial t}+f(u)+w\\&=(\alpha-au)\int_0^\infty \xi(c)\left[\int_{\mathbb{R}} K(x-y)H\left(u\left(y,t-\frac{1}{c}|x-y|\right)-\theta\right)\mathrm{d}y\right]\mathrm{d}c\\&\quad+(\beta-bu)\int_0^\infty \eta(\tau)\left[\int_{\mathbb{R}} W(x-y)H\big(u(y,t-\tau)-\Theta\big)\mathrm{d}y\right]\mathrm{d}\tau,\\&\frac{\partial w}{\partial t}=\varepsilon[g(u)-w],\end{aligned}$$

and the scalar integral differential equation

$$\begin{aligned}&\frac{\partial u}{\partial t}+f(u)\\&=(\alpha-au)\int_0^\infty \xi(c)\left[\int_{\mathbb{R}} K(x-y)H\left(u\left(y,t-\frac{1}{c}|x-y|\right)-\theta\right)\mathrm{d}y\right]\mathrm{d}c\\&\quad+(\beta-bu)\int_0^\infty \eta(\tau)\left[\int_{\mathbb{R}} W(x-y)H\big(u(y,t-\tau)-\Theta\big)\mathrm{d}y\right]\mathrm{d}\tau.\end{aligned}$$

There exist standing wave solutions to the nonlinear system. Similarly, there exist standing wave solutions to the scalar equation.

The author constructs Evans functions to establish stability of the standing wave solutions of the scalar equation and to establish bifurcations of the standing wave solutions of the nonlinear system.

Keywords Delayed synaptically coupled neuronal networks, standing wave solutions, existence, stability, instability, bifurcations, eigenvalue problems, Evans functions.

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1. Introduction

1.1. The Model Equations

When an action potential is generated across a neuron membrane, normally, sodium activation is considerably faster than potassium activation. This is reflected by a

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singular perturbation parameter ε in mathematical models. In this paper, we will show by using rigorous mathematical analysis that when the ratio of sodium activation and potassium activation reaches a certain value, which is closely related to the threshold above which an action potential is generated, bifurcations of standing wave solutions occur.

Consider the following nonlinear singular perturbed system of integral differential equations arising from delayed synaptically coupled neuronal networks

$$\begin{aligned}&\frac{\partial u}{\partial t}+f(u)+w\\ =&(\alpha-au)\int_0^\infty \xi(c)\left[\int_{\mathbb{R}}K(x-y)H\left(u\left(y,t-\frac{1}{c}|x-y|\right)-\theta\right)\mathrm{d}y\right]\mathrm{d}c\\ &+(\beta-bu)\int_0^\infty \eta(\tau)\left[\int_{\mathbb{R}}W(x-y)H\big(u(y,t-\tau)-\Theta\big)\mathrm{d}y\right]\mathrm{d}\tau,\end{aligned}\tag{1.1}$$

$$\frac{\partial w}{\partial t}=\varepsilon[g(u)-w],\tag{1.2}$$

and

$$\begin{aligned}&\frac{\partial u}{\partial t}+f(u)\\ =&(\alpha-au)\int_0^\infty \xi(c)\left[\int_{\mathbb{R}}K(x-y)H\left(u\left(y,t-\frac{1}{c}|x-y|\right)-\theta\right)\mathrm{d}y\right]\mathrm{d}c\\ &+(\beta-bu)\int_0^\infty \eta(\tau)\left[\int_{\mathbb{R}}W(x-y)H\big(u(y,t-\tau)-\Theta\big)\mathrm{d}y\right]\mathrm{d}\tau,\end{aligned}\tag{1.3}$$

where $u=u(x,t)$ represents the membrane potential of a neuron at position x and time t in a delayed synaptically coupled neuronal network, $w=w(x,t)$ represents the leaking current. The kernel functions K and W are defined on $\mathbb{R}$. They represent synaptic couplings between neurons. The gain function is the Heaviside step function: $H(u-\theta)=0$ for all $u<\theta$, $H(0)=\frac{1}{2}$, and $H(u-\theta)=1$ for all $u>\theta$. The probability density functions ξ and η are defined on $(0,\infty)$. The function ξ represents a statistical distribution of action potential speeds. Additionally, ξ may have a compact support $[c_1,c_2]$, where c_1 and c_2 are positive constants, denoting the lower bound and upper bound of biologically possible speeds, respectively. Moreover, a, b, α, β, ε, θ and Θ are nonnegative or positive constants, representing various biological mechanisms. The functions $f=f(u)$ and $g=g(u)$ are smooth functions. In addition, either $f(u)+g(u)=m(u-n)+k(u-l)$ is a linear function, where $k>0$ and $m>0$ are positive constants, l and n are real constants; or $f(u)+g(u)=u(u-1)(Du-1)$ is a cubic polynomial function, where $D>0$ is a positive constant. Biologically, to model the sodium current and the potassium current, a nonlinear function is much better than a linear function because sodium channels and potassium channels are voltage gated channels (in another word, sodium conductance and potassium conductance should be functions of voltage). On the other hand, mathematically, a linear function is much better than a nonlinear function because a linear function is easy to handle and in some sense it is a good approximation of the nonlinear function. We may interpret the constant m as the sodium conductance and the constant n as the sodium reversal potential. Similarly, we may interpret the constant k as the potassium conductance and the constant l as the potassium reversal potential. See

[2, 3, 4, 5, 7, 8, 9, 10, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24] for the same or very similar model equations.

The nonlinear singularly perturbed system of integral differential equations (1.1)-(1.2) becomes the scalar integral differential equation (1.3) if $\varepsilon = 0$ and $w = 0$. We will study existence, stability/instability and bifurcations of standing wave solutions, that is, solutions of the form $\big(u(x,t), w(x,t)\big) = \big(\phi(x), \psi(x)\big)$ to system (1.1)-(1.2) and solutions of the form $u(x,t) = \varphi(x)$ to equation (1.3). Note that the nonlinear system (1.1)-(1.2) is autonomous. Hence, if $\big(u(x,t), w(x,t)\big) = \big(\phi(x), \psi(x)\big)$ is a standing wave solution of system (1.1)-(1.2), then $\big(u(x,t), w(x,t)\big) = \big(\phi(x+h), \psi(x+h)\big)$ is also a standing wave solution of (1.1)-(1.2), for any real constant h.

There may exist both simple standing wave solutions (which cross only one threshold) and complicated standing wave solutions (which cross both thresholds). Our results show that there exist two complicated standing wave solutions and under additional conditions, there are four simple standing wave solutions. We will highlight complicated standing wave solutions. We call them standing wave solutions because they are special traveling wave solutions where the wave speed is equal to zero. We may also call them stationary solutions or steady state solutions.

Previously, Amari [1], Guo and Chow [6], Pinto and Ermentrout [9] and Zhang [19] have studied the existence, stability/instability of standing wave solutions of some integral differential equations arising from synaptically coupled neuronal networks. However, the existence, stability/instability and bifurcations of standing wave solutions of system (1.1)-(1.2) have been open for a long time. An interesting feature on the stability/instability analysis and the bifurcation analysis is that the eigenvalue problems derived from linearization of the nonlinear system are nonlinear in λ (this is the eigenvalue parameter). This difficulty arises because the system involves two kinds of delays and any of the two delays may cause such difficulty. We are able to overcome the difficulty to find the eigenvalues of the eigenvalue problems by the construction of Evans functions and by studying their properties.

1.2. The Mathematical Assumptions

Suppose that $a \geq 0$, $b \geq 0$, $\alpha \geq 0$, $\beta \geq 0$, $\varepsilon > 0$, $\theta > 0$ and $\Theta > 0$ are nonnegative or positive constants. Suppose that $\xi \geq 0$ and $\eta \geq 0$ are nonnegative probability density functions defined on $(0, \infty)$. Suppose that the kernel functions K and W are bounded, at least piecewise continuous functions defined on $\mathbb{R}$.

Suppose that either $f(u)+g(u) = m(u-n)+k(u-l)$, for some positive constants $k > 0$ and $m > 0$ and for some real constants l and n; or $f(u)+g(u) = u(u-1)(Du-1)$, for some positive constant $D > 0$. Suppose that

$$\alpha \geq a\Theta, \qquad \beta \geq b\theta, \tag{1.4}$$

$$K(-x) = K(x), \qquad \text{for all } x \in \mathbb{R}, \qquad \text{and } \int_{\mathbb{R}} K(x)\mathrm{d}x = 1, \tag{1.5}$$

$$W(-x) = W(x), \qquad \text{for all } x \in \mathbb{R}, \qquad \text{and } \int_{\mathbb{R}} W(x)\mathrm{d}x = 1, \tag{1.6}$$

$$\int_0^\infty \xi(c)\mathrm{d}c = 1, \qquad \int_0^\infty \eta(\tau)\mathrm{d}\tau = 1, \tag{1.7}$$

$$\int_0^\infty \frac{1}{c}\xi(c)\mathrm{d}c < \infty, \qquad \int_0^\infty \tau\eta(\tau)\mathrm{d}\tau < \infty. \tag{1.8}$$

Remark 1.1. The conditions

$$\int_0^\infty \xi(c)\mathrm{d}c = 1, \qquad \int_0^\infty \eta(\tau)\mathrm{d}\tau = 1,$$
$$\int_{\mathbb{R}} K(x)\mathrm{d}x = 1, \qquad \int_{\mathbb{R}} W(x)\mathrm{d}x = 1,$$

are not essential. Actually, if they are not equal to one, then we can always rescale the constants a, b, α and β so that the integrals are equal to one. For the existence and stability/instability analysis of traveling wave solutions, we must make the additional assumption

$$|K(x)| + |W(x)| \leq C \exp(-\rho|x|) \qquad \text{on} \qquad \mathbb{R},$$

for two positive constants $C > 0$ and $\rho > 0$. For example, $K(x) = \frac{\rho}{2}\exp(-\rho|x|)$ and $W(x) = \sqrt{\frac{\rho}{\pi}}\exp(-\rho x^2)$, where $\rho > 0$ is a parameter, satisfy this condition. For the existence and stability/instability analysis of standing wave solutions, we do not need such assumptions. For non-symmetric kernel functions K and W, the results obtained in this paper may not be true.

Let us find the constant solutions of the system. If $f(u) + g(u) = m(u - n) + k(u - l)$ and ϕ_0 is a constant solution, such that $\phi_0 < \theta$ and $\phi_0 < \Theta$, then $m(\phi_0 - n) + k(\phi_0 - l) = 0$. Thus

$$\phi_0 = \frac{kl + mn}{k + m} < \theta.$$

If $f(u) + g(u) = m(u - n) + k(u - l)$, $\theta < \Theta$ and ϕ_1 is a constant solution, such that $\theta < \phi_1 < \Theta$, then $m(\phi_1 - n) + k(\phi_1 - l) = \alpha - a\phi_1$. Hence

$$\theta < \phi_1 = \frac{\alpha + kl + mn}{a + k + m} < \Theta.$$

If $f(u) + g(u) = m(u - n) + k(u - l)$ and ϕ_2 is a constant solution, such that $\phi_2 > \theta$ and $\phi_2 > \Theta$, then $m(\phi_2 - n) + k(\phi_2 - l) = \alpha + \beta - a\phi_2 - b\phi_2$. Hence

$$\phi_2 = \frac{\alpha + \beta + kl + mn}{a + b + k + m} > \Theta.$$

The constant solution ϕ_1 does not exist if $\theta = \Theta$. All of them are stable constant solutions.

1.3. The Main Goal

In this paper, we will use rigorous mathematical analysis to establish the existence, stability/instability and bifurcations of the standing wave solutions to the nonlinear singularly perturbed system of integral differential equations with $\varepsilon > 0$. First of all, we will obtain explicit standing wave solutions for the nonlinear system. Then, we derive some eigenvalue problems. Then by constructing and making use of some complex analytic functions, called Evans functions, we will accomplish the stability/instability and the bifurcations of the standing wave solutions. Evans functions are complex analytic functions defined in some right half complex plane. Moreover, λ_0 is an eigenvalue of the eigenvalue problem if and only if λ_0 is a zero of the Evans function.

2. Existence of the Standing Wave Solutions

2.1. The nonlinear system (1.1)-(1.2)

First of all, we establish the existence of the standing wave solutions to the nonlinear singularly perturbed system of integral differential equations with $\varepsilon > 0$ and $f(u) + g(u) = m(u-n) + k(u-l)$, where $k > 0$ and $m > 0$ are positive constants, l and n are real constants. As mentioned before, the linear function is a good approximation of the nonlinear function.

