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STABILITY AND TRAVELING WAVES OF DIFFUSIVE PREDATOR-PREY MODEL WITH AGE-STRUCTURE AND NONLOCAL EFFECT*

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Abstract The paper is concerned with the dynamical behaviors of a stage-structured diffusive predator-prey model with nonlocal effect and harvesting. The linear stability of the equilibria is investigated by using the characteristic equation technique. By constructing a closed convex set bounded by a pair of upper-lower solutions and using Schauder fixed point theorem, the existence of traveling wave solution connecting two steady states is also derived. Finally, a pair of upper-lower solutions is constructed by using inequality technique and characteristic equations.

Keywords Diffusive predator-prey model, stage-structure, traveling wave, stability, Schauder fixed point theorem, upper-lower solutions.

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1. Introduction

One of the model for stage-structure population is proposed by Aiello and Freedman [1]:

$$\begin{cases} \dot{x}_1(t) = \alpha x_2(t) - r x_1(t) - a e^{-r\tau} x_2(t-\tau), \\ \dot{x}_2(t) = a e^{-r\tau} x_2(t-\tau) - m x_2^2(t), \end{cases} \tag{1.1}$$

where $x_1(t)$ and $x_2(t)$ are the immature and mature population densities, respectively; τ is the time for an individual taken from birth to maturity. The term $ae^{-r\tau}x_2(t-\tau)$ represents the immature individual who were born at time $t-\tau$ and survive at time t and therefore represents the transformation of immature to mature.

As we know that the predator-prey model is one of the important models for describing the interaction rule of multi-species, which have been studied for a long history by many authors, e.g. see [2, 3, 10, 14, 15]. Since the stage structure is also existing in predator-prey species, and species at different stage may have different behaviors, the investigation for predator-prey models with age structure seems necessary (see e.g. [8, 14, 16, 20]). In 2007, Qu and Wei [11] considered a

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stage-structure prey-predator model as follows:

$$\begin{cases} \dot{x}(t) = x(t)(r_1 - a_{11}x(t) - a_{13}Y(t)), \\ \dot{y}(t) = a_{31}x(t)Y(t) - r_2 y(t) - a_{31}e^{-\gamma\tau}x(t-\tau)Y(t-\tau), \\ \dot{Y}(t) = a_{31}e^{-\gamma\tau}x(t-\tau)Y(t-\tau) - r_3 Y(t), \end{cases} \tag{1.2}$$

where $x(t)$ denotes the density of prey, $y(t)$ and $Y(t)$ represent the immature and mature predator densities, respectively.

If predators and preys are spatially distributed, it is obvious that there is a temporal-spatial variation in the populations since the predators move to catch the food and the preys move to escape. Therefore, it is natural to consider the diffusive version of (1.2), which is a more realistic biological model in this world. On the other aspect, it is reasonable that the number of the predator individuals which leave the immature class and enter the mature class at time t and position x may depend on other position $y \in \mathbb{R}$. Therefore, in the present paper, we are concerned with the following delay predatorCprey model with nonlocal spatial effect and harvesting which is in fact a refinement of (1.2):

$$\begin{cases} \frac{\partial u}{\partial t} = D_0 \frac{\partial^2 u}{\partial x^2} + [r - a_{11}u(x,t)]u(x,t) - a_{12}u(x,t)v_2(x,t), \\ \frac{\partial v_1}{\partial t} = D_1 \frac{\partial^2 v_1}{\partial x^2} + au(x,t)v_2(x,t) - d_1 v_1(x,t) \\ \qquad - ae^{-d_1\tau} \int\limits_{-\infty}^{+\infty} G(\tau, x-y)u(y,t-\tau)v_2(y,t-\tau)dy, \\ \frac{\partial v_2}{\partial t} = D_2 \frac{\partial^2 v_2}{\partial x^2} + ae^{-d_1\tau} \int\limits_{-\infty}^{+\infty} G(\tau, x-y)u(y,t-\tau)v_2(y,t-\tau)dy \\ \qquad - (d_2 + qe)v_2(x,t) - b_{22}v_2^2(x,t), \end{cases} \tag{1.3}$$

where $u(x,t)$ denotes the density of prey, $v_1(x,t)$ and $v_2(x,t)$ represent the densities of immature and mature predators respectively, and

$$G(\tau, x-y) = \frac{1}{\sqrt{4\pi D_1 \tau}} e^{-\frac{(x-y)^2}{4D_1\tau}}.$$

This model is derived under the following assumptions. $r > 0$ is the intrinsic growth rate of the prey; $a_{11} > 0$ is the density-dependent coefficient of the prey, and $a_{12} > 0$ is the capturing rate of the mature predator; $a/a_{12} > 0$ is the rate of conversion of nutrients into the reproduction rate of the mature predator population; $d_1 > 0$ and $d_2 > 0$ are the death rate of the immature and mature predators respectively; $b_{22} > 0$ is the overcrowding rate of the mature predator population. Note that $qev_2(x,t)$ represents the catching rate function based on the catch-per-unit-effort hypothesis, where q is the catch-ability coefficients of the predator while e is the harvesting efforts for the mature predator. τ represents a constant time for a prey individual from immature to mature, and the term $ae^{-d_1\tau} \int\limits_{-\infty}^{+\infty} G(\tau, x-y)u(y,t-\tau)v_2(y,t-\tau)dy$ stands for the total number of the predator individuals which leave the immature class and enter the mature class at time t and position x. In fact, this term depends on the reproduction ability of the mature predators at time $t-\tau$ and any position $y \in \mathbb{R}$, and hence the capturing ability of the mature predators at time $t-\tau$ and any position $y \in \mathbb{R}$.

In (1.3), we consider the harvest behaviors of mankind, which is reasonable in the nature. That is to say, the mature predators will be hunted by hunters.

Because the first and the third equations of (1.3) do not depend on $v_1(x,t)$, for simplicity of notations, we denote $v_2(x,t)$ by $v(x,t)$, and then consider the following system

$$\begin{cases} \frac{\partial u}{\partial t} = D_0 \frac{\partial^2 u}{\partial x^2} + [r - a_{11}u(x,t)]u(x,t) - a_{12}u(x,t)v(x,t), \\ \frac{\partial v}{\partial t} = D_2 \frac{\partial^2 v}{\partial x^2} + ae^{-d_1\tau} \int\limits_{-\infty}^{+\infty} G(\tau, x-y)u(y,t-\tau)v(y,t-\tau)dy \\ \qquad - (d_2 + qe)v(x,t) - b_{22}v^2(x,t). \end{cases} \tag{1.4}$$

Accompanied with (1.4), we take the initial condition

$$\begin{matrix} u(x,\theta) = \delta_u(x,\theta) \geq 0, & \delta_u(x,0) > 0, \\ v(x,\theta) = \delta_v(x,\theta) \geq 0, & \delta_v(x,0) > 0, \end{matrix} \qquad x \in \mathbb{R}, \quad -\tau \leq \theta \leq 0. \tag{1.5}$$

Through this article, we assume that the solution of the initial value problem (1.3), (1.5) exists globally and remains nonnegative. Please see Redlinger[12] for some reference.

Stability is a classical topic for the study of dynamics[3, 4]. On the other hand, as a form of wave propagation for biological invasion (see [15]), traveling wave solution is a both interesting and significant topic in the study of population models. The theory and methods have been developed fast during the recent years. Here we refer the authors to [2, 5, 6, 7, 9, 10, 13, 17, 18, 19, 21] for more details. In the present article, we are mainly interested in the stability of equilibria and the existence of traveling waves connecting two equilibria.

This paper is organized as follows. In Section 2, we analyze the linear asymptotical stability of the equilibria for the system (1.4). In Section 3, by a pair of given upper and lower solutions satisfying some assumptions, we construct a closed convex set Γ. Using Schauder fixed point theorem, we prove the existence of traveling waves of the system (1.4) connecting the zero equilibrium and the positive equilibrium. In Section 4, we shall show that the upper and lower solutions demanded in Section 3 can be constructed, and thus the existence of traveling waves connecting two equilibria are guaranteed. We want to mention here that our argument for the upper-lower solutions is more delicate and convinced than what in the existing literatures.

2. Linear stability of equilibria

In this section, we shall discuss the linear stability of the equilibria of system (1.4). Firstly, it is obvious that the system (1.4) has three equilibria denoted by

$$E_0(0,0), \quad E_1(\frac{r}{a_{11}},0), \quad E_2(0,\frac{d_2+qe}{b_{22}}).$$

If the following hypothesis holds:

$(A1)$ $rae^{-d_1\tau} - d_2a_{11} - qea_{11} > 0$,

then system (1.4) has a unique positive steady state $E_3(u^+,v^+)$ where

$$u^+ = \frac{d_2a_{12} + qea_{12} + rb_{22}}{ae^{-d_1\tau}a_{12} + a_{11}b_{22}} > 0, \quad v^+ = \frac{rae^{-d_1\tau} - d_2a_{11} - qea_{11}}{ae^{-d_1\tau}a_{12} + a_{11}b_{22}} > 0.$$

The linearized system of (1.4) at (u^+, v^*) is

$$\begin{cases} \frac{\partial u}{\partial t} = D_0 \frac{\partial^2 u}{\partial x^2} + ru(x,t) - 2a_{11}u^+u(x,t) - a_{12}v^+u(x,t) - a_{12}u^+v(x,t), \\ \frac{\partial v}{\partial t} = D_2 \frac{\partial^2 v}{\partial x^2} + ae^{-d_1\tau}v^+ \int\limits_{-\infty}^{+\infty} G(\tau, x-y)u(y,t-\tau)dy + ae^{-d_1\tau}u \\ \qquad + \int\limits_{-\infty}^{+\infty} G(\tau, x-y)v(y,t-\tau)dy - (d_2+qe)v(x,t) - 2b_{22}v^+v(x,t). \end{cases} \tag{2.1}$$

We know that the linear equations (2.1) admits non-trivial solutions with the form (see Gourley & Kuang[4]):

$$\begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{\lambda t + i\sigma x}$$

if only and if

$$\begin{vmatrix} \kappa_1(\lambda,\sigma,u^+,v^+) & a_{12}u^+e^{-\lambda\tau} \\ -ae^{-d_1\tau-\lambda\tau}v^+e^{-D_1\sigma^2\tau} & \kappa_2(\lambda,\sigma,u^+,v^+) \end{vmatrix} = 0, \tag{2.2}$$

where

$$\begin{aligned} \kappa_1(\lambda,\sigma,u^+,v^+) : &= \lambda + D_0\sigma^2 - r + 2a_{11}u^+ + a_{12}v^+, \\ \kappa_2(\lambda,\sigma,u^+,v^+) : &= \lambda + D_2\sigma^2 - ae^{-d_1\tau-\lambda\tau}u^+e^{-D_1\sigma^2\tau} + d_2 + qe + 2b_{22}v^+. \end{aligned}$$

(2.2) is equivalent to

$$\kappa_1(\lambda,\sigma,u^+,v^+)\kappa_2(\lambda,\sigma,u^+,v^+) + aa_{12}e^{-d_1\tau-D_1\sigma^2\tau}u^+v^+e^{-2\lambda\tau} = 0, \tag{2.3}$$

where λ is a complex number and σ is a real number.