Theorem 2.1. *Suppose that $f(u) + g(u) = m(u-n) + k(u-l)$. Suppose that there exists a nonnegative constant $Z_0 \geq 0$, such that*

$$(\alpha - a\theta)K(0) \geq 0, \qquad (\beta - b\theta)W(-Z_0) \geq 0, \tag{2.1}$$

$$(\alpha - a\theta)K(0) + (\beta - b\theta)W(-Z_0) > 0, \tag{2.2}$$

$$(\alpha - a\Theta)K(Z_0) \geq 0, \qquad (\beta - b\Theta)W(0) \geq 0, \tag{2.3}$$

$$(\alpha - a\Theta)K(Z_0) + (\beta - b\Theta)W(0) > 0, \tag{2.4}$$

$$(\alpha - a\theta)K(x) + (\beta - b\theta)W(x - Z_0) \geq 0, \qquad \textit{for all } x \in \mathbb{R}, \tag{2.5}$$

$$(\alpha - a\Theta)K(x) + (\beta - b\Theta)W(x - Z_0) \geq 0, \qquad \textit{for all } x \in \mathbb{R}, \tag{2.6}$$

$$\int_{-\infty}^{-Z_0} W(z)\mathrm{d}z = \frac{(2k + 2m + a)\theta - (2kl + 2mn + \alpha)}{2(\beta - b\theta)}, \tag{2.7}$$

$$\int_{-\infty}^{Z_0} K(z)\mathrm{d}z = \frac{(2k + 2m + b)\Theta - (2kl + 2mn + \beta)}{2(\alpha - a\Theta)}, \tag{2.8}$$

$$k + m + a\int_{-\infty}^{x} K(z)\mathrm{d}z + b\int_{-\infty}^{x - Z_0} W(z)\mathrm{d}z > 0, \qquad \textit{for all } x \in \mathbb{R}, \tag{2.9}$$

$$\frac{kl + mn}{k + m} < \theta \leq \Theta < \frac{\alpha + \beta + kl + mn}{a + b + k + m}. \tag{2.10}$$

(I) There exist two complicated standing wave solutions

$$\begin{aligned}
\big(u_1(x,t), w_1(x,t)\big) &= \big(\phi_1(x), \psi_1(x)\big) = \big(\phi_1(x), g(\phi_1(x))\big),\\
\big(u_2(x,t), w_2(x,t)\big) &= \big(\phi_2(x), \psi_2(x)\big) = \big(\phi_2(x), g(\phi_2(x))\big),
\end{aligned}$$

to the nonlinear system of integral differential equations (1.1)-(1.2), where

$$\begin{aligned}
\phi_1(x) = &\left\{kl + mn + \alpha\int_{-\infty}^{x} K(z)\mathrm{d}z + \beta\int_{-\infty}^{x-Z_0} W(z)\mathrm{d}z\right\}\\
&\Big/\left\{k + m + a\int_{-\infty}^{x} K(z)\mathrm{d}z + b\int_{-\infty}^{x-Z_0} W(z)\mathrm{d}z\right\},\\
\phi_2(x) = &\left\{kl + mn + \alpha\int_{x}^{\infty} K(z)\mathrm{d}z + \beta\int_{x+Z_0}^{\infty} W(z)\mathrm{d}z\right\}\\
&\Big/\left\{k + m + a\int_{x}^{\infty} K(z)\mathrm{d}z + b\int_{x+Z_0}^{\infty} W(z)\mathrm{d}z\right\}.
\end{aligned}$$

Define the positive constants

$$P = \frac{\alpha + \beta + kl + mn}{a + b + k + m}, \qquad Q = \frac{kl + mn}{k + m}, \qquad R = \frac{\alpha + kl + mn}{a + k + m}.$$

The standing wave solutions satisfy the following boundary conditions

$$\begin{aligned}
\lim_{x\to-\infty}\big(u_1(x,t),w_1(x,t)\big) &= \big(Q,g(Q)\big),\\
\lim_{x\to\infty}\big(u_1(x,t),w_1(x,t)\big) &= \big(P,g(P)\big),\\
\lim_{x\to-\infty}\big(u_2(x,t),w_2(x,t)\big) &= \big(P,g(P)\big),\\
\lim_{x\to\infty}\big(u_2(x,t),w_2(x,t)\big) &= \big(Q,g(Q)\big).
\end{aligned}$$

(II) Suppose that the constants satisfy the additional conditions

$$\frac{\alpha+2kl+2mn}{a+2k+2m}=\theta<\frac{\alpha+kl+mn}{a+k+m}<\Theta=\frac{2\alpha+\beta+2kl+2mn}{2a+b+2k+2m}.$$

There exist four simple standing wave solutions

$$\big(u_i(x,t),w_i(x,t)\big)=\big(\phi_i(x),g\big(\phi_i(x)\big)\big)$$

to the system (1.1)-(1.2) of integral differential equations, where $i=3,4,5,6$ *and*

$$\begin{aligned}
\phi_3(x) &= \left\{kl+mn+\alpha\int_{-\infty}^{x}K(z)\mathrm{d}z\right\}\Big/\left\{k+m+a\int_{-\infty}^{x}K(z)\mathrm{d}z\right\},\\
\phi_4(x) &= \left\{kl+mn+\alpha\int_{x}^{\infty}K(z)\mathrm{d}z\right\}\Big/\left\{k+m+a\int_{x}^{\infty}K(z)\mathrm{d}z\right\},\\
\phi_5(x) &= \left\{kl+mn+\alpha+\beta\int_{-\infty}^{x}W(z)\mathrm{d}z\right\}\Big/\left\{k+m+a+b\int_{-\infty}^{x}W(z)\mathrm{d}z\right\},\\
\phi_6(x) &= \left\{kl+mn+\alpha+\beta\int_{x}^{\infty}W(z)\mathrm{d}z\right\}\Big/\left\{k+m+a+b\int_{x}^{\infty}W(z)\mathrm{d}z\right\}.
\end{aligned}$$

These standing wave solutions satisfy the following boundary conditions

$$\begin{aligned}
\lim_{x\to-\infty}\big(u_3(x,t),w_3(x,t)\big) &= \big(Q,g(Q)\big),\\
\lim_{x\to\infty}\big(u_3(x,t),w_3(x,t)\big) &= \big(R,g(R)\big),\\
\lim_{x\to-\infty}\big(u_4(x,t),w_4(x,t)\big) &= \big(R,g(R)\big),\\
\lim_{x\to\infty}\big(u_4(x,t),w_4(x,t)\big) &= \big(Q,g(Q)\big),\\
\lim_{x\to-\infty}\big(u_5(x,t),w_5(x,t)\big) &= \big(R,g(R)\big),\\
\lim_{x\to\infty}\big(u_5(x,t),w_5(x,t)\big) &= \big(P,g(P)\big),\\
\lim_{x\to-\infty}\big(u_6(x,t),w_6(x,t)\big) &= \big(P,g(P)\big),\\
\lim_{x\to\infty}\big(u_6(x,t),w_6(x,t)\big) &= \big(R,g(R)\big).
\end{aligned}$$

Remark 2.1. If the kernel functions K and W are positive functions, then the conditions (9)-(14) and (17) hold automatically. If one synaptic coupling represents a pure excitation and another denotes a lateral inhibition, then these conditions can be easily satisfied. If one synaptic coupling represents a pure excitation and another denotes a lateral excitation, then these conditions can also be easily satisfied.

Remark 2.2. If $\theta = \Theta$ and $(a+b+2k+2m)\theta = \alpha+\beta+2kl+2mn$, then $Z_0 = 0$. If $\theta < \Theta$, then $Z_0 > 0$.

Remark 2.3. By an intermediate value theorem, it is easy to show that there exists a real number ${Z_0}'$, such that

$$\int_{-\infty}^{-{Z_0}'} W(z)\mathrm{d}z = \frac{(2k+2m+a)\theta - (2kl+2mn+\alpha)}{2(\beta - b\theta)}.$$

Similarly, there exists a real number ${Z_0}''$, such that

$$\int_{-\infty}^{{Z_0}''} K(z)\mathrm{d}z = \frac{(2k+2m+b)\Theta - (2kl+2mn+\beta)}{2(\alpha - a\Theta)}.$$

For simplicity, we assume that the model parameters and the kernel functions are chosen appropriately, such that ${Z_0}' = {Z_0}'' \geq 0$.

Proof. Define

$$\omega_- = \int_{-\infty}^{-Z_0} W(z)\mathrm{d}z, \qquad \omega_+ = \int_{-\infty}^{Z_0} K(z)\mathrm{d}z.$$

Standing wave solutions satisfy $\dfrac{\partial u}{\partial t} = 0$ and $\dfrac{\partial w}{\partial t} = 0$. Substituting a solution of the form $\big(u(x,t), w(x,t)\big) = \big(\phi(x), \psi(x)\big)$ into the system (1.1)-(1.2), we get

$$\begin{aligned}
f\big(\phi(x)\big) + \psi(x) =& \big[\alpha - a\phi(x)\big] \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(x-y) H\big(\phi(y)-\theta\big)\mathrm{d}y\right] \mathrm{d}c \\
&+ \big[\beta - b\phi(x)\big] \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(x-y) H\big(\phi(y)-\Theta\big)\mathrm{d}y\right] \mathrm{d}\tau, \\
0 =& \varepsilon\big[g\big(\phi(x)\big) - \psi(x)\big].
\end{aligned}$$

(I) Suppose that the first standing wave solution satisfies the conditions $\phi < \theta$ on $(-\infty, 0)$, $\phi(0) = \theta$ and $\phi > \theta$ on $(0, \infty)$; $\phi < \Theta$ on $(-\infty, Z_0)$, $\phi(Z_0) = \Theta$ and $\phi > \Theta$ on (Z_0, ∞), for some nonnegative constant $Z_0 \geq 0$. Then the right hand side of the first equation in the last system becomes

$$\begin{aligned}
&\big[\alpha - a\phi(x)\big] \int_{\mathbb{R}} K(x-y) H\big(\phi(y)-\theta\big)\mathrm{d}y + \big[\beta - b\phi(x)\big] \int_{\mathbb{R}} W(x-y) H\big(\phi(y)-\Theta\big)\mathrm{d}y \\
=& \big[\alpha - a\phi(x)\big] \int_0^\infty K(x-y)\mathrm{d}y + \big[\beta - b\phi(x)\big] \int_{Z_0}^\infty W(x-y)\mathrm{d}y \\
=& \big[\alpha - a\phi(x)\big] \int_{-\infty}^x K(z)\mathrm{d}z + \big[\beta - b\phi(x)\big] \int_{-\infty}^{x-Z_0} W(z)\mathrm{d}z \\
=& \alpha \int_{-\infty}^x K(z)\mathrm{d}z + \beta \int_{-\infty}^{x-Z_0} W(z)\mathrm{d}z - \left[a \int_{-\infty}^x K(z)\mathrm{d}z + b \int_{-\infty}^{x-Z_0} W(z)\mathrm{d}z\right] \phi(x).
\end{aligned}$$

Hence we obtain the first standing wave solution

$$\big(u_1(x,t), w_1(x,t)\big) = \big(\phi_1(x), g\big(\phi_1(x)\big)\big)$$

to the nonlinear system (1.1)-(1.2) of integral differential equations, where $\phi_1 = \phi_1(x)$ is given by

$$\begin{aligned}\phi_1(x) &= \left\{kl + mn + \alpha\int_{-\infty}^{x} K(z)\mathrm{d}z + \beta\int_{-\infty}^{x-Z_0} W(z)\mathrm{d}z\right\} \\ &\quad / \left\{k + m + a\int_{-\infty}^{x} K(z)\mathrm{d}z + b\int_{-\infty}^{x-Z_0} W(z)\mathrm{d}z\right\}.\end{aligned}$$

The derivative of $\phi_1 = \phi_1(x)$ is given by

$$\begin{aligned}{\phi_1}'(x) &= \left\{\left[\alpha - a\phi_1(x)\right]K(x) + \left[\beta - b\phi_1(x)\right]W(x - Z_0)\right\} \\ &\quad / \left\{k + m + a\int_{-\infty}^{x} K(z)\mathrm{d}z + b\int_{-\infty}^{x-Z_0} W(z)\mathrm{d}z\right\}.\end{aligned}$$

In particular, we have

$$\begin{aligned}&{\phi_1}'(0) \\ =&\left\{(\alpha - a\theta)K(0) + (\beta - b\theta)W(-Z_0)\right\}/\left\{k + m + \frac{a}{2} + b\int_{-\infty}^{-Z_0} W(z)\mathrm{d}z\right\} > 0, \\ &{\phi_1}'(Z_0) \\ =&\left\{(\alpha - a\Theta)K(Z_0) + (\beta - b\Theta)W(0)\right\}/\left\{k + m + a\int_{-\infty}^{Z_0} K(z)\mathrm{d}z + \frac{b}{2}\right\} > 0.\end{aligned}$$

However, this is only a formal solution. We have to show that it is compatible, namely, we have to verify that the standing wave solution is below and above the threshold θ on $(-\infty, 0)$ and $(0, \infty)$, respectively; and it is below and above the threshold Θ on $(-\infty, Z_0)$ and (Z_0, ∞), respectively. The following inequalities are equivalent to one another (below, the symbol "<=>" means that either we always take "<" or we always take "=" or we always take ">"):

$$\begin{aligned}&\left\{kl + mn + \alpha\int_{-\infty}^{x} K(z)\mathrm{d}z + \beta\int_{-\infty}^{x-Z_0} W(z)\mathrm{d}z\right\} \\ &/\left\{k + m + a\int_{-\infty}^{x} K(z)\mathrm{d}z + b\int_{-\infty}^{x-Z_0} W(z)\mathrm{d}z\right\} <=> \theta; \\ &kl + mn + \alpha\int_{-\infty}^{x} K(z)\mathrm{d}z + \beta\int_{-\infty}^{x-Z_0} W(z)\mathrm{d}z \\ <=>&\left\{k + m + a\int_{-\infty}^{x} K(z)\mathrm{d}z + b\int_{-\infty}^{x-Z_0} W(z)\mathrm{d}z\right\}\theta; \\ &k(l-\theta) + m(n-\theta) + (\alpha - a\theta)\int_{-\infty}^{x} K(z)\mathrm{d}z \\ &+ (\beta - b\theta)\int_{-\infty}^{x-Z_0} W(z)\mathrm{d}z <=> 0; \\ &(\alpha - a\theta)\int_{-\infty}^{x} K(z)\mathrm{d}z + (\beta - b\theta)\int_{-\infty}^{x-Z_0} W(z)\mathrm{d}z <=> k(\theta - l) + m(\theta - n).\end{aligned}$$

Moreover, the following inequalities are equivalent to each other:

$$
\begin{aligned}
&\left\{kl + mn + \alpha \int_{-\infty}^{x} K(z)\mathrm{d}z + \beta \int_{-\infty}^{x-Z_0} W(z)\mathrm{d}z\right\} \\
&/\left\{k + m + a \int_{-\infty}^{x} K(z)\mathrm{d}z + b \int_{-\infty}^{x-Z_0} W(z)\mathrm{d}z\right\} <=> \Theta; \\
&kl + mn + \alpha \int_{-\infty}^{x} K(z)\mathrm{d}z + \beta \int_{-\infty}^{x-Z_0} W(z)\mathrm{d}z \\
<=> &\left\{k + m + a \int_{-\infty}^{x} K(z)\mathrm{d}z + b \int_{-\infty}^{x-Z_0} W(z)\mathrm{d}z\right\} \Theta; \\
&k(l-\Theta) + m(n-\Theta) + (\alpha - a\Theta) \int_{-\infty}^{x} K(z)\mathrm{d}z \\
&+ (\beta - b\Theta) \int_{-\infty}^{x-Z_0} W(z)\mathrm{d}z <=> 0; \\
&(\alpha - a\Theta) \int_{-\infty}^{x} K(z)\mathrm{d}z + (\beta - b\Theta) \int_{-\infty}^{x-Z_0} W(z)\mathrm{d}z <=> k(\Theta - l) + m(\Theta - n).
\end{aligned}
$$

Define the following auxiliary functions on $\mathbb{R}$:

$$
\begin{aligned}
A(x) =& (\alpha - a\theta) \int_{-\infty}^{x} K(z)\mathrm{d}z + (\beta - b\theta) \int_{-\infty}^{x-Z_0} W(z)\mathrm{d}z, \\
B(x) =& (\alpha - a\Theta) \int_{-\infty}^{x} K(z)\mathrm{d}z + (\beta - b\Theta) \int_{-\infty}^{x-Z_0} W(z)\mathrm{d}z.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
A(0) =& \frac{1}{2}(\alpha - a\theta) + (\beta - b\theta) \int_{-\infty}^{-Z_0} W(z)\mathrm{d}z = k(\theta - l) + m(\theta - n), \\
B(Z_0) =& (\alpha - a\Theta) \int_{-\infty}^{Z_0} K(z)\mathrm{d}z + \frac{1}{2}(\beta - b\Theta) = k(\Theta - l) + m(\Theta - n),
\end{aligned}
$$

and

$$
\begin{aligned}
A'(x) =& (\alpha - a\theta)K(x) + (\beta - b\theta)W(x - Z_0) \geq 0, \qquad \text{for all } x \in \mathbb{R}, \\
B'(x) =& (\alpha - a\Theta)K(x) + (\beta - b\Theta)W(x - Z_0) \geq 0, \qquad \text{for all } x \in \mathbb{R}, \\
A'(0) =& (\alpha - a\theta)K(0) + (\beta - b\theta)W(-Z_0) > 0, \\
B'(Z_0) =& (\alpha - a\Theta)K(Z_0) + (\beta - b\Theta)W(0) > 0.
\end{aligned}
$$

Hence, both $A(x)$ and $B(x)$ are increasing functions on $\mathbb{R}$. Therefore, we find that $\phi_1 < \theta$ on $(-\infty, 0)$, $\phi_1(0) = \theta$ and $\phi_1 > \theta$ on $(0, \infty)$. Similarly, $\phi_1 < \Theta$ on $(-\infty, Z_0)$, $\phi_1(Z_0) = \Theta$ and $\phi_1 > \Theta$ on (Z_0, ∞).

The existence of the second standing wave solution can be proved very similarly. Indeed, suppose that the second standing wave solution satisfies the conditions: $\phi_2 > \theta$ on $(-\infty, 0)$, $\phi_2(0) = \theta$ and $\phi_2 < \theta$ on $(0, \infty)$; $\phi_2 > \Theta$ on $(-\infty, -Z_0)$, $\phi_2(-Z_0) = \Theta$ and $\phi_2 < \Theta$ on $(-Z_0, \infty)$, for the same constant Z_0 as in the analysis of ϕ_1. Then the differential equations become

$$\begin{aligned} & m\big[\phi_2(x)-n\big]+k\big[\phi_2(x)-l\big]=f\big(\phi_2(x)\big)+g\big(\phi_2(x)\big) \\ = \quad & \big[\alpha-a\phi_2(x)\big]\int_x^{\infty}K(z)\mathrm{d}z+\big[\beta-b\phi_2(x)\big]\int_{x+Z_0}^{\infty}W(z)\mathrm{d}z. \end{aligned}$$

Therefore, we obtain the second standing wave solution

$$\big(u_2(x,t),w_2(x,t)\big)=\big(\phi_2(x),g\big(\phi_2(x)\big)\big),$$

where

$$\begin{aligned} \phi_2(x) \quad = \quad & \left\{kl+mn+\alpha\int_x^{\infty}K(z)\mathrm{d}z+\beta\int_{x+Z_0}^{\infty}W(z)\mathrm{d}z\right\} \\ / \quad & \left\{k+m+a\int_x^{\infty}K(z)\mathrm{d}z+b\int_{x+Z_0}^{\infty}W(z)\mathrm{d}z\right\}. \end{aligned}$$

The derivative of $\phi_2=\phi_2(x)$ is given by

$$\begin{aligned} {\phi_2}'(x) \quad = \quad & -\big\{\big[\alpha-a\phi_2(x)\big]K(x)+\big[\beta-b\phi_2(x)\big]W(x+Z_0)\big\} \\ / \quad & \left\{k+m+a\int_x^{\infty}K(z)\mathrm{d}z+b\int_{x+Z_0}^{\infty}W(z)\mathrm{d}z\right\}. \end{aligned}$$

In particular, we have

$$\begin{aligned} & {\phi_2}'(-Z_0) \\ = & -\big\{(\alpha-a\Theta)K(-Z_0)+(\beta-b\Theta)W(0)\big\}/\left\{k+m+a\int_{-Z_0}^{\infty}K(z)\mathrm{d}z+\frac{b}{2}\right\}<0, \\ & {\phi_2}'(0) \\ = & -\big\{(\alpha-a\theta)K(0)+(\beta-b\theta)W(Z_0)\big\}/\left\{k+m+\frac{a}{2}+b\int_{Z_0}^{\infty}W(z)\mathrm{d}z\right\}<0. \end{aligned}$$

It is easy to check that this standing wave solution also satisfies the prescribed conditions.

(II) To obtain the expressions of the standing wave solutions $\big(u_3(x,t),w_3(x,t)\big)=\big(\phi_3(x),\psi_3(x)\big)$ and $\big(u_4(x,t),w_4(x,t)\big)=\big(\phi_4(x),\psi_4(x)\big)$, we may simply let $b=0$, $Z_0=0$ and $\beta=0$ in the proof of Theorem 2.1, because $\big(\phi_3,\psi_3\big)$ and $\big(\phi_4,\psi_4\big)$ are below the big threshold Θ. To obtain the representations of the standing wave solutions $\big(u_5(x,t),w_5(x,t)\big)=\big(\phi_5(x),\psi_5(x)\big)$ and $\big(u_6(x,t),w_6(x,t)\big)=\big(\phi_6(x),\psi_6(x)\big)$, we may simply replace both $\int_{-\infty}^{x}K(z)\mathrm{d}z$ and $\int_x^{\infty}K(z)\mathrm{d}z$ with 1 in the proof of (I), because $\big(\phi_5,\psi_5\big)$ and $\big(\phi_6,\psi_6\big)$ are above the small threshold θ. All other details are the same as those in (I) and they are omitted. The proof of Theorem 2.1 is finished. □

2.2. The Scalar Integral Differential Equation (1.3)

Theorem 2.2. *Suppose that $f(u)=m(u-n)$, for some positive constant $m>0$ and for some real constant n. Suppose that there exists a nonnegative constant $Z_0\geq 0$, such that*

$$\alpha\geq a\Theta,\qquad \beta\geq b\theta,\qquad n<\theta\leq\Theta<\frac{\alpha+\beta+mn}{a+b+m},\tag{2.11}$$

$$(\alpha-a\theta)K(0)\geq 0,\qquad (\beta-b\theta)W(-Z_0)\geq 0,\tag{2.12}$$

$$(\alpha - a\theta)K(0) + (\beta - b\theta)W(-Z_0) > 0, \tag{2.13}$$

$$(\alpha - a\Theta)K(Z_0) \geq 0, \qquad (\beta - b\Theta)W(0) \geq 0, \tag{2.14}$$

$$(\alpha - a\Theta)K(Z_0) + (\beta - b\Theta)W(0) > 0, \tag{2.15}$$

$$(\alpha - a\theta)K(x) + (\beta - b\theta)W(x - Z_0) \geq 0, \qquad \textit{for all } x \in \mathbb{R}, \tag{2.16}$$

$$(\alpha - a\Theta)K(x) + (\beta - b\Theta)W(x - Z_0) \geq 0, \qquad \textit{for all } x \in \mathbb{R}, \tag{2.17}$$

$$\int_{-\infty}^{-Z_0} W(z)\mathrm{d}z = \frac{(2m+a)\theta - (2mn+\alpha)}{2(\beta - b\theta)}, \tag{2.18}$$

$$\int_{-\infty}^{Z_0} K(z)\mathrm{d}z = \frac{(2m+b)\Theta - (2mn+\beta)}{2(\alpha - a\Theta)}, \tag{2.19}$$

$$k + m + a\int_{-\infty}^{x} K(z)\mathrm{d}z + b\int_{-\infty}^{x-Z_0} W(z)\mathrm{d}z > 0, \ \textit{for all } x \in \mathbb{R}. \tag{2.20}$$

(I) There exist two complicated standing wave solutions $u_1(x,t) = \varphi_1(x)$ and $u_2(x,t) = \varphi_2(x)$ to the scalar integral differential equation (1.3), where

$$\begin{aligned}
\varphi_1(x) &= \left\{ mn + \alpha\int_{-\infty}^{x} K(z)\mathrm{d}z + \beta\int_{-\infty}^{x-Z_0} W(z)\mathrm{d}z \right\} \\
&\quad / \left\{ m + a\int_{-\infty}^{x} K(z)\mathrm{d}z + b\int_{-\infty}^{x-Z_0} W(z)\mathrm{d}z \right\}, \\
\varphi_2(x) &= \left\{ mn + \alpha\int_{x}^{\infty} K(z)\mathrm{d}z + \beta\int_{x+Z_0}^{\infty} W(z)\mathrm{d}z \right\} \\
&\quad / \left\{ m + a\int_{x}^{\infty} K(z)\mathrm{d}z + b\int_{x+Z_0}^{\infty} W(z)\mathrm{d}z \right\}.
\end{aligned}$$

The standing wave solutions satisfy the following boundary conditions

$$\begin{aligned}
\lim_{x\to-\infty} u_1(x,t) &= n, \qquad \lim_{x\to\infty} u_1(x,t) = \frac{\alpha+\beta+mn}{a+b+m}, \\
\lim_{x\to-\infty} u_2(x,t) &= \frac{\alpha+\beta+mn}{a+b+m}, \qquad \lim_{x\to\infty} u_2(x,t) = n.
\end{aligned}$$

(II) Suppose that the constants satisfy the additional conditions

$$\frac{\alpha+2mn}{a+2m} = \theta < \frac{\alpha+mn}{a+m} < \Theta = \frac{2\alpha+\beta+2mn}{2a+b+2m}.$$