Theorem 2.1. *$E_0(0,0)$ is linearly unstable.*

Proof. Letting $(u^+, v^+) = (0,0)$ in (2.3), it follows that

$$\kappa_1(\lambda,\sigma,0,0)\kappa_2(\lambda,\sigma,0,0) = 0. \tag{2.4}$$

It is easy to show that there exists at least a (λ_0, σ_0) such that $\lambda_0 > 0$ satisfying $\kappa_1(\lambda,\sigma,0,0) = 0$. The proof is complete. □

Theorem 2.2. *(1) If $\frac{rae^{-d_1\tau}}{a_{11}} - d_2 - qe < 0$, then $E_1(\frac{r}{a_{11}}, 0)$ is linearly asymptotically stable. (2) If $\frac{rae^{-d_1\tau}}{a_{11}} - d_2 - qe > 0$, then $E_1(\frac{r}{a_{11}}, 0)$ is linearly unstable.*

Proof. Let $(u^+, v^+) = (\frac{r}{a_{11}}, 0)$ in (2.3), it follows that

$$\kappa_1(\lambda,\sigma,\frac{r}{a_{11}},0)\kappa_2(\lambda,\sigma,\frac{r}{a_{11}},0) = 0, \tag{2.5}$$

Note that for any $\sigma \in \mathbb{R}$, the root of the equation $\kappa_1(\lambda,\sigma,\frac{r}{a_{11}},0) = 0$ is negative.

On the other hand, from $\kappa_2(\lambda,\sigma,\frac{r}{a_{11}},0) = 0$, we derive the following conclusions.

(1) If $\frac{rae^{-d_1\tau}}{a_{11}} - d_2 - qe < 0$, we claim that all roots of (2.5) satisfy $\mathrm{Re}\lambda < 0$. Otherwise, then there exists a root $\bar{\lambda}$ of (2.5) with $\mathrm{Re}\bar{\lambda} \geq 0$. Hence, from

$\kappa_2(\lambda, \sigma, \frac{r}{a_{11}}, 0) = 0$ we deduce that

$$\begin{aligned} \mathrm{Re}\bar{\lambda} &= -D_2\sigma^2 + \frac{rae^{-d_1\tau - D_1\sigma^2\tau}e^{-\tau\mathrm{Re}\bar{\lambda}}\cos(\tau\mathrm{Im}\bar{\lambda})}{a_{11}} - d_2 - qe \\ &\le \frac{rae^{-d_1\tau}}{a_{11}} - d_2 - qe < 0, \end{aligned} \tag{2.6}$$

a contradiction. That is, all roots of (2.5) satisfy $\mathrm{Re}\lambda < 0$. Therefore, the equilibrium $E_1(\frac{r}{a_{11}}, 0)$ is linearly asymptotically stable.

(2) If $\frac{rae^{-d_1\tau}}{a_{11}} - d_2 - qe > 0$, we claim that there exists at least a (λ_*, σ_*) such that $\lambda_* > 0$ satisfying (2.5). In fact, let

$$f(\lambda, \sigma) := -D_2\sigma^2 + \frac{rae^{-d_1\tau - \lambda\tau}e^{-D_1\sigma^2\tau}}{a_{11}} - d_2 - qe,$$

and we have from

$$f(0, 0) = \frac{rae^{-d_1\tau}}{a_{11}} - d_2 - qe > 0,$$

that there is a small σ_* such that

$$f(0, \sigma_*) = -D_2\sigma_*^2 + \frac{rae^{-d_1\tau}e^{-D_1\sigma_*^2\tau}}{a_{11}} - d_2 - qe > 0.$$

Note that $f(\infty, \sigma_*) < 0$. Therefore the equation $\lambda = f(\lambda, \sigma_*)$ has positive solution λ_*. That is, (λ_*, σ_*) satisfies (2.5). The proof is complete. □

Theorem 2.3. *(1) If $rb_{22} - a_{12}(d_2 + qe) > 0$, then the equilibrium point $E_2(0, \frac{d_2+qe}{b_{22}})$ is linearly asymptotically stable; (2) If $rb_{22} - a_{12}(d_2 + qe) < 0$, then $E_2(0, \frac{d_2+qe}{b_{22}})$ is linearly unstable.*

Proof. Let $(u^+, v^+) = (0, \frac{d_2+qe}{b_{22}})$ in (2.3), it follows that

$$\kappa_1(\lambda, \sigma, 0, \frac{d_2 + qe}{b_{22}})\kappa_2(\lambda, \sigma, 0, \frac{d_2 + qe}{b_{22}}) = 0. \tag{2.7}$$

A roots of the equation $\kappa_2(\lambda, \sigma, 0, \frac{d_2+qe}{b_{22}}) = 0$ are negative. On the other hand, from $\kappa_1(\lambda, \sigma, 0, \frac{d_2+qe}{b_{22}}) = 0$, we see that

$$\lambda = -D_0\sigma^2 - r + \frac{a_{12}(d_2 + qe)}{b_{22}}. \tag{2.8}$$

(1) If $\frac{a_{12}(d_2+qe)}{b_{22}} < r$, we claim that the only root of (2.7) satisfy $\mathrm{Re}\lambda < 0$. $E_2(0, \frac{d_2+qe}{b_{22}})$ is linearly asymptotically stable.

(2) If $\frac{a_{12}(d_2+qe)}{b_{22}} > r$, we see that there exists at least a (λ_*, σ_*) such that $\lambda_* > 0$ satisfying (2.8). Therefore, $E_2(0, \frac{d_2+qe}{b_{22}})$ is linearly unstable. The proof is complete. □

The following theorem give the linearly asymptotic stability of $E_3(u^+, v^+)$.

Theorem 2.4. *Assume that (A1) and $a_{11}b_{22} > aa_{12}e^{-d_1\tau}$ hold. Then the equilibrium point $E_3(u^+, v^+)$ is linearly asymptotically stable.*

Proof. For the epidemic steady state $E_3(u^+, v^+)$, (2.3) is of the form

$$\begin{aligned}&(\lambda + D_0\sigma^2 + a_{11}u^+)(\lambda + D_2\sigma^2 - ae^{-d_1\tau-\lambda\tau}u^+e^{-D_1\sigma^2\tau} + d_2 + qe + 2b_{22}v^+)\\&+ aa_{12}e^{-d_1\tau-D_1\sigma^2\tau}u^+v^+e^{-2\lambda\tau} = 0,\end{aligned} \tag{2.9}$$

where we used the fact that $-r + a_{11}u^+ + a_{12}v^+ = 0$. It is easy to see that $\lambda = -D_0\sigma^2 - a_{11}u^+ < 0$ is not the solution of (2.9), so we can rewrite (2.9) as

$$\begin{aligned}&\big(\lambda + D_0\sigma^2 + a_{11}u^+\big)\bigg(\lambda + D_2\sigma^2 - ae^{-d_1\tau-\lambda\tau}u^+e^{-D_1\sigma^2\tau} + d_2 + qe\\&+ 2b_{22}v^+ + \frac{aa_{12}e^{-d_1\tau-D_1\sigma^2\tau}u^+v^+e^{-2\lambda\tau}}{\lambda + D_0\sigma^2 + a_{11}u^+}\bigg) = 0.\end{aligned} \tag{2.10}$$

Therefore, looking for the solutions of (2.9) is equivalent to looking for the solutions of the following equation

$$\begin{aligned}&\lambda + D_2\sigma^2 - ae^{-d_1\tau-\lambda\tau}u^+e^{-D_1\sigma^2\tau} + d_2 + qe + 2b_{22}v^+\\&+ \frac{aa_{12}e^{-d_1\tau-D_1\sigma^2\tau}u^+v^+e^{-2\lambda\tau}}{\lambda + D_0\sigma^2 + a_{11}u^+} = 0.\end{aligned} \tag{2.11}$$

When $\tau = 0$, let $\triangle := aa_{12} + a_{11}b_{22}$. We have

$$\begin{aligned}&au^+ - d_2 - qe - b_{22}v^+\\=&\frac{a(d_2a_{12} + qea_{12} + rb_{22}) - (d_2 + qe)\triangle - b_{22}(ra - d_2a_{11} - qea_{11})}{\triangle} = 0.\end{aligned} \tag{2.12}$$

We suppose that $\mathrm{Re}\lambda \geq 0$. It follows from (2.12) that

$$\begin{aligned}\mathrm{Re}\lambda &= -D_2\sigma^2 + au^+ - d_2 - qe - 2b_{22}v^+ - \mathrm{Re}\frac{aa_{12}u^+v^+}{\lambda + D_0\sigma^2 + a_{11}u^+}\\&= -D_2\sigma^2 - b_{22}v^+ - \frac{aa_{12}u^+v^+(\mathrm{Re}\lambda + D_0\sigma^2 + a_{11}u^+)}{(\mathrm{Re}\lambda + D_0\sigma^2 + a_{11}u^+)^2 + (\mathrm{Im}\lambda)^2} \ < \ 0,\end{aligned}$$

which contradicts the above assumption $\mathrm{Re}\lambda \geq 0$. Therefore, $E_3(u^+, v^+)$ is locally asymptotically stable when $\tau = 0$.