There exist four additional standing wave solutions $u_i(x,t) = \varphi_i(x)$ to the scalar integral differential equation (1.3), where $i = 3,4,5,6$, and

$$\begin{aligned}
\varphi_3(x) &= \left\{ mn + \alpha\int_{-\infty}^{x} K(z)\mathrm{d}z \right\} / \left\{ m + a\int_{-\infty}^{x} K(z)\mathrm{d}z \right\}, \\
\varphi_4(x) &= \left\{ mn + \alpha\int_{x}^{\infty} K(z)\mathrm{d}z \right\} / \left\{ m + a\int_{x}^{\infty} K(z)\mathrm{d}z \right\}, \\
\varphi_5(x) &= \left\{ mn + \alpha + \beta\int_{-\infty}^{x} W(z)\mathrm{d}z \right\} / \left\{ m + a + b\int_{-\infty}^{x} W(z)\mathrm{d}z \right\}, \\
\varphi_6(x) &= \left\{ mn + \alpha + \beta\int_{x}^{\infty} W(z)\mathrm{d}z \right\} / \left\{ m + a + b\int_{x}^{\infty} W(z)\mathrm{d}z \right\}.
\end{aligned}$$

These standing wave solutions satisfy the following boundary conditions

$$
\begin{aligned}
\lim_{x\to-\infty} u_3(x,t) &= n, \qquad \lim_{x\to\infty} u_3(x,t) = \frac{\alpha+mn}{a+m},\\
\lim_{x\to-\infty} u_4(x,t) &= \frac{\alpha+mn}{a+m}, \qquad \lim_{x\to\infty} u_4(x,t) = n,\\
\lim_{x\to-\infty} u_5(x,t) &= \frac{\alpha+mn}{a+m}, \qquad \lim_{x\to\infty} u_5(x,t) = \frac{\alpha+\beta+mn}{a+b+m},\\
\lim_{x\to-\infty} u_6(x,t) &= \frac{\alpha+\beta+mn}{a+b+m}, \qquad \lim_{x\to\infty} u_6(x,t) = \frac{\alpha+mn}{a+m}.
\end{aligned}
$$

Proof. The proof follows from Theorem 2.1. □

Remark 2.4. It is not difficulty to show that

$$
\begin{aligned}
\phi_1(x) = \phi_2(-x), \quad \phi_3(x) = \phi_4(-x), \quad \phi_5(x) = \phi_6(-x),\\
\varphi_1(x) = \varphi_2(-x), \quad \varphi_3(x) = \varphi_4(-x), \quad \varphi_5(x) = \varphi_6(-x),
\end{aligned}
$$

for all $x \in \mathbb{R}$. Therefore, we obtain

$$
\begin{aligned}
{\phi_1}'(x) = -{\phi_2}'(-x), &\qquad {\phi_1}'(0) = -{\phi_2}'(0),\\
{\phi_3}'(x) = -{\phi_4}'(-x), &\qquad {\phi_3}'(0) = -{\phi_4}'(0),\\
{\phi_5}'(x) = -{\phi_6}'(-x), &\qquad {\phi_5}'(0) = -{\phi_6}'(0),
\end{aligned}
$$

and ${\phi_1}'(Z_0) = -{\phi_2}'(-Z_0)$. Moreover, we have

$$
\begin{aligned}
{\varphi_1}'(x) = -{\varphi_2}'(-x), &\qquad {\varphi_1}'(0) = -{\varphi_2}'(0),\\
{\varphi_3}'(x) = -{\varphi_4}'(-x), &\qquad {\varphi_3}'(0) = -{\varphi_4}'(0),\\
{\varphi_5}'(x) = -{\varphi_6}'(-x), &\qquad {\varphi_5}'(0) = -{\varphi_6}'(0).
\end{aligned}
$$

These relationships will play important roles in the construction of Evans functions and to establish the stability/instability of the standing wave solutions.

Remark 2.5. Both standing wave solutions of system (1.1)-(1.2) enjoy the estimates

$$
\begin{aligned}
&\frac{1}{|\phi'(0)|}\\
=\ &\left\{k+m+\frac{a}{2}+b\int_{-\infty}^{-Z_0} W(z)\mathrm{d}z\right\} / \big\{(\alpha-a\theta)K(0)+(\beta-b\theta)W(-\gamma Z_0)\big\} > 0,\\
&\frac{1}{|\phi'(\gamma Z_0)|}\\
=\ &\left\{k+m+a\int_{-\infty}^{Z_0} K(z)\mathrm{d}z+\frac{b}{2}\right\} / \big\{(\alpha-a\Theta)K(\gamma Z_0)+(\beta-b\Theta)W(0)\big\} > 0.
\end{aligned}
$$

where $\gamma = 1$ for the first standing wave solution and $\gamma = -1$ for the second standing wave solution.

Both standing wave solutions of equation (1.3) enjoy the estimates

$$
\begin{aligned}
\frac{1}{|\varphi'(0)|} &= \left\{m+\frac{a}{2}+b\int_{-\infty}^{-Z_0} W(z)\mathrm{d}z\right\} / \big\{(\alpha-a\theta)K(0)+(\beta-b\theta)W(-\gamma Z_0)\big\} > 0,\\
\frac{1}{|\varphi'(\gamma Z_0)|} &= \left\{m+a\int_{-\infty}^{Z_0} K(z)\mathrm{d}z+\frac{b}{2}\right\} / \big\{(\alpha-a\Theta)K(\gamma Z_0)+(\beta-b\Theta)W(0)\big\} > 0.
\end{aligned}
$$

where $\gamma = 1$ for the first standing wave solution and $\gamma = -1$ for the second standing wave solution.

3. Stability/Instability of the Standing Wave Solutions

In this section, our goal is to establish the stability/instability of the standing wave solutions. First of all, we derive some eigenvalue problems. Then we construct complex analytic functions (namely, the Evans functions) corresponding to the eigenvalue problems. Then we study properties of the Evans functions. Finally, we finish the stability/instability analysis.

We will focus on the mathematical analysis of the eigenvalue problems and the Evans functions of the first two standing wave solutions because they cross both thresholds θ and Θ. For the next two standing wave solutions $\big(\phi_3, \psi_3\big)$ and $\big(\phi_4, \psi_4\big)$, simply let $b = 0$, $Z_0 = 0$ and $\beta = 0$ because $\big(\phi_3, \psi_3\big)$ and $\big(\phi_4, \psi_4\big)$ do not cross the big threshold Θ. For the last two standing wave solutions $\big(\phi_5, \psi_5\big)$ and $\big(\phi_6, \psi_6\big)$, simply let $Z_0 = 0$ and $\dfrac{K(x)}{|\phi'(0)|} = 0$ because $\big(\phi_5, \psi_5\big)$ and $\big(\phi_6, \psi_6\big)$ do not cross the small threshold θ.

3.1. Derivation of the Eigenvalue Problems

Recall that all standing wave solutions satisfy

$$\begin{aligned}
\frac{\partial \phi}{\partial t} + f\big(\phi\big) + \psi = \big[\alpha - a\phi(x)\big] \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(x-y) H\big(\phi(y) - \theta\big) \mathrm{d}y\right] \mathrm{d}c \\
+ \big[\beta - b\phi(x)\big] \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(x-y) H\big(\phi(y) - \Theta\big) \mathrm{d}y\right] \mathrm{d}\tau, \\
\frac{\partial \psi}{\partial t} = \varepsilon[g\big(\phi\big) - \psi].
\end{aligned}$$

Linearizing the nonlinear system of integral differential equations

$$\begin{aligned}
&\frac{\partial u}{\partial t} + f(u) + w \\
=&(\alpha - au) \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(x-y) H\left(u\left(y, t - \frac{1}{c}|x-y|\right) - \theta\right) \mathrm{d}y\right] \mathrm{d}c \\
&+ (\beta - bu) \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(x-y) H\big(u(y, t-\tau) - \Theta\big) \mathrm{d}y\right] \mathrm{d}\tau, \\
&\frac{\partial w}{\partial t} = \varepsilon[g(u) - w],
\end{aligned}$$

about the standing wave solutions, we obtain the linear system of integral differential equations

$$\frac{\partial w}{\partial t} = \varepsilon\big[g'\big(\phi(x)\big)u - w\big],$$

$$\begin{aligned}
\frac{\partial u}{\partial t}+f'\big(\phi(x)\big)u+w \quad = \quad & \big[\alpha-a\phi(x)\big]\frac{K(x)}{|\phi'(0)|}\int_0^\infty \xi(c)u\left(0,t-\frac{1}{c}|x|\right)\mathrm{d}c \\
& +\big[\beta-b\phi(x)\big]\frac{W(x-\gamma Z_0)}{|\phi'(\gamma Z_0)|}\int_0^\infty \eta(\tau)u(\gamma Z_0,t-\tau)\mathrm{d}\tau \\
& -au\int_0^\infty \xi(c)\left[\int_{\mathbb{R}} K(x-y)H\big(\phi(y)-\theta\big)\mathrm{d}y\right]\mathrm{d}c \\
& -bu\int_0^\infty \eta(\tau)\left[\int_{\mathbb{R}} W(x-y)H\big(\phi(y)-\Theta\big)\mathrm{d}y\right]\mathrm{d}\tau,
\end{aligned}$$

where $\gamma=1$ for the first standing wave solution $\big(\phi_1,\psi_1\big)$ and $\gamma=-1$ for the second standing wave solution $\big(\phi_2,\psi_2\big)$.

Suppose that $\big(u(x,t),w(x,t)\big)=\exp(\lambda t)\big(\psi_1(x),\psi_2(x)\big)$ is a solution of this system, where λ is a complex number and $\psi_1(x)$ and $\psi_2(x)$ are bounded complex functions. Then

$$\begin{aligned}
&\lambda\psi_1(x)\exp(\lambda t)+f'\big(\phi(x)\big)\psi_1(x)\exp(\lambda t)+\psi_2(x)\exp(\lambda t) \\
=&\big[\alpha-a\phi(x)\big]\frac{K(x)}{|\phi'(0)|}\int_0^\infty \xi(c)\psi_1(0)\exp\left(\lambda t-\frac{\lambda}{c}|x|\right)\mathrm{d}c \\
&+\big[\beta-b\phi(x)\big]\frac{W(x-\gamma Z_0)}{|\phi'(\gamma Z_0)|}\int_0^\infty \eta(\tau)\psi_1(\gamma Z_0)\exp(\lambda t-\lambda\tau)\mathrm{d}\tau \\
&-a\psi_1(x)\exp(\lambda t)\int_0^\infty \xi(c)\left[\int_{\mathbb{R}} K(x-y)H\big(\phi(y)-\theta\big)\mathrm{d}y\right]\mathrm{d}c \\
&-b\psi_1(x)\exp(\lambda t)\int_0^\infty \eta(\tau)\left[\int_{\mathbb{R}} W(x-y)H\big(\phi(y)-\Theta\big)\mathrm{d}y\right]\mathrm{d}\tau, \\
&\lambda\psi_2(x)\exp(\lambda t)=\varepsilon\big[g'\big(\phi(x)\big)\psi_1(x)\exp(\lambda t)-\psi_2(x)\exp(\lambda t)\big].
\end{aligned}$$

By canceling out the exponential function $\exp(\lambda t)$, we obtain the eigenvalue problems

$$\begin{aligned}
&\lambda\psi_1(x)+f'\big(\phi(x)\big)\psi_1(x)+\psi_2(x) \\
=&\big[\alpha-a\phi(x)\big]\frac{K(x)}{|\phi'(0)|}\left[\int_0^\infty \xi(c)\exp\left(-\frac{\lambda}{c}|x|\right)\mathrm{d}c\right]\psi_1(0) \\
&+\big[\beta-b\phi(x)\big]\frac{W(x-\gamma Z_0)}{|\phi'(\gamma Z_0)|}\psi_1(\gamma Z_0) \\
&-a\left\{\int_0^\infty \xi(c)\left[\int_{\mathbb{R}} K(x-y)H\big(\phi(y)-\theta\big)\mathrm{d}y\right]\mathrm{d}c\right\}\psi_1(x) \\
&-b\left\{\int_0^\infty \eta(\tau)\left[\int_{\mathbb{R}} W(x-y)H\big(\phi(y)-\Theta\big)dy\right]\mathrm{d}\tau\right\}\psi_1(x), \\
&\lambda\psi_2(x)=\varepsilon\big[g'\big(\phi(x)\big)\psi_1(x)-\psi_2(x)\big].
\end{aligned}$$

The eigenvalue problems may be written as

$$\mathcal{L}(\lambda)\psi=\lambda\psi, \qquad \psi(x)=\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix}\in[L^\infty(\mathbb{R})]^2, \tag{3.1}$$

where $\mathcal{L}(\lambda):[L^\infty(\mathbb{R})]^2\to[L^\infty(\mathbb{R})]^2$ is a family of linear operators. As we can easily see, due to the presence of any of the delays, these eigenvalue problems are nonlinear in λ. This new feature makes the eigenvalue problems more difficult to solve. We will construct Evans functions to investigate the eigenvalues of this family of linear operators.

Definition 3.1. If there exists a complex number λ_0 and there exists a nontrivial bounded continuous function ψ_0 defined on $\mathbb{R}$, such that $\mathcal{L}(\lambda_0)\psi_0 = \lambda_0\psi_0$, then λ_0 is called an eigenvalue and ψ_0 is called an eigenfunction of the eigenvalue problem.

3.2. Derivation of the Evans Functions

Define a real constant γ: $\gamma = 1$ for the first standing wave solution and $\gamma = -1$ for the second standing wave solution. Note that for the standing wave solutions, we have

$$\begin{aligned}
& \int_0^\infty \xi(c)\left[\int_{\mathbb{R}} K(x-y)H\big(\phi(y)-\theta\big)\mathrm{d}y\right]\mathrm{d}c \\
= \; & \int_{\mathbb{R}} K(x-y)H\big(\phi(y)-\theta\big)\mathrm{d}y = \int_{-\infty}^{\gamma x} K(z)\mathrm{d}z, \\
& \int_0^\infty \eta(\tau)\left[\int_{\mathbb{R}} W(x-y)H\big(\phi(y)-\Theta\big)\mathrm{d}y\right]\mathrm{d}\tau \\
= \; & \int_{\mathbb{R}} W(x-y)H\big(\phi(y)-\Theta\big)\mathrm{d}y = \int_{-\infty}^{\gamma x - Z_0} W(z)\mathrm{d}z.
\end{aligned}$$

Now we are ready to solve the eigenvalue problems and construct the Evans functions to study the stability/instability of the standing wave solutions.

Define the positive constants

$$\xi_0 = \int_0^\infty \frac{1}{c}\xi(c)\mathrm{d}c, \qquad \eta_0 = \int_0^\infty \tau\eta(\tau)\mathrm{d}\tau.$$

Define two complex analytic functions

$$\begin{aligned}
\xi_1(\lambda) &= \int_0^\infty \xi(c)\exp\left(-\frac{\lambda}{c}|Z_0|\right)\mathrm{d}c, \qquad \eta_1(\lambda) = \int_0^\infty \eta(\tau)\exp(-\lambda\tau)\mathrm{d}\tau, \\
\xi_2(\lambda) &= \int_0^\infty \frac{|Z_0|}{c}\xi(c)\exp\left(-\frac{\lambda}{c}|Z_0|\right)\mathrm{d}c, \qquad \eta_2(\lambda) = \int_0^\infty \tau\eta(\tau)\exp(-\lambda\tau)\mathrm{d}\tau.
\end{aligned}$$

Letting $x = 0$ and $x = \gamma Z_0$, respectively, in the eigenvalue problems, we have

$$\begin{aligned}
& \lambda\psi_1(0) + f'(\theta)\psi_1(0) + \psi_2(0) \\
= & (\alpha - a\theta)\frac{K(0)}{|\phi'(0)|}\psi_1(0) + (\beta - b\theta)\frac{W(-\gamma Z_0)}{|\phi'(\gamma Z_0)|}\eta_1(\lambda)\psi_1(\gamma Z_0) \\
& - \frac{a}{2}\psi_1(0) - b\omega_-\psi_1(0), \\
& \lambda\psi_2(0) = \varepsilon\big[g'(\theta)\psi_1(0) - \psi_2(0)\big],
\end{aligned}$$

and

$$\begin{aligned}
&\lambda\psi_1(\gamma Z_0) + f'(\Theta)\psi_1(\gamma Z_0) + \psi_2(\gamma Z_0)\\
=&(\alpha - a\Theta)\frac{K(\gamma Z_0)}{|\phi'(0)|}\xi_1(\lambda)\psi_1(0) + (\beta - b\Theta)\frac{W(0)}{|\phi'(\gamma Z_0)|}\eta_1(\lambda)\psi_1(\gamma Z_0)\\
&- a\omega_+\psi_1(\gamma Z_0) - \frac{b}{2}\psi_1(\gamma Z_0),\\
&\lambda\psi_2(\gamma Z_0) = \varepsilon\big[g'(\Theta)\psi_1(\gamma Z_0) - \psi_2(\gamma Z_0)\big].
\end{aligned}$$

For the standing wave solutions (ϕ_3, ψ_3) and (ϕ_4, ψ_4), we only need the first system to construct the Evans function $\mathcal{E}_2(\lambda, \varepsilon)$, where $b = 0$, $Z_0 = 0$ and $\beta = 0$. For the standing wave solutions (ϕ_5, ψ_5) and (ϕ_6, ψ_6), we only need the second system to construct the Evans function $\mathcal{E}_3(\lambda, \varepsilon)$, where we must let $Z_0 = 0$ and $\dfrac{K(0)}{|\phi'(0)|} = 0$.

In each of the above systems, it is easy to see that

$$\psi_2(0) = \frac{\varepsilon}{\lambda + \varepsilon}g'(\theta)\psi_1(0), \qquad \psi_2(\gamma Z_0) = \frac{\varepsilon}{\lambda + \varepsilon}g'(\Theta)\psi_1(\gamma Z_0).$$

If we plug ψ_2 back into the first equation, then we get the following equations

$$\begin{aligned}
&\left\{\lambda + f'(\theta) + \frac{\varepsilon}{\lambda + \varepsilon}g'(\theta) + \frac{a}{2} + b\omega_- - (\alpha - a\theta)\frac{K(0)}{|\phi'(0)|}\right\}\psi_1(0)\\
=\ &\left\{(\beta - b\theta)\frac{W(-\gamma Z_0)}{|\phi'(\gamma Z_0)|}\eta_1(\lambda)\right\}\psi_1(\gamma Z_0),\\
&\left\{\lambda + f'(\Theta) + \frac{\varepsilon}{\lambda + \varepsilon}g'(\Theta) + a\omega_+ + \frac{b}{2} - (\beta - b\Theta)\frac{W(0)}{|\phi'(\gamma Z_0)|}\eta_1(\lambda)\right\}\psi_1(\gamma Z_0)\\
=\ &\left\{(\alpha - a\Theta)\frac{K(\gamma Z_0)}{|\phi'(0)|}\xi_1(\lambda)\right\}\psi_1(0).
\end{aligned}$$

It is not difficult to find that $\psi_1(0) = 0$ if and only if $\psi_1(\gamma Z_0) = 0$. If $\psi_1(0) = 0$ or if $\psi_1(\gamma Z_0) = 0$, then

$$\begin{aligned}
&\left[\lambda + f'\big(\phi(x)\big) + \frac{\varepsilon}{\lambda + \varepsilon}g'\big(\phi(x)\big) + a\int_{-\infty}^{\gamma x} K(z)\mathrm{d}z + b\int_{-\infty}^{\gamma x - Z_0} W(z)\mathrm{d}z\right]\psi_1(x)\\
=&\big[\alpha - a\phi(x)\big]\frac{K(x)}{|\phi'(0)|}\left[\int_0^\infty \xi(c)\exp\left(-\frac{\lambda}{c}|x|\right)\mathrm{d}c\right]\psi_1(0)\\
&+ \big[\beta - b\phi(x)\big]\frac{W(x - \gamma Z_0)}{|\phi'(\gamma Z_0)|}\eta_1(\lambda)\psi_1(\gamma Z_0) = 0.
\end{aligned}$$

Hence $\psi_1(x) = 0$, for all $x \in \mathbb{R}$. Therefore, if λ_0 is an eigenvalue and $\psi_0 = \begin{pmatrix} \psi_{01} \\ \psi_{02} \end{pmatrix}$ is an eigenfunction of the eigenvalue problem $\mathcal{L}(\lambda)\psi = \lambda\psi$, then $\psi_{01}(0) \neq 0$ and $\psi_{01}(\gamma Z_0) \neq 0$.

If we multiply these two equations together and cancel out $\psi_1(0)\psi_1(\gamma Z_0)$, then we find that

$$\left\{\lambda + f'(\theta) + \frac{\varepsilon}{\lambda+\varepsilon} g'(\theta) + \frac{a}{2} + b\omega_- - (\alpha - a\theta)\frac{K(0)}{|\phi'(0)|}\right\}$$
$$\cdot \left\{\lambda + f'(\Theta) + \frac{\varepsilon}{\lambda+\varepsilon} g'(\Theta) + a\omega_+ + \frac{b}{2} - (\beta - b\Theta)\frac{W(0)}{|\phi'(\gamma Z_0)|}\eta_1(\lambda)\right\}$$
$$= \left\{(\beta - b\theta)\frac{W(-\gamma Z_0)}{|\phi'(\gamma Z_0)|}\eta_1(\lambda)\right\}\left\{(\alpha - a\Theta)\frac{K(\gamma Z_0)}{|\phi'(0)|}\xi_1(\lambda)\right\}.$$

Definition 3.2. (I) Define the domain $\Omega = \{\lambda \in \mathbb{C}\colon\ \lambda \neq -\varepsilon\}$.
(II) Define the domain $\Omega_0 = \{\lambda \in \mathbb{C}\colon\ \lambda \text{ satisfies the following conditions}\}$,

$$\operatorname{Re}\lambda > -f'(\theta) - \frac{a}{2} - b\omega_-, \quad \operatorname{Re}\lambda > -f'(\Theta) - a\omega_+ - \frac{b}{2}.$$

(III) Define the Evans function $\mathcal{E} = \mathcal{E}_1(\lambda, \varepsilon)$ for the first two standing wave solutions (ϕ_1, ψ_1) and (ϕ_2, ψ_2) of the nonlinear singularly perturbed system (1.1)-(1.2) by

$$\mathcal{E}_1(\lambda,\varepsilon) = \left\{\lambda + f'(\theta) + \frac{\varepsilon}{\lambda+\varepsilon} g'(\theta) + \frac{a}{2} + b\omega_- - (\alpha - a\theta)\frac{K(0)}{|\phi'(0)|}\right\}$$
$$\cdot \quad \left\{\lambda + f'(\Theta) + \frac{\varepsilon}{\lambda+\varepsilon} g'(\Theta) + a\omega_+ + \frac{b}{2} - (\beta - b\Theta)\frac{W(0)}{|\phi'(\gamma Z_0)|}\eta_1(\lambda)\right\}$$
$$- \quad \left\{(\beta - b\theta)\frac{W(-\gamma Z_0)}{|\phi'(\gamma Z_0)|}\eta_1(\lambda)\right\}\left\{(\alpha - a\Theta)\frac{K(\gamma Z_0)}{|\phi'(0)|}\xi_1(\lambda)\right\},$$

for all $\lambda \in \Omega$.
(IV) Define the Evans function $\mathcal{E} = \mathcal{E}_2(\lambda, \varepsilon)$ for the next two standing wave solutions (ϕ_3, ψ_3) and (ϕ_4, ψ_4) of the nonlinear singularly perturbed system (1.1)-(1.2) by

$$\mathcal{E}_2(\lambda,\varepsilon) = \lambda + f'(\theta) + \frac{\varepsilon}{\lambda+\varepsilon} g'(\theta) + \frac{a}{2} - (\alpha - a\theta)\frac{K(0)}{|\phi'(0)|},$$

for all $\lambda \in \Omega$.
(V) Define the Evans function $\mathcal{E} = \mathcal{E}_3(\lambda, \varepsilon)$ for the last two standing wave solutions (ϕ_5, ψ_5) and (ϕ_6, ψ_6) of the nonlinear singularly perturbed system (1.1)-(1.2) by

$$\mathcal{E}_3(\lambda,\varepsilon) = \lambda + f'(\Theta) + \frac{\varepsilon}{\lambda+\varepsilon} g'(\Theta) + a + \frac{b}{2} - (\beta - b\Theta)\frac{W(0)}{|\phi'(0)|}\eta_1(\lambda),$$

for all $\lambda \in \Omega$.
(VI) Define the Evans function $\mathcal{E} = \mathcal{E}_1(\lambda)$ for the first two standing wave solutions φ_1 and φ_2 of the scalar integral differential equation (1.3) by

$$\mathcal{E}_1(\lambda) = \left\{\lambda + f'(\theta) + \frac{a}{2} + b\omega_- - (\alpha - a\theta)\frac{K(0)}{|\phi'(0)|}\right\}$$
$$\cdot \quad \left\{\lambda + f'(\Theta) + a\omega_+ + \frac{b}{2} - (\beta - b\Theta)\frac{W(0)}{|\phi'(\gamma Z_0)|}\eta_1(\lambda)\right\}$$
$$- \quad \left\{(\beta - b\theta)\frac{W(-\gamma Z_0)}{|\phi'(\gamma Z_0)|}\eta_1(\lambda)\right\}\left\{(\alpha - a\Theta)\frac{K(\gamma Z_0)}{|\phi'(0)|}\xi_1(\lambda)\right\},$$

for all $\lambda \in \Omega_0$.

(VII) Define the Evans function $\mathcal{E} = \mathcal{E}_2(\lambda)$ for the next two standing wave solutions φ_3 and φ_4 of the scalar integral differential equation (1.3) by

$$\mathcal{E}_2(\lambda) = \lambda + f'(\theta) + \frac{a}{2} - (\alpha - a\theta)\frac{K(0)}{|\phi'(0)|},$$

for all $\lambda \in \Omega_0$.