When $\tau > 0$, one can also show that all roots of (2.9) satisfy $\mathrm{Re}\lambda < 0$. In fact, $\mathrm{Re}\lambda$ is continuous on τ. If there is a $\lambda = \lambda(\tau)$ $(\tau > 0)$ such that $\mathrm{Re}\lambda \geq 0$, then $\mathrm{Re}\lambda$ will pass zero value as τ is increasing from $\tau = 0$ to $\tau > 0$. Assume that $\mathrm{i}\theta$ is a solution of (2.9), separating real and imaginary parts, we derive from (2.9) that

$$\begin{cases}\theta^2 - p_0 = q_0\theta\cos(\theta\tau) - q_1\theta\sin(\theta\tau),\\ p_1\theta = q_0\sin(\theta\tau) - q_1\theta\cos(\theta\tau),\end{cases} \tag{2.13}$$

where the coefficients $p_i, q_i (i = 0, 1)$ are

$$\begin{aligned}&p_0 = \omega(\mu_1 + \mu_2), \quad p_1 = \omega + \mu_1 + \mu_2, \quad q_0 = \xi u^+(a_{12}v^+ - \omega), \quad q_1 = \xi u^+,\\&\omega := D_0\sigma^2 + a_{11}u^+, \quad \mu_1 := D_2\sigma^2 + b_{22}v^+,\\&\mu_2 := d_2 + qe + b_{22}v^+, \quad \xi := ae^{-d_1\tau-D_1\sigma^2\tau}.\end{aligned}$$

Squaring and adding both equations of (2.13), it follows that

$$\theta^4 + (p_1^2 - 2p_0 - q_1^2)\theta^2 + p_0^2 - q_0^2 = 0. \tag{2.14}$$

Let $z = \theta^2$, then Eq. (2.14) becomes

$$z^2 + (p_1^2 - 2p_0 - q_1^2)z + p_0^2 - q_0^2 = 0. \tag{2.15}$$

Note that $ae^{-d_1\tau}u^+ = \mu_2$ and $\mu_2^2 - \xi^2(u^+)^2 > 0$. By calculation it follows that

$$\begin{aligned} p_1^2 - 2p_0 - q_1^2 &= (\omega + \mu_1 + \mu_2)^2 - 2\omega(\mu_1 + \mu_2) - \xi^2(u^+)^2 \\ &\geq \omega^2 + \mu_1^2 + 2\mu_1\mu_2 > 0. \end{aligned}$$

On the other hand, by using the assumption $a_{11}b_{22} > aa_{12}e^{-d_1\tau}$, we can derive that

$$\begin{aligned} p_0^2 - q_0^2 &= (p_0 + q_0)(p_0 - q_0) \\ &= [\omega(\mu_1 + \mu_2) + \xi(u^+)(a_{12}v^+ - \omega)][\omega(\mu_1 + \mu_2) - \xi(u^+)(a_{12}v^+ - \omega)] \\ &\geq (\omega\mu_1 + a_{12}\xi u^+ v^+)[\omega(D_2\sigma^2 + ae^{-d_1\tau}u^+) + b_{22}v^+ D_0\sigma^2] > 0. \end{aligned}$$

By Routh-Hurwitz criterion we know that all the roots of (2.15) have negative real parts. This is a contradiction. Therefore, if $(A1)$ and $a_{11}b_{22} > aa_{12}e^{-d_1\tau}$ hold, then the endemic steady state $E_3(u^+, v^+)$ is locally asymptotically stable for all $\tau \geq 0$. The proof is complete. □

Remark 2.1. The condition $a_{11}b_{22} > aa_{12}e^{-d_1\tau}$ implies that the role of intra-action of each species is much important than the inter-action between two species, which guarantee the asymptotic stability of the positive equilibrium.

3. Existence of traveling waves

In this section, we consider the possibility of a transition between the equilibria E_0 and E_3 in the form of a traveling wave solution. We shall apply Schauder's fixed point theorem, the method of cross iteration scheme associated with upper-lower solutions to establish the existence of traveling wave solutions for system (1.4). From this section, we always assume $(A1)$ holds, and thus the positive equilibrium E_3 exists.

A traveling wave solution of (1.4) is a translation invariant solution of the form $(u(x,t), v(x,t)) = (\phi(x+ct), \psi(x+ct))$, where the profile of the wave propagates through one-dimensional spatial domain at a constant speed $c > 0$. Substituting $u(x,t) = \phi(x+ct), v(x,t) = \psi(x+ct)$ into the system (1.4) and denote the traveling wave coordinate $x + ct$ by t, we derive the wave profile system

$$\begin{cases} D_0\phi''(t) - c\phi'(t) + f_1(\phi, \psi)(t) = 0, \\ D_2\psi''(t) - c\psi'(t) + f_2(\phi, \psi)(t) = 0, \end{cases} \tag{3.1}$$

where

$$\begin{aligned} f_1(\phi, \psi)(t) &= (r - a_{11}\phi(t))\phi(t) - a_{12}\phi(t)\psi(t), \\ f_2(\phi, \psi)(t) &= ae^{-d_1\tau} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi D_1\tau}} e^{-\frac{y^2}{4D_1\tau}} \phi(t - y - c\tau)\psi(t - y - c\tau)dy \\ &\quad -(d_2 + qe)\psi(t) - b_{22}\psi^2(t). \end{aligned} \tag{3.2}$$

Eq.(3.1) will be solved subject to the following boundary value conditions:

$$\begin{aligned} &\lim_{t\to-\infty}\phi(t)=0, \quad \lim_{t\to+\infty}\phi(t)=u^+,\\ &\lim_{t\to-\infty}\psi(t)=0, \quad \lim_{t\to+\infty}\psi(t)=v^+. \end{aligned} \tag{3.3}$$

We introduce the concept of desirable pair of upper-lower solutions of system (3.1) as follows.

Definition 3.1. A pair of continuous functions $\overline{\rho}=(\overline{\phi},\overline{\psi})$ and $\underline{\rho}=(\underline{\phi},\underline{\psi})$ for $t\in\mathbb{R}$ is called a pair of upper-lower solutions of (3.1), if there exists a finite set of points $S=\{t_i\in\mathbb{R}, i=1,2,\cdots,n\}$ with $t_1<t_2<\cdots<t_n$ such that $\overline{\rho}$ and $\underline{\rho}$ are twice continuously differentiable on $\mathbb{R}\backslash S$ and satisfy

$$D_0\overline{\phi}''(t)-c\overline{\phi}'(t)+f_1(\overline{\phi},\underline{\psi})(t)\le 0,$$

$$D_2\overline{\psi}''(t)-c\overline{\psi}'(t)+f_2(\overline{\phi},\overline{\psi})(t)\le 0,$$

and

$$D_0\underline{\phi}''(t)-c\underline{\phi}'(t)+f_1(\underline{\phi},\overline{\psi})(t)\ge 0,$$
$$D_2\underline{\psi}''(t)-c\underline{\psi}'(t)+f_2(\underline{\phi},\underline{\psi})(t)\ge 0,$$

for $t\in\mathbb{R}\backslash S$.

It is not difficult to verify the following equality:

$$\int_{-\infty}^{\infty}G(\tau,y)e^{-\lambda(y+c\tau)}dy=\int_{-\infty}^{\infty}\frac{1}{\sqrt{4D_1\pi\tau}}e^{-\frac{y^2}{4D_1\tau}}e^{-\lambda(y+c\tau)}dy=e^{(D_1\lambda^2-c\lambda)\tau}. \tag{3.4}$$

Let

$$M_1\ge u^+, \quad M_2\ge v^+. \tag{3.5}$$

Define a set of functions

$$\begin{aligned} C_{[0,M]}(\mathbb{R},\mathbb{R}^2):=\{(\phi,\psi)\in C(\mathbb{R},\mathbb{R}^2)|\ &0\le\phi(t)\le M_1, 0\le\psi(t)\le M_2,\\ &\text{for}\ \ i=1,2,\ t\in\mathbb{R}\}. \end{aligned}$$

Taking

$$\beta_1\ge 2a_{11}M_1+a_{12}M_2-r, \quad \beta_2\ge d_{2+}qe+2bM_2, \tag{3.6}$$

we define two operators $H=(H_1,H_2)$ and $F=(F_1,F_2)$ from $C_{[0,M]}(\mathbb{R},\mathbb{R}^2)$ to $C(\mathbb{R},\mathbb{R}^2)$ by

$$H_1(\phi,\psi)(t)=f_1(\phi,\psi)(t)+\beta_1\phi(t), \quad H_2(\phi,\psi)(t)=f_2(\phi,\psi)(t)+\beta_2\psi(t),$$

$$F_1(\phi,\psi)(t)=\tfrac{1}{D_0(\lambda_{12}-\lambda_{11})}\left[\int_{-\infty}^{t}e^{\lambda_{11}(t-s)}H_1(\phi,\psi)(s)ds+\int_{t}^{+\infty}e^{\lambda_{12}(t-s)}H_1(\phi,\psi)(s)ds\right],$$

$$F_2(\phi,\psi)(t)=\tfrac{1}{D_2(\lambda_{22}-\lambda_{21})}\left[\int_{-\infty}^{t}e^{\lambda_{21}(t-s)}H_2(\phi,\psi)(s)ds+\int_{t}^{+\infty}e^{\lambda_{22}(t-s)}H_2(\phi,\psi)(s)ds\right],$$

where

$$\lambda_{11}=\frac{c-\sqrt{c^2+4\beta_1D_0}}{2D_0}<0, \quad \lambda_{12}=\frac{c+\sqrt{c^2+4\beta_1D_0}}{2D_0}>0,$$

$$\lambda_{21} = \frac{c - \sqrt{c^2 + 4\beta_2 D_2}}{2D_2} < 0, \quad \lambda_{22} = \frac{c + \sqrt{c^2 + 4\beta_2 D_2}}{2D_2} > 0.$$

It is obvious that a fixed point of F is a solution of (3.1), and vice verse.

For $\mu > 0$, define

$$B_\mu(\mathbb{R}, \mathbb{R}^2) = \{(\phi, \psi) \in C(\mathbb{R}, \mathbb{R}^2) : \sup_{t \in \mathbb{R}} |(\phi, \psi)(t)| e^{-\mu|t|} < \infty\}$$

and

$$|(\phi, \psi)|_\mu = \sup_{t \in \mathbb{R}} |(\phi, \psi)(t)| e^{-\mu|t|}.$$

Then it is easy to check that $(B_\mu(\mathbb{R}, \mathbb{R}^2), |\cdot|_\mu)$ is a Banach space. For our purpose, we will take μ such that

$$0 < \mu < \min\{-\lambda_{11}, \lambda_{12}, -\lambda_{21}, \lambda_{22}\}. \tag{3.7}$$

Now we explore some basic properties of H and F. In view of the boundedness and continuity of f_1, f_2 on $C_{[0,M]}(\mathbb{R}, \mathbb{R}^2)$, the following conclusion is obvious.

Lemma 3.1. *For sufficiently large β_1, β_2 satisfying* (3.6)*, we have*

$$H_1(\phi_1, \psi_1)(t) \geq H_1(\phi_2, \psi_1)(t), \quad H_1(\phi_1, \psi_1)(t) \leq H_1(\phi_1, \psi_2)(t),$$

$$H_2(\phi_1, \psi_1)(t) \geq H_2(\phi_2, \psi_1)(t), \quad H_2(\phi_1, \psi_1)(t) \geq H_2(\phi_1, \psi_2)(t),$$

for $t \in \mathbb{R}$ with $0 \leq \phi_2(t) \leq \phi_1(t) \leq M_1, 0 \leq \psi_2(t) \leq \psi_1(t) \leq M_2$.

In view of the definition of F, we can easily see that $F = (F_1, F_2)$ also enjoys the same properties as those for $H = (H_1, H_2)$ stated in Lemmas 3.1.

Lemma 3.2. *For sufficiently large β_1, β_2 satisfying* (3.6)*, we have*

$$F_1(\phi_1, \psi_1)(t) \geq F_1(\phi_2, \psi_1)(t), \quad F_1(\phi_1, \psi_1)(t) \leq F_1(\phi_1, \psi_2)(t),$$

$$F_2(\phi_1, \psi_1)(t) \geq F_2(\phi_2, \psi_1)(t), \quad F_2(\phi_1, \psi_1)(t) \geq F_2(\phi_1, \psi_2)(t),$$

for $t \in \mathbb{R}$ with $0 \leq \phi_2(t) \leq \phi_1(t) \leq M_1, 0 \leq \psi_2(t) \leq \psi_1(t) \leq M_2$.

In what follows, we assume that (3.1) has a pair of upper solution $(\overline{\phi}(t), \overline{\psi}(t))$ and lower solution $(\underline{\phi}(t), \underline{\psi}(t))$ satisfying

- $(P1)$ $(0,0) \leq \big(\underline{\phi}(t), \underline{\psi}(t)\big) \leq \big(\overline{\phi}(t), \overline{\psi}(t)\big) \leq (M_1, M_2)$.
- $(P2)$ $\lim_{t \to -\infty} \big(\overline{\phi}(t), \overline{\psi}(t)\big) = (0,0)$, $\lim_{t \to +\infty} \big(\underline{\phi}(t), \underline{\psi}(t)\big) = \lim_{t \to +\infty} \big(\overline{\phi}(t), \overline{\psi}(t)\big) = (u^+, v^+)$.
- $(P3)$ $\overline{\phi}'(t+) \leq \overline{\phi}'(t-)$ and $\underline{\phi}'(t+) \geq \underline{\phi}'(t-)$ for $t \in \mathbb{R}$.

We shall look for traveling wave solution of system (3.1) in the following profile set:

$$\Gamma = \left\{(\phi, \psi) \in C_{[0,M]}(\mathbb{R}, \mathbb{R}^2) \,\middle|\, (\underline{\phi}(t), \underline{\psi}(t)) \leq (\phi(t), \psi(t)) \leq (\overline{\phi}(t), \overline{\psi}(t)) \text{ for } t \in \mathbb{R} \right\}.$$

Obviously, Γ is non-empty, convex, closed, and bounded.

Lemma 3.3. *$F = (F_1, F_2)$ is continuous with respective to the norm $|\cdot|_\mu$ in $B_\mu(\mathbb{R}, \mathbb{R}^2)$.*

Proof. We first prove that $H : B_\mu(\mathbb{R},\mathbb{R}^2) \to B_\mu(\mathbb{R},\mathbb{R}^2)$ is continuous with respect to the norm $|\cdot|_\mu$. Note that we have

$$\int_{-\infty}^{+\infty} G(\tau,y)e^{\mu|y+c\tau|}dy \le \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4D_1\pi\tau}} e^{-\frac{y^2}{4D_1\tau}} e^{\mu(|y|+c\tau)}dy$$
$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4D_1\pi\tau}} e^{-\frac{(|y|-2D_1\mu\tau)^2}{4D_1\tau}} e^{(c\mu+D_1\mu^2)\tau}dy = e^{(c\mu+D_1\mu^2)\tau}.$$

If $\Phi = (\phi_1,\psi_1), \Psi = (\phi_2,\psi_2) \in B_\mu(\mathbb{R},\mathbb{R}^2)$, we have

$$\begin{aligned}
&|H_2(\phi_1,\psi_1)(t) - H_2(\phi_2,\psi_2)(t)|e^{-\mu|t|}\\
\le&|f_2(\phi_1,\psi_1)(t) - f_2(\phi_2,\psi_2)(t)|e^{-\mu|t|} + \beta_2|\psi_1(t)-\psi_2(t)|e^{-\mu|t|}\\
\le&|ae^{-d_1\tau}\int_{-\infty}^{+\infty} G(\tau,y)[\phi_1(t-y-c\tau)\psi_1(t-y-c\tau) - \phi_2(t-y-c\tau)\psi_2(t-y-c\tau)]dy\\
&-(d_2+q_2e_2)[\psi_1(t)-\psi_2(t)] - a_{22}[\psi_1^2(t)-\psi_2^2(t)]|e^{-\mu|t|} + \beta_2||\Phi-\Psi|_\mu\\
=&|ae^{-d_1\tau}\int_{-\infty}^{+\infty} G(\tau,y)[\phi_1(t-y-c\tau)\psi_1(t-y-c\tau) - \phi_2(t-y-c\tau)\psi_1(t-y-c\tau)\\
&+\phi_2(t-y-c\tau)\psi_1(t-y-c\tau) - \phi_2(t-y-c\tau)\psi_2(t-y-c\tau)]dy\\
&-(d_2+q_2e_2)[\psi_1(t)-\psi_2(t)] - a_{22}[\psi_1^2(t)-\psi_2^2(t)]|e^{-\mu|t|} + \beta_2||\Phi-\Psi|_\mu\\
\le&\{ae^{-d_1\tau}\int_{-\infty}^{+\infty} G(\tau,y)|\phi_1(t-y-c\tau)-\phi_2(t-y-c\tau)|\cdot|\psi_1(t-y-c\tau)|dy\\
&+ae^{-d_1\tau}\int_{-\infty}^{+\infty} G(\tau,y)|\psi_1(t-y-c\tau)-\psi_2(t-y-c\tau)|\cdot|\phi_2(t-y-c\tau)|dy\\
&+(d_2+q_2e_2)|\psi_1(t)-\psi_2(t)| + a_{22}|\psi_1(t)+\psi_2(t)||\psi_1(t)-\psi_2(t)|\}e^{-\mu|t|}\\
&+\beta_2|\Phi-\Psi|_\mu\\
\le&ae^{-d_1\tau}M_2\int_{-\infty}^{+\infty} G(\tau,y)e^{\mu|y+c\tau|}dy|\phi_1-\phi_2|_\mu\\
&+ae^{-d_1\tau}M_1\int_{-\infty}^{+\infty} G(\tau,y)e^{\mu|y+c\tau|}dy|\psi_1-\psi_2|_\mu\\
&+(d_2+q_2e_2)|\Phi-\Psi|_\mu + 2a_{22}M_2|\Phi-\Psi|_\mu + \beta_2|\Phi-\Psi|_\mu \quad\le\quad \bar\kappa_2|\Phi-\Psi|_\mu,
\end{aligned}$$

where

$$\bar\kappa_2 := ae^{-d_1\tau}(M_1+M_2)e^{(D_1\mu^2+c\mu)\tau} + d_2 + qe + 2b_{22}M_2 + \beta_2.$$

Therefore, that implies $H_2 : B_\mu(\mathbb{R},\mathbb{R}^2) \to B_\mu(\mathbb{R},\mathbb{R}^2)$ is continuous with respect to the norm $|\cdot|_\mu$. Similarly, we can show that

$$|H_1(\phi_1,\psi_1)(t) - H_1(\phi_2,\psi_2)(t)|e^{-\mu|t|} \le \bar\kappa_1|\Phi-\Psi|_\mu$$

where

$$\bar{\kappa}_1 := r + 2a_{11}M_1 + a_{12}M_1 + a_{12}M_2 + \beta_1,$$

and thus $H_1 : B_\mu(\mathbb{R},\mathbb{R}^2) \to B_\mu(\mathbb{R},\mathbb{R}^2)$ is continuous with respect to the norm $|\cdot|_\mu$.