(VIII) Define the Evans function $\mathcal{E} = \mathcal{E}_3(\lambda)$ for the last two standing wave solutions φ_5 and φ_6 of the scalar integral differential equation (1.3) by

$$\mathcal{E}_3(\lambda) = \lambda + f'(\Theta) + a + \frac{b}{2} - (\beta - b\Theta)\frac{W(0)}{|\phi'(0)|}\eta_1(\lambda),$$

for all $\lambda \in \Omega_0$.

Remark 3.1. The definitions of the Evans functions $\mathcal{E}_i(\lambda, \varepsilon)$ for the last four standing wave solutions of (1.1)-(1.2) make sense only if

$$\frac{\alpha + 2kl + 2mn}{a + 2k + 2m} = \theta < \frac{\alpha + kl + mn}{a + k + m} < \Theta = \frac{2\alpha + \beta + 2kl + 2mn}{2a + b + 2k + 2m}.$$

Similarly, the definitions of the Evans functions $\mathcal{E}_i(\lambda)$ for the last four standing wave solutions of (1.3) make sense only if

$$\frac{\alpha + 2mn}{a + 2m} = \theta < \frac{\alpha + mn}{a + m} < \Theta = \frac{2\alpha + \beta + 2mn}{2a + b + 2m}.$$

Theorem 3.1. *(I) The Evans function $\mathcal{E} = \mathcal{E}_1(\lambda, \varepsilon)$ is a complex analytic function of λ and it is real valued if the eigenvalue parameter λ is real.*

(II) The complex number λ_0 is an eigenvalue of the eigenvalue problem $\mathcal{L}(\lambda)\psi = \lambda\psi$ if and only if $\mathcal{E}_1(\lambda_0, \varepsilon) = 0$.

(III) The Evans functions enjoy the following limit

$$\lim_{|\lambda|\to\infty} \frac{\mathcal{E}_1(\lambda, \varepsilon)}{\lambda^2} = 1.$$

Proof. (I) Obviously, the statement is true.

(II) If $\lambda_0 \in \Omega$ is an eigenvalue, then there exists a nontrivial bounded continuous solution $\psi_0 = \begin{pmatrix} \psi_{01} \\ \psi_{02} \end{pmatrix}$ to the eigenvalue problem $\mathcal{L}(\lambda)\psi = \lambda\psi$, such that $\psi_{01}(0)\psi_{01}(\gamma Z_0) \neq 0$. Therefore, $\mathcal{E}_1(\lambda_0, \varepsilon) = 0$. On the other hand, if $\mathcal{E}_1(\lambda_0, \varepsilon) = 0$, then there exists a nontrivial vector $\big(\psi_{01}(0), \psi_{01}(\gamma Z_0)\big)$, such that

$$\begin{pmatrix} a_{11}(\lambda_0, \varepsilon) & a_{12}(\lambda_0, \varepsilon) \\ a_{21}(\lambda_0, \varepsilon) & a_{22}(\lambda_0, \varepsilon) \end{pmatrix} \begin{pmatrix} \psi_{01}(0) \\ \psi_{01}(\gamma Z_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where

$$\mathcal{E}_1(\lambda, \varepsilon) \equiv \det \begin{pmatrix} a_{11}(\lambda, \varepsilon) & a_{12}(\lambda, \varepsilon) \\ a_{21}(\lambda, \varepsilon) & a_{22}(\lambda, \varepsilon) \end{pmatrix}.$$

Moreover, there exists a nontrivial bounded continuous function $\psi_0 = \begin{pmatrix} \psi_{01}(x) \\ \psi_{02}(x) \end{pmatrix}$, such that $\mathcal{L}(\lambda_0)\psi_0 = \lambda_0\psi_0$. Therefore, λ_0 is an eigenvalue of the eigenvalue problem $\mathcal{L}(\lambda)\psi = \lambda\psi$.

(III) Clearly, the conclusion is correct.

The proof of Theorem 3.1 is finished. $\square$

Remark 3.2. We believe that the algebraic multiplicity of any eigenvalue λ_0 of the eigenvalue problem $\mathcal{L}(\lambda)\psi = \lambda\psi$ is equal to the order of λ_0 as a zero of the corresponding Evans function $\mathcal{E} = \mathcal{E}_1(\lambda, \varepsilon)$. However, it may be very complicated to prove.

Let us find the derivatives of the Evans functions. For the nonlinear singularly perturbed system (1.1)-(1.2), by using Definition 3.2, we find that

$$\begin{aligned}
\frac{\partial \mathcal{E}_1}{\partial \lambda}(\lambda, \varepsilon) = &\left\{1 - \frac{\varepsilon}{(\lambda+\varepsilon)^2} g'(\theta)\right\} \\
&\cdot \left\{\lambda + f'(\Theta) + \frac{\varepsilon}{\lambda+\varepsilon} g'(\Theta) + a\omega_+ + \frac{b}{2} - (\beta - b\Theta)\frac{W(0)}{|\phi'(\gamma Z_0)|}\eta_1(\lambda)\right\} \\
&+ \left\{\lambda + f'(\theta) + \frac{\varepsilon}{\lambda+\varepsilon} g'(\theta) + \frac{a}{2} + b\omega_- - (\alpha - a\theta)\frac{K(0)}{|\phi'(0)|}\right\} \\
&\cdot \left\{1 - \frac{\varepsilon}{(\lambda+\varepsilon)^2} g'(\Theta) + (\beta - b\Theta)\frac{W(0)}{|\phi'(\gamma Z_0)|}\eta_2(\lambda)\right\} \\
&+ \left\{(\beta - b\theta)\frac{W(-\gamma Z_0)}{|\phi'(\gamma Z_0)|}\eta_2(\lambda)\right\}\left\{(\alpha - a\Theta)\frac{K(\gamma Z_0)}{|\phi'(0)|}\xi_1(\lambda)\right\} \\
&+ \left\{(\beta - b\theta)\frac{W(-\gamma Z_0)}{|\phi'(\gamma Z_0)|}\eta_1(\lambda)\right\}\left\{(\alpha - a\Theta)\frac{K(\gamma Z_0)}{|\phi'(0)|}\xi_2(\lambda)\right\}.
\end{aligned}$$

Moreover, for the scalar integral differential equation (1.3), we have

$$\begin{aligned}
{\mathcal{E}_1}'(\lambda) = &\left\{\lambda + f'(\Theta) + a\omega_+ + \frac{b}{2} - (\beta - b\Theta)\frac{W(0)}{|\phi'(\gamma Z_0)|}\eta_1(\lambda)\right\} \\
&+ \left\{\lambda + f'(\theta) + \frac{a}{2} + b\omega_- - (\alpha - a\theta)\frac{K(0)}{|\phi'(0)|}\right\} \\
&\cdot \left\{1 + (\beta - b\Theta)\frac{W(0)}{|\phi'(\gamma Z_0)|}\eta_2(\lambda)\right\} \\
&+ \left\{(\beta - b\theta)\frac{W(-\gamma Z_0)}{|\phi'(\gamma Z_0)|}\eta_2(\lambda)\right\} \cdot \left\{(\alpha - a\Theta)\frac{K(\gamma Z_0)}{|\phi'(0)|}\xi_1(\lambda)\right\} \\
&+ \left\{(\beta - b\theta)\frac{W(-\gamma Z_0)}{|\phi'(\gamma Z_0)|}\eta_1(\lambda)\right\} \\
&\cdot \left\{(\alpha - a\Theta)\frac{K(\gamma Z_0)}{|\phi'(0)|}\left[\int_0^\infty \frac{|Z_0|}{c}\xi(c)\exp\left(-\frac{\lambda}{c}|Z_0|\right)\mathrm{d}c\right]\right\}.
\end{aligned}$$

In particular, for $\lambda = 0$, we have

$$\begin{aligned}
\frac{\partial \mathcal{E}_1}{\partial \lambda}(0, \varepsilon) = &\left\{1 - \frac{1}{\varepsilon} g'(\theta)\right\} \cdot \left\{f'(\Theta) + g'(\Theta) + a\omega_+ + \frac{b}{2} - (\beta - b\Theta)\frac{W(0)}{|\phi'(\gamma Z_0)|}\right\} \\
&+ \left\{f'(\theta) + g'(\theta) + \frac{a}{2} + b\omega_- - (\alpha - a\theta)\frac{K(0)}{|\phi'(0)|}\right\} \\
&\cdot \left\{1 - \frac{1}{\varepsilon} g'(\Theta) + (\beta - b\Theta)\frac{W(0)}{|\phi'(\gamma Z_0)|}\eta_0\right\} \\
&+ \left\{(\beta - b\theta)\frac{W(-\gamma Z_0)}{|\phi'(\gamma Z_0)|}\eta_0\right\}\left\{(\alpha - a\Theta)\frac{K(\gamma Z_0)}{|\phi'(0)|}\right\}
\end{aligned}$$

$$+\left\{(\beta-b\theta)\frac{W(-\gamma Z_0)}{|\phi'(\gamma Z_0)|}\right\}\left\{(\alpha-a\Theta)\frac{K(\gamma Z_0)}{|\phi'(0)|}|Z_0|\xi_1(\lambda)\right\}.$$

Moreover

$$\begin{aligned}
{\mathcal{E}_1}'(0)=&\left\{f'(\Theta)+a\omega_++\frac{b}{2}-(\beta-b\Theta)\frac{W(0)}{|\phi'(\gamma Z_0)|}\right\}\\
&+\left\{f'(\theta)+\frac{a}{2}+b\omega_--(\alpha-a\theta)\frac{K(0)}{|\phi'(0)|}\right\}\left\{1+(\beta-b\Theta)\frac{W(0)}{|\phi'(\gamma Z_0)|}\eta_0\eta_0\right\}\\
&+\left\{(\beta-b\theta)\frac{W(-\gamma Z_0)}{|\phi'(\gamma Z_0)|}\eta_0\right\}\left\{(\alpha-a\Theta)\frac{K(\gamma Z_0)}{|\phi'(0)|}\left[\int_0^\infty\xi(c)\mathrm{d}c\right]\right\}\\
&+\left\{(\beta-b\theta)\frac{W(-\gamma Z_0)}{|\phi'(\gamma Z_0)|}\right\}\left\{(\alpha-a\Theta)\frac{K(\gamma Z_0)}{|\phi'(0)|}x|Z_0|\xi_0\right\}>0.
\end{aligned}$$

Let us also find the partial derivatives of the Evans functions $\mathcal{E}_1(\lambda,\varepsilon)$ with respect to ε:

$$\begin{aligned}
&\frac{\partial\mathcal{E}_1}{\partial\varepsilon}(\lambda,\varepsilon)\\
=&\frac{\lambda}{(\lambda+\varepsilon)^2}g'(\theta)\cdot\left\{\lambda+f'(\Theta)+\frac{\varepsilon}{\lambda+\varepsilon}g'(\Theta)+a\omega_++\frac{b}{2}-(\beta-b\Theta)\frac{W(0)}{|\phi'(\gamma Z_0)|}\eta_1(\lambda)\right\}\\
&+\frac{\lambda}{(\lambda+\varepsilon)^2}g'(\Theta)\left\{\lambda+f'(\theta)+\frac{\varepsilon}{\lambda+\varepsilon}g'(\theta)+\frac{a}{2}+b\omega_--(\alpha-a\theta)\frac{K(0)}{|\phi'(0)|}\right\}.
\end{aligned}$$

For the next four standing wave solutions of the nonlinear system (1.1)-(1.2), we find that

$$\begin{aligned}
\frac{\partial\mathcal{E}_2}{\partial\lambda}(\lambda,\varepsilon)&=1-\frac{\varepsilon}{(\lambda+\varepsilon)^2}g'(\theta),\\
\frac{\partial\mathcal{E}_2}{\partial\lambda}(0,\varepsilon)&=1-\frac{1}{\varepsilon}g'(\theta),
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial\mathcal{E}_3}{\partial\lambda}(\lambda,\varepsilon)&=1-\frac{\varepsilon}{(\lambda+\varepsilon)^2}g'(\Theta)+(\beta-b\Theta)\frac{W(0)}{|\phi'(0)|}\eta_2(\lambda),\\
\frac{\partial\mathcal{E}_3}{\partial\lambda}(0,\varepsilon)&=1-\frac{1}{\varepsilon}g'(\Theta)+(\beta-b\Theta)\frac{W(0)}{|\phi'(0)|}\eta_0.
\end{aligned}$$

For the next four standing wave solutions of the scalar integral differential equation (1.3), we have

$$\begin{aligned}
\frac{\partial\mathcal{E}_2}{\partial\lambda}(\lambda)=1,\qquad\frac{\partial\mathcal{E}_2}{\partial\lambda}(0)=1,\\
\frac{\partial\mathcal{E}_3}{\partial\lambda}(\lambda)=1,\qquad\frac{\partial\mathcal{E}_3}{\partial\lambda}(0)=1.
\end{aligned}$$

Definition 3.3. Define the positive numbers ε_1, ε_2 and ε_3 for the three pairs of s-tanding wave solutions $\big[(\phi_1,\psi_1),(\phi_2,\psi_2)\big]$, $\big[(\phi_3,\psi_3),(\phi_4,\psi_4)\big]$, $\big[(\phi_5,\psi_5),(\phi_6,\psi_6)\big]$ of the system (1.1)-(1.2) by using the conditions

$$\frac{\partial\mathcal{E}_1}{\partial\lambda}(0,\varepsilon_1)=0,\qquad\frac{\partial\mathcal{E}_2}{\partial\lambda}(0,\varepsilon_2)=0,\qquad\frac{\partial\mathcal{E}_3}{\partial\lambda}(0,\varepsilon_3)=0.$$

Therefore, we find that

$$\begin{aligned}\varepsilon_1 &= O(1), \quad \varepsilon_2 = g'(\theta) > 0,\\ \varepsilon_3 &= g'(\Theta) / \left\{ 1 + (\beta - b\Theta) \frac{W(0)}{|\phi'(0)|} \eta_0 \right\} > 0.\end{aligned}$$

Theorem 3.2. *(I) $\lambda = 0$ is a simple eigenvalue of the eigenvalue problem $\mathcal{L}(\lambda)\psi = \lambda\psi$ except for $\varepsilon = \varepsilon_1$, that is*

$$\begin{aligned}&\mathcal{E}_1(0,\varepsilon) = 0, \qquad \frac{\partial \mathcal{E}_1}{\partial \lambda}(0,\varepsilon) > 0, \qquad \text{for all } \varepsilon \in (\varepsilon_1, \infty),\\ &\mathcal{E}_1(0,\varepsilon) = 0, \qquad \frac{\partial \mathcal{E}_1}{\partial \lambda}(0,\varepsilon) < 0, \qquad \text{for all } \varepsilon \in (0, \varepsilon_1).\end{aligned}$$

(II) For any $\varepsilon \in (0, \varepsilon_1)$, there exist two positive numbers $\lambda_0(\varepsilon)$ and $\lambda_1(\varepsilon)$, such that $\lambda_1(\varepsilon) > \lambda_0(\varepsilon) > 0$, and

$$\frac{\partial \mathcal{E}_1}{\partial \lambda}\big(\lambda_0(\varepsilon), \varepsilon\big) = 0, \qquad \mathcal{E}_1\big(\lambda_1(\varepsilon), \varepsilon\big) = 0.$$

(III) In the unbounded domain Ω, the Evans functions

$$\mathcal{E}_1(\lambda, \varepsilon) \neq 0, \qquad \text{for all } \varepsilon \in (\varepsilon_1, \infty), \qquad \text{for all } \lambda \neq 0, \text{with } \mathrm{Re}\lambda \geq 0.$$

Proof. (I) Differentiating the standing wave equations with respect to x, we get

$$\begin{aligned}f'\big(\phi(x)\big)\phi'(x) + \psi'(x) &= \big[\alpha - a\phi(x)\big]K(\gamma x) + \big[\beta - b\phi(x)\big]W(\gamma x - Z_0)\\ &\quad - a\phi'(x)\int_{-\infty}^{\gamma x} K(z)\mathrm{d}z - b\phi'(x)\int_{-\infty}^{\gamma x - Z_0} W(z)\mathrm{d}z,\\ 0 &= \varepsilon\big[g'\big(\phi(x)\big)\phi'(x) - \psi'(x)\big].\end{aligned}$$

From this system, we see that $\lambda = 0$ is an eigenvalue and $\big(\phi'(x), \psi'(x)\big)$ is an eigenfunction of the eigenvalue problem $\mathcal{L}(\lambda)\psi = \lambda\psi$.

By definition, $\dfrac{\partial \mathcal{E}_1}{\partial \lambda}(0,\varepsilon) = 0$. By the expression of the derivatives of the Evans functions with respect to λ, we find that if $\varepsilon \in (0, \varepsilon_1)$, then $\dfrac{\partial \mathcal{E}_1}{\partial \lambda}(0,\varepsilon) < 0$ and if $\varepsilon \in (\varepsilon_1, \infty)$, then $\dfrac{\partial \mathcal{E}_1}{\partial \lambda}(0,\varepsilon) > 0$.

(II) Fix $\varepsilon \in (0, \varepsilon_1)$. Then for all positive, sufficiently large $\lambda > 0$, we find that

$$\frac{\partial \mathcal{E}_1}{\partial \lambda}(\lambda, \varepsilon) > 0.$$

By using intermediate value theorem, we know that there exists a positive number $\lambda_0(\varepsilon) > 0$, such that

$$\frac{\partial \mathcal{E}_1}{\partial \lambda}\big(\lambda_0(\varepsilon), \varepsilon\big) = 0.$$

Clearly, we know that $\mathcal{E}_1(0,\varepsilon) = 0$ and that

$$\frac{\partial \mathcal{E}_1}{\partial \lambda}(\lambda, \varepsilon) < 0,$$

for all real number $\lambda \in \big(0, \lambda_0(\varepsilon)\big)$. Therefore,

$$\mathcal{E}_1\big(\lambda_0(\varepsilon), \varepsilon\big) < 0.$$

Very similar to before, for all positive, sufficiently large $\lambda > 0$, we find that $\mathcal{E}_1(\lambda, \varepsilon) > 0$.

By using intermediate value theorem, we know that there exists a positive number $\lambda_1(\varepsilon)$ with $\lambda_1(\varepsilon) > \lambda_0(\varepsilon) > 0$, such that

$$\mathcal{E}_1\big(\lambda_1(\varepsilon), \varepsilon\big) = 0.$$

Additionally, we find that

$$\lambda_0 = O(\varepsilon), \qquad \lambda_1 = O(1).$$

Lemma 3.1. *Suppose that the nonnegative function $\omega \geq 0$ is defined on $(0, \infty)$ and suppose that $0 < \int_0^\infty \omega(x)\mathrm{d}x < \infty$. For any complex number $\lambda \neq 0$, if $\mathrm{Re}\lambda \geq 0$, then*

$$\left|\int_0^\infty \exp(-\lambda x)\omega(x)\mathrm{d}x\right| < \int_0^\infty \omega(x)\mathrm{d}x.$$

Proof. See [14]. □

(III) For all $\varepsilon \in (\varepsilon_1, \infty)$ and for all $\lambda \in \Omega$, if $\mathrm{Re}\lambda \geq 0$ and $\lambda \neq 0$, then by using Lemma 3.1, we have the following estimates

$$\begin{aligned}
& \left|\lambda + f'(\theta) + \frac{\varepsilon}{\lambda + \varepsilon} g'(\theta) + \frac{a}{2} + b\omega_- - (\alpha - a\theta)\frac{K(0)}{|\phi'(0)|}\right| \\
> & \left|f'(\theta) + g'(\theta) + \frac{a}{2} + b\omega_- - (\alpha - a\theta)\frac{K(0)}{|\phi'(0)|}\right| \\
= & \left\{k + m + \frac{a}{2} + b\omega_-\right\} \frac{(\beta - b\theta)W(-\gamma Z_0)}{(\alpha - a\theta)K(0) + (\beta - b\theta)W(-\gamma Z_0)} \geq 0, \\
& \left|\lambda + f'(\Theta) + \frac{\varepsilon}{\lambda + \varepsilon} g'(\Theta) + a\omega_+ + \frac{b}{2} - (\beta - b\Theta)\frac{W(0)}{|\phi'(\gamma Z_0)|}\eta_1(\lambda)\right| \\
> & \left|f'(\Theta) + g'(\Theta) + a\omega_+ + \frac{b}{2} - (\beta - b\Theta)\frac{W(0)}{|\phi'(\gamma Z_0)|}\right| \\
= & \left\{k + m + a\omega_+ + \frac{b}{2}\right\} \frac{(\alpha - a\Theta)K(\gamma Z_0)}{(\alpha - a\Theta)K(\gamma Z_0) + (\beta - b\Theta)W(0)} \geq 0, \\
& \left|(\beta - b\theta)\frac{W(-\gamma Z_0)}{|\phi'(\gamma Z_0)|}\eta_1(\lambda)\right| \\
\leq & \left|(\beta - b\theta)\frac{W(-\gamma Z_0)}{|\phi'(\gamma Z_0)|}\right| = (\beta - b\theta)\frac{W(-\gamma Z_0)}{|\phi'(\gamma Z_0)|}, \\
& \left|(\alpha - a\Theta)\frac{K(\gamma Z_0)}{|\phi'(0)|}\xi_1(\lambda)\right| \\
\leq & \left|(\alpha - a\Theta)\frac{K(\gamma Z_0)}{|\phi'(0)|}\left[\int_0^\infty \xi(c)\mathrm{d}c\right]\right| = (\alpha - a\Theta)\frac{K(\gamma Z_0)}{|\phi'(0)|}.
\end{aligned}$$

Therefore, we obtain the desired estimate

$$|\mathcal{E}_1(\lambda, \varepsilon)| > |\mathcal{E}_1(0, \varepsilon)| = \mathcal{E}_1(0, \varepsilon) = 0.$$

The proof of Theorem 3.2 is finished. □

Corollary 3.1. *For the standing wave solutions of the integral differential equation (1.3), there hold the following results*

$$\mathcal{E}_1(0) = 0, \qquad {\mathcal{E}_1}'(0) > 0.$$

Moreover, for all complex number λ with $\lambda \neq 0$ and $\mathrm{Re}\lambda \geq 0$*, we have*

$$\mathcal{E}_1(\lambda) \neq 0.$$

Proof. The first half is easy to prove. Let us establish the estimates in the second half. We have the following estimates

$$\begin{aligned}
& \left|\lambda + f'(\theta) + \frac{a}{2} + b\omega_- - (\alpha - a\theta)\frac{K(0)}{|\phi'(0)|}\right| \\
> \ & \left|f'(\theta) + \frac{a}{2} + b\omega_- - (\alpha - a\theta)\frac{K(0)}{|\phi'(0)|}\right| \\
= \ & \left\{m + \frac{a}{2} + b\omega_-\right\}\frac{(\beta - b\theta)W(-\gamma Z_0)}{(\alpha - a\theta)K(0) + (\beta - b\theta)W(-\gamma Z_0)} \geq 0, \\
& \left|\lambda + f'(\Theta) + a\omega_+ + \frac{b}{2} - (\beta - b\Theta)\frac{W(0)}{|\phi'(\gamma Z_0)|}\eta_1(\lambda)\right| \\
> \ & \left|f'(\Theta) + a\omega_+ + \frac{b}{2} - (\beta - b\Theta)\frac{W(0)}{|\phi'(\gamma Z_0)|}\right| \\
= \ & \left\{m + a\omega_+ + \frac{b}{2}\right\}\frac{(\alpha - a\Theta)K(\gamma Z_0)}{(\alpha - a\Theta)K(\gamma Z_0) + (\beta - b\Theta)W(0)} \geq 0, \\
& \left|(\beta - b\theta)\frac{W(-\gamma Z_0)}{|\phi'(\gamma Z_0)|}\eta_1(\lambda)\right| \\
\leq \ & \left|(\beta - b\theta)\frac{W(-\gamma Z_0)}{|\phi'(\gamma Z_0)|}\right| = (\beta - b\theta)\frac{W(-\gamma Z_0)}{|\phi'(\gamma Z_0)|}, \\
& \left|(\alpha - a\Theta)\frac{K(\gamma Z_0)}{|\phi'(0)|}\xi_1(\lambda)\right| \\
\leq \ & \left|(\alpha - a\Theta)\frac{K(\gamma Z_0)}{|\phi'(0)|}\left[\int_0^\infty \xi(c)\mathrm{d}c\right]\right| = (\alpha - a\Theta)\frac{K(\gamma Z_0)}{|\phi'(0)|}.
\end{aligned}$$

Therefore, we obtain

$$|\mathcal{E}_1(\lambda)| > |\mathcal{E}_1(0)| = \mathcal{E}_1(0) = 0.$$

The proof of Corollary 3 is finished. □

Theorem 3.3. *There hold the following results*

$$\lambda_0(\varepsilon_1) = 0, \qquad \lambda_1(\varepsilon_1) = 0.$$

For all $\varepsilon < \varepsilon_1$, there hold

$$\begin{aligned}
\lambda_0(\varepsilon) > 0, \qquad {\lambda_0}'(\varepsilon) > 0, \\
\lambda_1(\varepsilon) > 0, \qquad {\lambda_1}'(\varepsilon) > 0.
\end{aligned}$$

For all $\varepsilon > \varepsilon_1$, there hold

$$\begin{aligned}
\lambda_0(\varepsilon) < 0, \qquad {\lambda_0}'(\varepsilon) < 0, \\
\lambda_1(\varepsilon) < 0, \qquad {\lambda_1}'(\varepsilon) < 0.
\end{aligned}$$

Remark 3.3. The Evans function $\mathcal{E}_1(\lambda)$ for the first two standing wave solutions of equation (1.3) enjoys the same properties as those of $\mathcal{E} = \mathcal{E}_1(\lambda, \varepsilon)$ mentioned in Theorem 3.1, Theorem 3.2 and Corollary 3. All other Evans functions $\mathcal{E} = \mathcal{E}_i(\lambda, \varepsilon)$ for the standing wave solutions of the system (1.1)-(1.2) and the Evans functions $\mathcal{E}_i = \mathcal{E}_i(\lambda)$ for the standing wave solutions of equation (1.3) enjoy similar properties, where $i = 2, 3$.