Now, we show that $F_2 : B_\mu(\mathbb{R},\mathbb{R}^2) \to B_\mu(\mathbb{R},\mathbb{R}^2)$ is continuous with respect to the norm $|\cdot|_\mu$. For $t > 0$, we have from the choice of μ that

$$\begin{aligned}
&|F_2(\phi,\psi)(t) - F_2(\phi,\psi)(t)|e^{-\mu|t|}\\
=&\frac{e^{-\mu t}}{D_0(\lambda_{22}-\lambda_{21})}\left[\int_{-\infty}^{t} e^{\lambda_{21}(t-s)} + \int_{t}^{+\infty} e^{\lambda_{22}(t-s)}\right]|H_2(\phi_1,\psi_1) - H_2(\phi_2,\psi_2)(s)|ds\\
=&\frac{\bar{\kappa}_2 e^{-\mu t}}{D_0(\lambda_{22}-\lambda_{21})}\left[e^{\lambda_{21}t}\int_{-\infty}^{0} e^{-(\lambda_{21}+\mu)s}ds + e^{\lambda_{21}t}\int_{0}^{t} e^{(-\lambda_{21}+\mu)s}ds\right.\\
&\left. + e^{\lambda_{22}t}\int_{t}^{+\infty} e^{(-\lambda_{22}+\mu)s}ds\right]|\Phi-\Psi|_\mu\\
=&\frac{\bar{\kappa}_2}{D_0(\lambda_{22}-\lambda_{21})}\left[\frac{2\mu}{\lambda_{21}^2-\mu^2}e^{(\lambda_{21}-\mu)t} + \frac{\lambda_{22}-\lambda_{21}}{(\mu-\lambda_{21})(\lambda_{22}-\mu)}\right]|\Phi-\Psi|_\mu\\
\leq&\frac{\bar{\kappa}_2}{D_0(\lambda_{22}-\lambda_{21})}\left[\frac{2\mu}{\lambda_{21}^2-\mu^2} + \frac{\lambda_{22}-\lambda_{21}}{(\mu-\lambda_{21})(\lambda_{22}-\mu)}\right]|\Phi-\Psi|_\mu.
\end{aligned}$$

Similarly, for $t \leq 0$, we have

$$\begin{aligned}
&|F_2(\phi,\psi)(t) - F_2(\phi,\psi)(t)|e^{-\mu|t|}\\
\leq&\frac{\bar{\kappa}_2}{D_0(\lambda_{22}-\lambda_{21})}\left[\frac{2\mu}{\lambda_{22}^2-\mu^2} - \frac{\lambda_{22}-\lambda_{21}}{(\mu+\lambda_{21})(\lambda_{22}+\mu)}\right]|\Phi-\Psi|_\mu,
\end{aligned}$$

which implies that $F_2 : B_\mu(\mathbb{R},\mathbb{R}^2) \to B_\mu(\mathbb{R},\mathbb{R}^2)$ is continuous with respect to the norm $|\cdot|_\mu$ in $B_\mu(\mathbb{R},\mathbb{R}^2)$.

By using a similar argument as above, we can also prove that $F_1 : B_\mu(\mathbb{R},\mathbb{R}^2) \to B_\mu(\mathbb{R},\mathbb{R}^2)$ is continuous with respect to the norm $|\cdot|_\mu$ in $B_\mu(\mathbb{R},\mathbb{R}^2)$. The proof is complete. □

Lemma 3.4. $F(\Gamma) \subset \Gamma$.

Proof. For any (ϕ,ψ) with $(\underline{\phi},\underline{\psi}) \leq (\phi,\psi) \leq (\overline{\phi},\overline{\psi})$, It follows from Lemma 3.2 that

$$F_1(\underline{\phi},\overline{\psi}) \leq F_1(\phi,\psi) \leq F_1(\overline{\phi},\underline{\psi}),$$

$$F_2(\underline{\phi},\underline{\psi}) \leq F_2(\phi,\psi) \leq F_2(\overline{\phi},\overline{\psi}).$$

Now we show $F_1(\overline{\phi},\underline{\psi}) \leq \overline{\phi}$. Without loss of generality, we assume that $t_1 < t_2 < \cdots < t_n$ and denote $t_0 = -\infty, t_{n+1} = +\infty$, By the definition of upper-lower solutions, we have

$$H_1(\overline{\phi},\underline{\psi})(s) \leq -D_0\overline{\phi}''(t) + c\overline{\phi}'(t) + \beta_1\overline{\phi}(t) \text{ for } t \in \mathbb{R}\backslash S.$$

If $t \in \mathbb{R}\backslash S$, then

$$\begin{aligned}
&F_1(\overline{\phi}, \underline{\psi})(t)\\
=&\frac{1}{D_0(\lambda_{12}-\lambda_{11})}\Big[\int_{-\infty}^{t} e^{\lambda_{11}(t-s)} + \int_{t}^{+\infty} e^{\lambda_{12}(t-s)}\Big] H_1(\overline{\phi}, \underline{\psi})(s)ds\\
=&\frac{1}{D_0(\lambda_{12}-\lambda_{11})} \times \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \min\left\{e^{\lambda_{11}(t-s)}, e^{\lambda_{12}(t-s)}\right\} H_1(\overline{\phi}, \underline{\psi})(s)ds\\
\leq&\frac{1}{(D_0\lambda_{12}-\lambda_{11})} \times \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \min\left\{e^{\lambda_{11}(t-s)}, e^{\lambda_{12}(t-s)}\right\}\\
&\Big[-D_0\overline{\phi}''(s) + c\overline{\phi}'(s) + \beta_1\overline{\phi}(s)\Big] ds\\
=&\overline{\phi}(t) + \frac{1}{\lambda_{12}-\lambda_{11}} \times \left\{\sum_{j=0}^{n} \min\left\{e^{\lambda_{11}(t-t_j)}, e^{\lambda_{12}(t-t_j)}\right\}\Big[\overline{\phi}'(t_{j^+}) - \overline{\phi}'(t_{j^-})\Big]\right\}\\
\leq&\overline{\phi}(t) \quad \text{(by the assumption } (P3)).
\end{aligned}$$

In fact, the above inequality holds for $t \in \mathbb{R}$ by the continuity of $F_1(\overline{\phi}, \underline{\psi})(t)$ and $\overline{\phi}(t)$. Similar arguments lead to $F_1(\underline{\phi}, \overline{\psi}) \geq \underline{\phi}$, $F_2(\underline{\phi}, \underline{\psi}) \geq \underline{\psi}$, $F_2(\overline{\phi}, \overline{\psi}) \leq \overline{\psi}$, but we omit the details. The proof is complete. □

Lemma 3.5. *$F : \Gamma \to \Gamma$ is compact.*

Proof. For any $(\phi, \psi) \in \Gamma$,

$$\begin{aligned}
&F_1'(\phi, \psi)(t)\\
=&\frac{\lambda_{11}e^{\lambda_{11}t}}{D_0(\lambda_{12}-\lambda_{11})} \int_{-\infty}^{t} e^{-\lambda_{11}s} H_1(\phi, \psi)(s)ds + \frac{\lambda_{12}e^{\lambda_{12}t}}{D_0(\lambda_{12}-\lambda_{11})} \int_{t}^{+\infty} e^{-\lambda_{12}s} H_1(\phi, \psi)(s)ds.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&|F_1'(\phi, \psi)(t)|_\mu\\
=&\sup_{t\in\mathbb{R}}\Bigg|\frac{\lambda_{11}e^{\lambda_{11}t}}{D_0(\lambda_{12}-\lambda_{11})} \int_{-\infty}^{t} e^{-\lambda_{11}s} H_1(\phi, \psi)(s)ds\\
&+ \frac{\lambda_{12}e^{\lambda_{12}t}}{D_0(\lambda_{12}-\lambda_{11})} \int_{t}^{+\infty} e^{-\lambda_{12}s} H_1(\phi, \psi)(s)ds\Bigg| e^{-\mu|t|}\\
\leq&\frac{|\lambda_{11}|}{D_0(\lambda_{12}-\lambda_{11})} \sup_{t\in\mathbb{R}}\left\{e^{\lambda_{11}t-\mu|t|} \int_{-\infty}^{t} e^{-\lambda_{11}s} e^{\mu|s|} e^{-\mu|s|} H_1(\phi, \psi)(s)ds\right\}\\
&+ \frac{\lambda_{12}}{D_0(\lambda_{12}-\lambda_{11})} \sup_{t\in\mathbb{R}}\left\{e^{\lambda_{12}t-\mu|t|} \int_{t}^{+\infty} e^{-\lambda_{12}s} e^{\mu|s|} e^{-\mu|s|} H_1(\phi, \psi)(s)ds\right\}
\end{aligned}$$

$$\leq \frac{|\lambda_{11}|}{D_0(\lambda_{12}-\lambda_{11})}|H_1(\phi,\psi)|_\mu \sup_{t\in\mathbb{R}}\left\{e^{\lambda_{11}t-\mu|t|}\int_{-\infty}^{t} e^{-\lambda_{11}s}e^{\mu|s|}ds\right\}$$
$$+\frac{\lambda_{12}}{D_0(\lambda_{12}-\lambda_{11})}|H_1(\phi,\psi)|_\mu \sup_{t\in\mathbb{R}}\left\{e^{\lambda_{12}t-\mu|t|}\int_{t}^{+\infty} e^{-\lambda_{12}s}e^{\mu|s|}ds\right\}.$$

If $t>0$, we obtain

$$|F_1'(\phi,\psi)(t)|_\mu$$
$$\leq \frac{|\lambda_{11}|}{D_0(\lambda_{12}-\lambda_{11})}|H_1(\phi,\psi)|_\mu \sup_{t\in\mathbb{R}} e^{(\lambda_{11}-\mu)t}\left[\int_{-\infty}^{0} e^{-(\lambda_{11}+\mu)s}ds+\int_{0}^{t} e^{(\mu-\lambda_{11})s}ds\right]$$
$$+\frac{\lambda_{12}}{D_0(\lambda_{12}-\lambda_{11})}|H_1(\phi,\psi)|_\mu \sup_{t\in\mathbb{R}}\left\{e^{(\lambda_{12}-\mu)t}\int_{t}^{+\infty} e^{(\mu-\lambda_{12})s}ds\right\}.$$
$$\leq \frac{\lambda_{11}}{D_0(\lambda_{12}-\lambda_{11})(\mu+\lambda_{11})}|H_1(\phi,\psi)|_\mu+\frac{\lambda_{12}}{D_0(\lambda_{12}-\lambda_{11})(-\mu+\lambda_{12})}|H_1(\phi,\psi)|_\mu$$
$$=\frac{1}{D_0(\lambda_{12}-\lambda_{11})}\left[\frac{\lambda_{11}}{\mu+\lambda_{11}}+\frac{\lambda_{12}}{-\mu+\lambda_{12}}\right]|H_1(\phi,\psi)|_\mu.$$

If $t<0$, we have

$$|F_1'(\phi,\psi)(t)|_\mu$$
$$\leq \frac{|\lambda_{11}|}{D_0(\lambda_{12}-\lambda_{11})}|H_1(\phi,\psi)|_\mu \sup_{t\in\mathbb{R}}\left\{e^{(\lambda_{11}+\mu)t}\int_{-\infty}^{t} e^{-(\lambda_{11}+\mu)s}ds\right\}$$
$$+\frac{\lambda_{12}}{D_0(\lambda_{12}-\lambda_{11})}|H_1(\phi,\psi)|_\mu \sup_{t\in\mathbb{R}}\left\{e^{(\lambda_{12}+\mu)t}\Big[\int_{t}^{0} e^{-(\lambda_{11}+\mu)s}ds+\int_{0}^{\infty} e^{(\mu-\lambda_{12})s}ds\Big]\right\}$$
$$\leq \frac{\lambda_{11}}{D_0(\lambda_{12}-\lambda_{11})(\mu+\lambda_{11})}|H_1(\phi,\psi)|_\mu+\frac{\lambda_{12}}{D_0(\lambda_{12}-\lambda_{11})(-\mu+\lambda_{12})}|H_1(\phi,\psi)|_\mu$$
$$=\frac{1}{D_0(\lambda_{12}-\lambda_{11})}\left[\frac{\lambda_{11}}{\mu+\lambda_{11}}+\frac{\lambda_{12}}{-\mu+\lambda_{12}}\right]|H_1(\phi,\psi)|_\mu.$$

Since $H_1: B_\mu(\mathbb{R},\mathbb{R}^2)\to B_\mu(\mathbb{R},\mathbb{R}^2)$ is continuous with respect to the norm $|\cdot|_\mu$ and the set Γ is uniformly bounded, there exists a constant N_1 such that $|F_1'(\phi,\psi)(t)|_\mu \leq N_1$. In the similar way, there exists a constant N_2 such that $|F_2'(\phi,\psi)(t)|_\mu \leq N_2$. Hence F is equicontinuous on Γ and $F(\Gamma)$ is uniformly bounded with respect to the norm $|\cdot|_\mu$.