Definition 3.4. Denote by (ϕ, ψ) a standing wave solution of system (1.1)-(1.2). If there exist positive constants $C > 0$, $M > 0$ and $\delta > 0$, such that if the initial data $\big(u_0, w_0\big)$ of system (1.1)-(1.2) satisfies the condition

$$\left\| \begin{pmatrix} u_0 \\ w_0 \end{pmatrix} - \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\|_{L^\infty(\mathbb{R})} \le C,$$

then there exists a real constant h, such that the global solution $(u, w) = \big(u(x,t), w(x,t)\big)$ of system (1.1)-(1.2) corresponding to the initial data $\big(u_0, w_0\big)$ satisfies the estimate

$$\left\| \begin{pmatrix} u(\cdot,t) \\ w(\cdot,t) \end{pmatrix} - \begin{pmatrix} \phi(\cdot+h) \\ \psi(\cdot+h) \end{pmatrix} \right\|_{L^\infty(\mathbb{R})} \le C \left\| \begin{pmatrix} u_0 \\ w_0 \end{pmatrix} - \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\|_{L^\infty(\mathbb{R})} \exp(-\delta t),$$

for all $t > 0$, where the constant h satisfies the condition

$$|h| \le M \left\| \begin{pmatrix} u_0 \\ w_0 \end{pmatrix} - \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\|_{L^\infty(\mathbb{R})},$$

then we say the standing wave solution (ϕ, ψ) is a stable. Otherwise, we say it is unstable.

The Linearized Stability Criterion. (I) If there exists no nonzero eigenvalue to the eigenvalue problem $\mathcal{L}(\lambda)\psi = \lambda\psi$ in $\big\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \ge 0\big\}$ and if $\lambda = 0$ is a simple eigenvalue of the eigenvalue problem, then the standing wave solution is stable.

(II) If there exists a positive eigenvalue to the eigenvalue problem $\mathcal{L}(\lambda)\psi = \lambda\psi$, then the standing wave solution is unstable.

(III) If $\lambda = 0$ is not a simple eigenvalue of the eigenvalue problem $\mathcal{L}(\lambda)\psi = \lambda\psi$, then the standing wave solution is unstable.

The proof of the linearized stability criterion may be found in [22].

4. Bifurcations of the Standing Wave Solutions

Theorem 4.1. *There exist three positive constants $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and $\varepsilon_3 > 0$, such that for all $\varepsilon < \varepsilon_i$, the standing wave solutions $\big(\phi_{2i-1}, \psi_{2i-1}\big)$ and $\big(\phi_{2i}, \psi_{2i}\big)$ are unstable and for all $\varepsilon > \varepsilon_i$, the standing wave solutions $\big(\phi_{2i-1}, \psi_{2i-1}\big)$ and $\big(\phi_{2i}, \psi_{2i}\big)$ are stable, where $i = 1, 2, 3$.*

Proof. The proof follows from Theorem 4.2 and Theorem 4.3. □

Remark 4.1. The bifurcations are caused by the singular perturbation, not by delays. In another word, even if the delays are not present, the bifurcations exist for the nonlinear system (1.1)-(1.2).

4.1. Stability on the interval (ε_i, ∞)

Theorem 4.2. *Let $f(u)+g(u)=m(u-n)+k(u-l)$ in the nonlinear singularly perturbed system of integral differential equations (1.1)-(1.2).*
(I) For all $\varepsilon\in(\varepsilon_1,\infty)$, the standing wave solutions $\big(\phi_1,\psi_1\big)$ and $\big(\phi_2,\psi_2\big)$ of system (1.1)-(1.2) are exponentially stable.
(II) For all $\varepsilon\in(\varepsilon_2,\infty)$, the standing wave solutions $\big(\phi_3,\psi_3\big)$ and $\big(\phi_4,\psi_4\big)$ of system (1.1)-(1.2) are exponentially stable.
(III) For all $\varepsilon\in(\varepsilon_3,\infty)$, the standing wave solutions $\big(\phi_5,\psi_5\big)$ and $\big(\phi_6,\psi_6\big)$ of system (1.1)-(1.2) are exponentially stable.

Proof. For each of the six standing wave solutions $\big(\phi_1,\psi_1\big)$, $\big(\phi_2,\psi_2\big)$, $\big(\phi_3,\psi_3\big)$, $\big(\phi_4,\psi_4\big)$, $\big(\phi_5,\psi_5\big)$ and $\big(\phi_6,\psi_6\big)$, there exists no nonzero eigenvalue to the eigenvalue problem $\mathcal{L}(\lambda)\psi=\lambda\psi$ in the region $\big\{\lambda\in\mathbb{C}:\mathrm{Re}\lambda\geq 0\big\}$. Moreover, the neutral eigenvalue $\lambda=0$ is simple. By using the linearized stability criterion, we find that the standing wave solutions of the system (1.1)-(1.2) are exponentially stable. The proof of Theorem 4.2 is finished. □

Corollary 4.1. *Let $f(u)=m(u-n)$ in the scalar integral differential equation (1.3). (I) The standing wave solutions φ_1 and φ_2 of (1.3) are exponentially stable. (II) The standing wave solutions φ_3, φ_4, φ_5 and φ_6 of (1.3) are exponentially stable if $\theta<\frac{\alpha+mn}{a+m}<\Theta$.*

Proof. The proof of Corollary 4 is very similar to that of Theorem 4.2. □

4.2. Instability on the interval $(0,\varepsilon_i)$

Theorem 4.3. *Let $f(u)+g(u)=m(u-n)+k(u-l)$ in the nonlinear singularly perturbed system of integral differential equations (1.1)-(1.2).*
(I) For all $\varepsilon\in(0,\varepsilon_1)$, the standing wave solutions $\big(\phi_1,\psi_1\big)$ and $\big(\phi_2,\psi_2\big)$ of system (1.1)-(1.2) are exponentially unstable.
(II) For all $\varepsilon\in(0,\varepsilon_2)$, the standing wave solutions $\big(\phi_3,\psi_3\big)$ and $\big(\phi_4,\psi_4\big)$ of system (1.1)-(1.2) are exponentially unstable.
(III) For all $\varepsilon\in(0,\varepsilon_3)$, the standing wave solutions $\big(\phi_5,\psi_5\big)$ and $\big(\phi_6,\psi_6\big)$ of system (1.1)-(1.2) are exponentially unstable.

Proof. For each of the six standing wave solutions $\big(\phi_1,\psi_1\big)$, $\big(\phi_2,\psi_2\big)$, $\big(\phi_3,\psi_3\big)$, $\big(\phi_4,\psi_4\big)$, $\big(\phi_5,\psi_5\big)$ and $\big(\phi_6,\psi_6\big)$, there exists a positive eigenvalue $\lambda_1(\varepsilon)>0$ to the eigenvalue problem $\mathcal{L}(\lambda)\psi=\lambda\psi$ in the region $\big\{\lambda\in\mathbb{C}:\mathrm{Re}\lambda>-\varepsilon\big\}$. By the linearized stability criterion, we see the standing wave solutions of the system (1.1)-(1.2) are exponentially unstable. The proof of Theorem 4.3 is finished. □

5. Concluding Remarks

5.1. Summary

Consider the following nonlinear singularly perturbed system of integral differential equations

$$\begin{aligned}&\frac{\partial u}{\partial t}+f(u)+w\\=&(\alpha-au)\int_0^\infty\xi(c)\left[\int_{\mathbb{R}}K(x-y)H\left(u\left(y,t-\frac{1}{c}|x-y|\right)-\theta\right)\mathrm{d}y\right]\mathrm{d}c\end{aligned}$$

$$
\begin{aligned}
&+ (\beta - bu) \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(x-y) H\big(u(y,t-\tau) - \Theta\big) \mathrm{d}y \right] \mathrm{d}\tau, \\
&\frac{\partial w}{\partial t} = \varepsilon [g(u) - w],
\end{aligned}
$$

and the scalar integral differential equation

$$
\begin{aligned}
&\frac{\partial u}{\partial t} + f(u) \\
=&(\alpha - au) \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(x-y) H\left(u\left(y, t - \frac{1}{c}|x-y|\right) - \theta\right) \mathrm{d}y \right] \mathrm{d}c \\
&+ (\beta - bu) \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(x-y) H\big(u(y,t-\tau) - \Theta\big) \mathrm{d}y \right] \mathrm{d}\tau.
\end{aligned}
$$

These model equations generalize many important integral differential equations arising from delayed synaptically coupled neuronal networks.

For the nonlinear singularly perturbed system of integral differential equations, let $f(u) + g(u) = m(u-n) + k(u-l)$, for some positive constants $k > 0$ and $m > 0$ and for some real constants l and n, we have obtained the following results.
(I-1) There exist two complicated standing wave solutions $\big(u_1(x,t), w_1(x,t)\big) = \big(\phi_1(x), g\big(\phi_1(x)\big)\big)$ and $\big(u_2(x,t), w_2(x,t)\big) = \big(\phi_2(x), g\big(\phi_2(x)\big)\big)$, where

$$
\begin{aligned}
\phi_1(x) = &\left\{ kl + mn + \alpha \int_{-\infty}^{x} K(z) \mathrm{d}z + \beta \int_{-\infty}^{x - Z_0} W(z) \mathrm{d}z \right\} \\
&/ \left\{ k + m + a \int_{-\infty}^{x} K(z) \mathrm{d}z + b \int_{-\infty}^{x - Z_0} W(z) \mathrm{d}z \right\}, \\
\phi_2(x) = &\left\{ kl + mn + \alpha \int_{x}^{\infty} K(z) \mathrm{d}z + \beta \int_{x + Z_0}^{\infty} W(z) \mathrm{d}z \right\} \\
&/ \left\{ k + m + a \int_{x}^{\infty} K(z) \mathrm{d}z + b \int_{x + Z_0}^{\infty} W(z) \mathrm{d}z \right\}.
\end{aligned}
$$

(I-2) There exist four simple standing wave solutions under additional conditions.
(I-3) There exists three positive constants $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and $\varepsilon_3 > 0$. For all $\varepsilon \in (0, \varepsilon_1)$, the standing wave solutions $\big(\phi_1(x), \psi_1(x)\big)$ and $\big(\phi_2(x), \psi_2(x)\big)$ are unstable. For all $\varepsilon \in (\varepsilon_1, \infty)$, the standing wave solutions $\big(\phi_1(x), \psi_1(x)\big)$ and $\big(\phi_2(x), \psi_2(x)\big)$ are stable.
For all $\varepsilon \in (0, \varepsilon_2)$, the standing wave solutions $\big(\phi_3(x), \psi_3(x)\big)$ and $\big(\phi_4(x), \psi_4(x)\big)$ are unstable. For all $\varepsilon \in (\varepsilon_2, \infty)$, the standing wave solutions $\big(\phi_3(x), \psi_3(x)\big)$ and $\big(\phi_4(x), \psi_4(x)\big)$ are stable.
For all $\varepsilon \in (0, \varepsilon_3)$, the standing wave solutions $\big(\phi_5(x), \psi_5(x)\big)$ and $\big(\phi_6(x), \psi_6(x)\big)$ are unstable. For all $\varepsilon \in (\varepsilon_3, \infty)$, the standing wave solutions $\big(\phi_5(x), \psi_5(x)\big)$ and $\big(\phi_6(x), \psi_6(x)\big)$ are stable.
For the scalar integral differential equation, let $f(u) = m(u-n)$, we have obtained the following results.
(II-1) There exist two complicated standing wave solutions $u_1(x,t) = \varphi_1(x)$ and $u_2(x,t) = \varphi_2(x)$.
(II-2) There exist four simple standing wave solutions under additional conditions.
(II-3) The standing wave solutions of the scalar integral differential equation (1.3) are stable.

It is worth of mentioning that for the scalar integral differential equation, the standing wave solutions are always stable. However, for the nonlinear singularly perturbed system of integral differential equations, even though the parameter ε plays no role in the existence analysis of the standing wave solutions, it does play a very important role in the stability/instability analysis and the bifurcation analysis. The main idea to establish the stability/instability and the bifurcations is to construct Evans functions corresponding to several associated eigenvalue problems to find the eigenvalues.

5.2. Open Problems

Under different conditions on the model parameters and functions, there may exist standing pulse solutions rather than standing front solutions. But these problems have not been investigated rigorously.

The results for the nonlinear system are surprisingly interesting in mathematical neuroscience.

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