We next prove that $F(\Gamma)\to\Gamma$ is compact. Define $F^n(\phi,\psi)$ by

$$F^n(\phi,\psi)=\begin{cases} F(\phi,\psi)(t), & t\in[-n,n],\\ F(\phi,\psi)(n), & t\in(n,\infty),\\ F(\phi,\psi)(-n), & t\in(-\infty,-n).\end{cases}$$

Then, for any $n\geq 1$, $F^n(\Gamma)$ is equicontinuous and uniformly bounded. Now, in the interval $[-n,n]$, it follows from Ascoli-Arzela Theorem that $F^n(\Gamma)$ is compact. On the other hand, $F^n \to F$ in $B_\mu(\mathbb{R},\mathbb{R}^2)$ as $n \to \infty$, since

$$\begin{aligned}&\sup_{t\in\mathbb{R}} |F^n(\phi,\psi)(t) - F(\phi,\psi)(t)|e^{-\mu|t|}\\ =&\sup_{t\in(-\infty,-n)\cup(n,\infty)} |F^n(\phi,\psi)(t) - F(\phi,\psi)(t)|e^{-\mu|t|}\\ \le& 2(M_1+M_2)e^{-\mu n} \to 0 \quad as \quad n\to\infty,\end{aligned}$$

for any $(\phi,\psi) \in \Gamma$, Proposition 2.12 in [22], we have that $F : \Gamma \to \Gamma$ is compact. The proof is complete. □

Now, we give the existence theorem of this section.

Theorem 3.1. *Suppose there is a pair of upper-lower solutions* $\overline{\rho} = (\overline{\phi},\overline{\psi})$ *and* $\underline{\rho} = (\underline{\phi},\underline{\psi})$ *for* (3.1)*, satisfying* $(P1)$*-*$(P3)$*. Then, system* (3.1) *has a traveling wave solution.*

4. Upper-lower solutions for system (3.1)

From Theorem 3.1, we see that the existence of traveling wave solutions for system (3.1) follows from the existence of a pair of upper-lower solutions $(\overline{\phi},\overline{\psi})$ and $(\underline{\phi},\underline{\psi})$ of (3.1) satisfying the conditions $(P1)$-$(P3)$. Let (A1) holds, which ensures that E_3 exists. In this section, we further assume that $a_{11}b_{22} > aa_{12}e^{-d_1\tau}$, and this implies from Theorem 2.4 that E_3 is linearly asymptotically stable.

Denote

$$\begin{cases}\Delta_1(\lambda,c) = D_0\lambda^2 - c\lambda + r,\\ \Delta_2(\lambda,c) = D_0\lambda^2 - c\lambda + ae^{-d_1\tau}M_1e^{D_1\tau\lambda^2 - c\tau\lambda} - d_2 - qe.\end{cases} \tag{4.1}$$

By analyzing the properties of functions $\Delta_1(\lambda,c)$ and $\Delta_2(\lambda,c)$, one can see that $\Delta_1(\lambda,c)$ and $\Delta_2(\lambda,c)$ satisfy the following:

$$\begin{aligned}\Delta_1(\lambda,0) >& 0 \text{ for } \lambda\in\mathbb{R}, \quad \Delta_1(0,c) = r > 0,\\ \Delta_1'(\lambda,c) =& 2D_0\lambda - c, \quad \Delta_1''(\lambda,c) = 2D_0 > 0,\\ \Delta_2(\lambda,0) =& D_0\lambda^2 + ae^{-d_1\tau}M_1e^{D_1\tau\lambda^2} - d_2 - qe > 0 \text{ for } \lambda\in\mathbb{R},\\ \Delta_2(0,c) =& ae^{-d_1\tau}M_1 - d_2 - qe > 0,\\ \Delta_2'(\lambda,c) =& 2D_0\lambda - c + ae^{-d_1\tau}M_1(2D_1\tau\lambda - c\tau)e^{D_1\tau\lambda^2 - c\tau\lambda},\\ \Delta_2''(\lambda,c) =& 2D_0 + ae^{-d_1\tau}M_1(2D_1\tau\lambda - c\tau)^2e^{D_1\tau\lambda^2 - c\tau\lambda}\\ &+ 2D_1\tau ae^{-d_1\tau}M_1e^{D_1\tau\lambda^2 - c\tau\lambda} > 0.\end{aligned}$$

Therefore, we obtain a lemma as follows.

Lemma 4.1. *There exists* $c_1 > 0, c_2 > 0$ *such that the following four conclusions hold.*

(*i*) *For any given* $c > c_1$*,* $\Delta_1(\lambda,c) = 0$ *has two distinct positive roots* $\lambda_1(c)$*and* $\lambda_2(c)$*. Moreover, assume that* $0 < \lambda_1(c) < \lambda_2(c)$ *hold. Then*

$$\Delta_1(\lambda,c)\begin{cases} > 0, & for\ 0 < \lambda < \lambda_1(c),\\ < 0, & for\ \lambda_1(c) < \lambda < \lambda_2(c),\\ > 0, & for\ \lambda > \lambda_2(c).\end{cases}$$

(ii) *For any given* $c > c_2$, $\Delta_2(\lambda, c) = 0$ *has two distinct positive roots* $\lambda_3(c)$*and* $\lambda_4(c)$. *Moreover, assume that* $0 < \lambda_3(c) < \lambda_4(c)$ *hold. Then*

$$\Delta_2(\lambda, c) \begin{cases} > 0, & for\ 0 < \lambda < \lambda_3(c), \\ < 0, & for\ \lambda_3(c) < \lambda < \lambda_4(c), \\ > 0, & for\ \lambda > \lambda_4(c). \end{cases}$$

(iii) *If* $c < c_1$, *then* $\Delta_1(\lambda, c) = 0$ *has no real root.*

(iv) *If* $c < c_2$, *then* $\Delta_2(\lambda, c) = 0$ *has no real root.*

In the following , we denote $\lambda_i = \lambda_i(c)$, $i = 1, 2, 3, 4$.

Since $a_{11}b_{22} > aa_{12}e^{-d_1\tau}$, there exist $\varepsilon_i > 0 (i = 1, 2, 3, 4)$ satisfying $\varepsilon_3 < u^+, \varepsilon_4 < v^+$ and

$$\begin{array}{ll} a_{11}\varepsilon_1 - a_{12}\varepsilon_4 > 0, & b_{22}\varepsilon_2 - ae^{-d_1\tau}\varepsilon_1 > 0, \\ a_{11}\varepsilon_3 - a_{12}\varepsilon_2 > 0, & b_{22}\varepsilon_4 - ae^{-d_1\tau}\varepsilon_3 > 0. \end{array} \tag{4.2}$$

Using above constants, define continuous functions $(\underline{\phi}(t), \underline{\psi}(t))$ and $(\underline{\phi}(t), \underline{\psi}(t))$ as follows

$$\overline{\phi}(t) = \begin{cases} u^+e^{\lambda_1 t}, & t \leq t_1, \\ u^+ + \varepsilon_1 e^{-\lambda t}, & t \geq t_1, \end{cases} \qquad \overline{\psi}(t) = \begin{cases} v^+e^{\lambda_3 t}, & t \leq t_2, \\ v^+ + \varepsilon_2 e^{-\lambda t}, & t \geq t_2, \end{cases}$$

$$\underline{\phi}(t) = \begin{cases} 0, & t \leq t_3, \\ u^+ - \varepsilon_3 e^{-\lambda t}, & t \geq t_3, \end{cases} \qquad \underline{\psi}(t) = \begin{cases} 0, & t \leq t_4, \\ v^+ - \varepsilon_4 e^{-\lambda t}, & t \geq t_4, \end{cases}$$

where $\lambda > 0$ is small to be chosen later. It is easy to know that $M_1 := \sup\limits_{t\in\mathbb{R}} \overline{\phi}(t) > u^+, M_2 := \sup\limits_{t\in\mathbb{R}} \overline{\psi}(t) > v^+$. On the other hand, we have $t_1 > 0, t_2 > 0, t_3 < 0, t_4 < 0$.

We can see $\overline{\phi}, \overline{\psi}, \underline{\phi}$ and $\underline{\psi}$ satisfy the conditions $(P1)$, $(P2)$ and $(P3)$. We now prove that the continuous function $(\overline{\phi}(t), \overline{\psi}(t))$ and $(\underline{\phi}(t), \underline{\psi}(t))$ are an upper solution and a lower solution of (3.1), respectively.

Lemma 4.2. *Assume that (A1),* $a_{11}b_{22} > aa_{12}e^{-d_1\tau}$ *and* $c \geq \max\{c_1, c_2, D_1(\lambda_1 + 2\lambda_3)\}$ *hold. Then* $(\overline{\phi}(t), \overline{\psi}(t))$ *and* $(\underline{\phi}(t), \underline{\psi}(t))$ *is a pair of upper-lower solutions of* (3.1).

Proof. It suffices to prove that $(\overline{\phi}(t), \overline{\psi}(t))$ and $(\underline{\phi}(t), \underline{\psi}(t))$ satisfy the definition of upper-lower solutions.

We first consider $\overline{\phi}(t)$. If $t \leq t_1$, then $\overline{\phi}(t) = u^+e^{\lambda_1 t}$. It therefore follows that:

$$\begin{aligned} & D_0\overline{\phi}''(t) - c\overline{\phi}'(t) + (r - a_{11}\overline{\phi}(t))\overline{\phi}(t) - a_{12}\overline{\phi}(t)\underline{\psi}(t) \\ \leq\ & D_0\overline{\phi}''(t) - c\overline{\phi}'(t) + r\overline{\phi}(t) \\ =\ & u^+e^{\lambda_1 t}(D_0\lambda_1^2 - c\lambda_1 + r) \ = \ 0. \end{aligned}$$

If $t > t_1$, then $\overline{\phi}(t) = u^+ + \varepsilon_1 e^{-\lambda t}$, $\underline{\psi}(t) \geq v^+ - \varepsilon_4 e^{-\lambda t}$, It therefore follows that:

$$\begin{aligned}
&D_0\overline{\phi}''(t) - c\overline{\phi}'(t) + [r - a_{11}\overline{\phi}(t)]\overline{\phi}(t) - a_{12}\overline{\phi}(t)\underline{\psi}(t) \\
\leq\ & D_0(u^+ + \varepsilon_1 e^{-\lambda t})'' - c(u^+ + \varepsilon_1 e^{-\lambda t})' + [r - a_{11}(u^+ + \varepsilon_1 e^{-\lambda t})](u^+ + \varepsilon_1 e^{-\lambda t}) \\
&- a_{12}(u^+ + \varepsilon_1 e^{-\lambda t})(v^+ - \varepsilon_4 e^{-\lambda(t)}) \\
=\ & D_0\varepsilon_1\lambda^2 e^{-\lambda t} + c\varepsilon_1\lambda e^{-\lambda t} + [r - a_{11}(u^+ + \varepsilon_1 e^{-\lambda t})](u^+ + \varepsilon_1 e^{-\lambda t}) \\
&- a_{12}(u^+ + \varepsilon_1 e^{-\lambda t})(v^+ - \varepsilon_4 e^{-\lambda t}) \\
=\ & D_0\varepsilon_1\lambda^2\varrho + c\varepsilon_1\lambda\varrho + [r - a_{11}(u^+ + \varepsilon_1\varrho)](u^+ + \varepsilon_1\varrho) - a_{12}(u^+ + \varepsilon_1\varrho)(v^+ - \varepsilon_4\varrho) \\
=\ & (D_0\varepsilon_1\lambda^2 + c\varepsilon_1\lambda)\varrho + [(a_{12}\varepsilon_4 - a_{11}\varepsilon_1)\varrho](u^+ + \varepsilon_1\varrho) \\
=:\ & I_1(\lambda, \varrho),
\end{aligned}$$

where $\varrho := e^{-\lambda t} \in (0, e^{-\lambda t_1}) \subset (0,1)$ for $t \in (t_1, \infty)$ and $\lambda > 0$. Since $a_{12}\varepsilon_4 - a_{11}\varepsilon_1 < 0$, we have, for any $\varrho \in (0,1)$

$$I_1(0, \varrho) = [(a_{12}\varepsilon_4 - a_{11}\varepsilon_1)\varrho](u^+ + \varepsilon_1\varrho) < 0.$$

Then there exists a constant $\lambda_1^* > 0$ such that $I_1(\lambda, \varrho) < 0$ for any $\lambda \in (0, \lambda_1^*)$, $\varrho \in (0,1)$. Thus, $\overline{\phi}$ satisfies the definition of the upper solution.

We now consider $\overline{\psi}(t)$. If $t \leq t_2$, then $\overline{\psi}(t) = v^+ e^{\lambda_3 t}$, $\overline{\phi}(t) \leq u^+ e^{\lambda_1 t}$. Note that we always have $\overline{\psi}(t - y - c\tau) \leq v^+ e^{\lambda_3(t-y-c\tau)}$ and $\overline{\phi}(t - y - c\tau) \leq u^+ e^{\lambda_1(t-y-c\tau)}$. It therefore follows that

$$\begin{aligned}
&D_0\overline{\psi}''(t) - c\overline{\psi}'(t) + ae^{-d_1\tau}\int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi D_1\tau}} e^{-\frac{y^2}{4D_1\tau}} \overline{\phi}(t - y - c\tau)\overline{\psi}(t - y - c\tau)dy \\
&- (d_2 + qe)\overline{\psi}(t) - b_{22}\overline{\psi}^2(t) \\
\leq\ & ae^{-d_1\tau}\int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi D_1\tau}} e^{-\frac{y^2}{4D_1\tau}} u^+ e^{\lambda_1(t-y-c\tau)} v^+ e^{\lambda_3(t-y-c\tau)} dy \\
&+ D_0(v^+ e^{\lambda_3 t})'' - c(v^+ e^{\lambda_3 t})' - (d_2 + qe)(v^+ e^{\lambda_3 t}) - b_{22}(v^+ e^{\lambda_3 t})^2 \\
\leq\ & v^+ e^{\lambda_3 t}\Bigg(ae^{-d_1\tau}\int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi D_1\tau}} e^{-\frac{y^2}{4D_1\tau}} u^+ e^{\lambda_1(t-y-c\tau)} e^{\lambda_3(-y-c\tau)} dy \\
&+ D_0\lambda_3^2 - c\lambda_3 - (d_2 + qe)\Bigg) \\
\leq\ & v^+ e^{\lambda_3 t}\Bigg(ae^{-d_1\tau} u^+ e^{\lambda_1 t}\int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi D_1\tau}} e^{-\frac{y^2}{4D_1\tau}} e^{(\lambda_1+\lambda_3)(-y-c\tau)} dy \\
&+ D_0\lambda_3^2 - c\lambda_3 - (d_2 + qe)\Bigg) \\
=\ & v^+ e^{\lambda_3 t}\left(D_0\lambda_3^2 - c\lambda_3 + ae^{-d_1\tau} M_1 e^{D_1\tau(\lambda_1+\lambda_3)^2 - c\tau(\lambda_1+\lambda_3)} - (d_2 + qe)\right) \\
\leq\ & v^+ e^{\lambda_3 t}\left(D_0\lambda_3^2 - c\lambda_3 + ae^{-d_1\tau} M_1 e^{D_1\tau\lambda_3^2 - c\tau\lambda_3} - (d_2 + qe)\right) = 0.
\end{aligned}$$

Here, in the last inequality, we have used the assumption $c \geq D_1(\lambda_1 + 2\lambda_3)$.

If $t > t_2$, then $\overline{\psi}(t) = v^+ + \varepsilon_2 e^{-\lambda t}$, $\overline{\phi}(t) \leq u^+ + \varepsilon_1 e^{-\lambda t}$. Note that we always have $\overline{\psi}(t-y-c\tau) \leq v^+ + \varepsilon_2 e^{-\lambda(t-y-c\tau)}$ and $\overline{\psi}(t-y-c\tau) \leq u^+ + \varepsilon_1 e^{-\lambda(t-y-c\tau)}$. It therefore follows that

$$
\begin{aligned}
& D_0\overline{\psi}''(t) - c\overline{\psi}'(t) + ae^{-d_1\tau}\int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi D_1\tau}} e^{-\frac{y^2}{4D_1\tau}} \overline{\phi}(t-y-c\tau)\overline{\psi}(t-y-c\tau)dy \\
& - (d_2+qe)\overline{\psi}(t) - b_{22}\overline{\psi}^2(t) \\
\leq\ & D_0(v^+ + \varepsilon_2 e^{-\lambda t})'' - c(v^+ + \varepsilon_2 e^{-\lambda t})' \\
& + ae^{-d_1\tau}\int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi D_1\tau}} e^{-\frac{y^2}{4D_1\tau}} (u^+ + \varepsilon_1 e^{-\lambda(t-y-c\tau)})(v^+ + \varepsilon_2 e^{-\lambda(t-y-c\tau)})dy \\
& - (d_2+qe)(v^+ + \varepsilon_2 e^{-\lambda t}) - b_{22}(v^+ + \varepsilon_2 e^{-\lambda t})^2 \\
\leq\ & D_0\varepsilon_2\lambda^2 e^{-\lambda t} + c\varepsilon_2\lambda e^{-\lambda t} \\
& + ae^{-d_1\tau}\int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi D_1\tau}} e^{-\frac{y^2}{4D_1\tau}} (u^+ + \varepsilon_1 e^{-\lambda(t-y-c\tau)})(v^+ + \varepsilon_2 e^{-\lambda(t-y-c\tau)})dy \\
& - (d_2+qe)(v^+ + \varepsilon_2 e^{-\lambda t}) - b_{22}(v^+ + \varepsilon_2 e^{-\lambda t})^2 \\
=\ & D_0\varepsilon_2\lambda^2 e^{-\lambda t} + c\varepsilon_2\lambda e^{-\lambda t} \\
& + ae^{-d_1\tau}\Bigg(u^+v^+ + u^+\varepsilon_2 e^{-\lambda t + D_1\tau\lambda^2 + c\tau\lambda} + v^+\varepsilon_1 e^{-\lambda t + D_1\tau\lambda^2 + c\tau\lambda} \\
& + \varepsilon_1\varepsilon_2 e^{-2\lambda t + 4D_1\tau\lambda^2 + 2c\tau\lambda}\Bigg) - (d_2+qe)(v^+ + \varepsilon_2 e^{-\lambda t}) - b_{22}(v^+ + \varepsilon_2 e^{-\lambda t})^2 \\
=\ & (D_0\varepsilon_2\lambda^2 + c\varepsilon_2\lambda)\varrho \\
& + ae^{-d_1\tau}\left(u^+v^+ + u^+\varepsilon_2\varrho e^{D_1\tau\lambda^2 + c\tau\lambda} + v^+\varepsilon_1\varrho e^{D_1\tau\lambda^2 + c\tau\lambda} + \varepsilon_1\varepsilon_2\varrho^2 e^{4D_1\tau\lambda^2 + 2c\tau\lambda}\right) \\
& - (d_2+qe)(v^+ + \varepsilon_2\varrho) - b_{22}(v^+ + \varepsilon_2\varrho)^2 \\
=:\ & I_2(\lambda, \varrho),
\end{aligned}
$$

where $\varrho = e^{-\lambda t}$ is defined as above. Note from the fact $ae^{-d_1\tau}\varepsilon_1 - b_{22}\varepsilon_2 < 0$ that

$$
\begin{aligned}
I_2(0,\varrho) &= ae^{-d_1\tau}(u^+ + \varepsilon_1\varrho)(v^+ + \varepsilon_2\varrho) - (d_2+qe)(v^+ + \varepsilon_2\varrho) - b_{22}(v^+ + \varepsilon_2\varrho)^2 \\
&= (v^+ + \varepsilon_2\varrho)(ae^{-d_1\tau}\varepsilon_1\varrho - b_{22}\varepsilon_2\varrho) < 0.
\end{aligned}
$$

Similar to the argument of $I_1(\lambda,\varrho) < 0$, there exists a constant $\lambda_2^* > 0$ such that $I_2(\lambda,\varrho) < 0$ for any $\lambda \in (0, \lambda_2^*)$, $\varrho \in (0,1)$. Thus, $\overline{\psi}$ satisfies the definition of the upper solution.

If $t \leq t_3$, then $\underline{\phi}(t) = 0$. It follows that

$$
D_0\underline{\phi}''(t) - c\underline{\phi}'(t) + (r - a_{11}\underline{\phi}(t))\underline{\phi}(t) - a_{12}\underline{\phi}(t)\overline{\psi}(t) = 0.
$$

If $t > t_3$, then $\underline{\phi}(t) = u^+ - \varepsilon_3 e^{-\lambda t}$, $\overline{\psi}(t) \leq v^+ + \varepsilon_2 e^{-\lambda t}$, and we derive that:

$$
\begin{aligned}
& D_0\underline{\phi}''(t) - c\underline{\phi}'(t) + (r - a_{11}\underline{\phi}(t))\underline{\phi}(t) - a_{12}\underline{\phi}(t)\overline{\psi}(t) \\
\geq\ & D_0(u^+ - \varepsilon_3 e^{-\lambda t})'' - c(u^+ - \varepsilon_3 e^{-\lambda t})' + (r - a_{11}(u^+ - \varepsilon_3 e^{-\lambda t}))(u^+ - \varepsilon_3 e^{-\lambda t})
\end{aligned}
$$

$$
\begin{aligned}
& - a_{12}(u^+ - \varepsilon_3 e^{-\lambda t})(v^+ + \varepsilon_2 e^{-\lambda t}) \\
= & - D_0\varepsilon_3\lambda^2 e^{-\lambda t} - c\varepsilon_3\lambda e^{-\lambda t} + [r - a_{11}(u^+ - \varepsilon_3 e^{-\lambda t})](u^+ - \varepsilon_3 e^{-\lambda t}) \\
& - a_{12}(u^+ - \varepsilon_3 e^{-\lambda t})(v^+ + \varepsilon_2 e^{-\lambda t}) \\
= & - (D_0\varepsilon_3\lambda^2 + c\varepsilon_3\lambda)\varrho + [r - a_{11}(u^+ - \varepsilon_3\varrho)](u^+ - \varepsilon_3\varrho) - a_{12}(u^+ - \varepsilon_3\varrho)(v^+ + \varepsilon_2\varrho) \\
=: & \ I_3(\lambda, \varrho).
\end{aligned}
$$

Note that

$$
\begin{aligned}
I_3(0, \varrho) &= (r - a_{11}(u^+ - \varepsilon_3\varrho))(u^+ - \varepsilon_3\varrho) - a_{12}(u^+ - \varepsilon_3\varrho)(v^+ + \varepsilon_2\varrho) \\
&= (u^+ - \varepsilon_3\varrho)(a_{11}\varepsilon_3\varrho - a_{12}\varepsilon_2\varrho) > 0.
\end{aligned}
$$

Note $\varrho = e^{-\lambda t}$ and $t_3 = \frac{1}{\lambda}\ln\frac{\varepsilon_3}{u^+}$, thus $\varrho \in (0, e^{-\lambda t_3}) = (0, \frac{u^+}{\varepsilon_3})$. Then there exists a constant $\lambda_3^* > 0$ such that $I_3(\lambda, \varrho) > 0$ for any $\lambda \in (0, \lambda_3^*)$ and $\varrho \in (0, \frac{u^+}{\varepsilon_3})$. Thus, $\underline{\phi}$ satisfies the definition of the lower solution.

If $t \le t_4$, then $\underline{\psi}(t) = 0$, $\underline{\phi}(t) \le u^+ e^{\lambda_1 t}$, and thus

$$
\begin{aligned}
& D_0\underline{\psi}''(t) - c\underline{\psi}'(t) + ae^{-d_1\tau}\underline{\phi}(t)\int_{-\infty}^{+\infty}\frac{1}{\sqrt{4\pi D_1\tau}}e^{-\frac{y^2}{4D_1\tau}}\underline{\psi}(t - y - c\tau)dy \\
& - (d_2 + qe)\underline{\psi}(t) - b_{22}\underline{\psi}^2(t) \ \ge \ 0.
\end{aligned}
$$

If $t > t_4$, then $\underline{\psi}(t) = v^+ - \varepsilon_4 e^{-\lambda t}$, $\underline{\phi}(t) \le u^+ - \varepsilon_3 e^{-\lambda t}$. Note that we always have $\underline{\psi}(t - y - c\tau) \ge v^+ - \varepsilon_4 e^{-\lambda(t-y-c\tau)}$ and $\underline{\phi}(t - y - c\tau) \ge u^+ - \varepsilon_3 e^{-\lambda(t-y-c\tau)}$. It then follows that

$$
\begin{aligned}
& D_0\underline{\psi}''(t) - c\underline{\psi}'(t) + ae^{-d_1\tau}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{4\pi D_1\tau}}e^{-\frac{y^2}{4D_1\tau}}\underline{\phi}(t - y - c\tau)\underline{\psi}(t - y - c\tau)dy \\
& - (d_2 + qe)\underline{\psi}(t) - b_{22}\underline{\psi}^2(t) \\
\ge & \ D_0(v^+ - \varepsilon_4 e^{-\lambda t})'' - c(v^+ - \varepsilon_4 e^{-\lambda t})' \\
& + ae^{-d_1\tau}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{4\pi D_1\tau}}e^{-\frac{y^2}{4D_1\tau}}(u^+ - \varepsilon_3 e^{-\lambda(t-y-c\tau)})(v^+ - \varepsilon_4 e^{-\lambda(t-y-c\tau)})dy \\
& - (d_2 + qe)(v^+ - \varepsilon_4 e^{-\lambda t}) - b_{22}(v^+ - \varepsilon_4 e^{-\lambda t})^2 \\
\ge & \ - D_0\varepsilon_4\lambda^2 e^{-\lambda t} - c\varepsilon_4\lambda e^{-\lambda t} \\
& + ae^{-d_1\tau}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{4\pi D_1\tau}}e^{-\frac{y^2}{4D_1\tau}}(u^+ - \varepsilon_3 e^{-\lambda(t-y-c\tau)})(v^+ - \varepsilon_4 e^{-\lambda(t-y-c\tau)})dy \\
& - (d_2 + qe)(v^+ - \varepsilon_4 e^{-\lambda t}) - b_{22}(v^+ - \varepsilon_4 e^{-\lambda t})^2 \\
= & \ - D_0\varepsilon_4\lambda^2 e^{-\lambda t} - c\varepsilon_4\lambda e^{-\lambda t} \\
& + ae^{-d_1\tau}\Bigg(u^+v^+ - u^+\varepsilon_4 e^{-\lambda t + D_1\tau\lambda^2 + c\tau\lambda} - v^+\varepsilon_3 e^{-\lambda t + D_1\tau\lambda^2 + c\tau\lambda} \\
& + \varepsilon_3\varepsilon_4 e^{-2\lambda t + 4D_1\tau\lambda^2 + 2c\tau\lambda}\Bigg) - (d_2 + qe)(v^+ - \varepsilon_4 e^{-\lambda t}) - b_{22}(v^+ - \varepsilon_4 e^{-\lambda t})^2
\end{aligned}
$$

$$
\begin{aligned}
&= -\,(D_0\varepsilon_4\lambda^2 - c\varepsilon_4\lambda)\varrho \\
&\quad + ae^{-d_1\tau}\left(u^+v^+ - u^+\varepsilon_4\varrho e^{D_1\tau\lambda^2+c\tau\lambda} - v^+\varepsilon_3\varrho e^{D_1\tau\lambda^2+c\tau\lambda} + \varepsilon_3\varepsilon_4\varrho^2 e^{4D_1\tau\lambda^2+2c\tau\lambda}\right) \\
&\quad - (d_2+qe)(v^+ - \varepsilon_4\varrho) - b_{22}(v^+ - \varepsilon_4\varrho)^2 \\
&=: I_4(\lambda,\varrho).
\end{aligned}
$$

Note the fact

$$
\begin{aligned}
I_4(0,\varrho) \quad &= ae^{-d_1\tau}(u^+ - \varepsilon_3\varrho)(v^+ - \varepsilon_4\varrho) - (d_2+qe)(v^+ - \varepsilon_4\varrho) - b_{22}(v^+ - \varepsilon_4\varrho)^2 \\
&= (v^+ - \varepsilon_4\varrho)(b_{22}\varepsilon_4\varrho - ae^{-d_1\tau}\varepsilon_3\varrho) > 0.
\end{aligned}
$$

Similarly to the situation of $t > t_3$, there exists a constant $\lambda_4^* > 0$ such that $I_4(\lambda,\varrho) > 0$ for any $\lambda \in (0,\lambda_4^*)$ and $\varrho \in (0, \frac{v^+}{\varepsilon_4})$. Thus, $\underline{\psi}$ satisfies the definition of the lower solution.

Choose $\lambda^* = \min_{i=1,2,3,4}\{\lambda_i^*\}$. Then for $\lambda \in (0,\lambda^*)$, the conclusion of Lemma 4.2 is true. The proof is complete. □

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