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HYERS-ULAM-RASSIAS STABILITY OF FUNCTIONAL EQUATIONS IN MENGER PROBABILISTIC NORMED SPACES

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Abstract We introduce the approximately quadratic functional equation in Menger probabilistic normed spaces. More precisely, we show under some suitable conditions that an approximately quadratic functional equation can be approximated by a quadratic function in above mentioned spaces. Also we consider the stability problem for approximately pexiderized functional equation in Menger probabilistic normed spaces.

Keywords Menger probabilistic normed space, quadratic functional equation, pexiderized quadratic functional equation, stability.

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1. Introduction

The notion of probabilistic metric spaces was introduced by Menger [10]. Menger proposed transferring the probabilistic notions of quantum mechanic from physics to the underlying geometry. Probabilistic normed spaces are real linear spaces in which the norm of each vector is an appropriate probability distribution function rather than a number. The theory of probabilistic normed spaces was introduced by Šerstnev in 1963 [20]. In [1] Alsina, Schweizer and Sklar gave a new definition of probabilistic normed spaces which includes Šerstnev's as a special case and leads naturally to the identification of the principle class of probabilistic normed spaces, the Menger spaces.

The idea of Menger was to use distribution function instead of nonnegtive real numbers as values of the metric. It corresponds to the situation when we do not know exactly the distance between two points, we know only probabilities of possible values of this distance. The probabilistic generalization of metric spaces appears to be well adapted for the investigation of quantum particle physics particulary in connections with both string and ε^{∞} theory which were given and studied by El-Naschie [11, 12].

Stability problem of a functional equation was first posed in [22] which was answered in [6] and then generalized in [2, 18] for additive mappings and linear mappings respectively. Since then several stability problems for various functional equations have been investigated in [7, 8, 9, 19].

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Recently, several results for the Hyers-Ulam-Rassias stability of many functional equations on Menger probabilistic normed space have been proved by several researchers[5, 21]. Our goal is to determine some stability results concerning the quadratic and pexiderized quadratic functional equations in probabilistic normed spaces.

2. Preliminaries

For reader's convenience, in this section we briefly recall some concepts and results from probabilistic metric spaces theory used in the paper.

Definition 2.1. A function $F : \mathbb{R} \to [0,1]$ is called a distribution function if it is nondecreasing and left-continuous, with $sup_{t\in\mathbb{R}}F(t) = 1$ and $inf_{t\in\mathbb{R}}F(t) = 0$.

The class of all distribution functions F with $F(0) = 0$ is denoted by D_+. ε_0 is the element of D_+ defined by

$$\varepsilon_0 = \begin{cases} 1, & t > 0, \\ 0, & t \leq 0. \end{cases}$$

Definition 2.2. A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is said to be a t-norm if it satisfies the following conditions:

(1) $*$ is commutative and associative;
(2) $*$ is continuous;
(3) $a * 1 = a$ for all $a \in [0,1]$;
(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1]$.

Definition 2.3. [3] Let X be a real vector space, F a mapping from X to D_+ (for any $x \in X$, $F(x)$ is denoted by F_x) and $*$ a t-norm. The triple $(X, F, *)$ is called a Menger probabilistic normed space (briefly Menger PN-space) if the following conditions are satisfied:

1) $F_x(0) = 0$, for all $x \in X$;
2) $F_x = \varepsilon_0$ iff $x = \theta$;
3) $F_{\alpha x}(t) = F_x(\frac{t}{|\alpha|})$ for all $\alpha \in \mathbb{R}$, $\alpha \neq 0$ and $x \in X$;
4) $F_{x+y}(t_1 + t_2) \geq F_x(t_1) * F_x(t_2)$, for all $x, y \in X$ and $t_1, t_2 > 0$.

Definition 2.4. Let $(X, F, *)$ be a Menger PN-space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that

$$\lim_{n\to\infty} F_{x_n - x}(t) = 1$$

for all $t > 0$. In this case x is called the limit of $\{x_n\}$.

Definition 2.5. The sequence $\{x_n\}$ in Menger PN-space $(X, F, *)$ is called Cauchy if for each $\epsilon > 0$ and $\delta > 0$, there exists some n_0 such that $F_{x_n - x_m}(\delta) > 1 - \epsilon$ for all $m, n \geq n_0$.

Clearly, every convergent sequence in Menger PN-space is Cauchy. If each Cauchy sequence is convergent sequence in a Menger PN-space $(X, F, *)$, then $(X, F, *)$ is called Menger probabilistic Banach space (briefly, Menger PB-space).

3. Stability of quadratic functional equation in Menger PN-spaces

In this section, we define an approximately quadratic mapping in Menger PN-spaces.

Definition 3.1. Let $(X, F, *)$ be a Menger PN-space and $(Y, G, *)$ be a Menger PB-space. A mapping $f : X \to Y$ is said to be P-approximately quadratic if

$$\begin{aligned} &G_{f(x+y)+f(x-y)-2f(x)-2f(y)}(t+s) \geq F_x(t) * F_y(s), \\ &\forall x, y \in X, \quad t, s \in [0, \infty). \end{aligned} \tag{3.1}$$

The following result gives a Hyers-Ulam-Rassias stability of the P-approximately quadratic functional equation.

Theorem 3.1. *Let $f : X \to Y$ be a P-approximately quadratic functional equation. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$G_{Q(x)-f(x)}(t) \geq F_x(t), \qquad \forall x \in X, t > 0. \tag{3.2}$$

Proof. Put $x = y$ and $s = t$ in (3.1) to obtain

$$G_{f(2x)-4f(x)}(2t) \geq F_x(t). \tag{3.3}$$

Replacing x by $2^n x$ in (3.3), we see that

$$G_{f(2^{n+1}x)-4f(2^n x)}(2t) \geq F_{2^n x}(t).$$

It follows that

$$G_{f(2^{n+1}x)-4f(2^n x)}(2^{n+1}t) \geq F_x(t).$$

Whence

$$G_{\frac{f(2^{n+1}x)}{4^{n+1}}-\frac{f(2^n x)}{4^n}}(2^{-n-1}t) \geq F_x(t).$$

If $n > m > 0$, then

$$\begin{aligned} &G_{\frac{f(2^n x)}{4^n}-\frac{f(2^m x)}{4^m}}\Big(\sum_{k=m+1}^{n} 2^{-k-1}t\Big) \\ \geq &G_{\sum_{k=m+1}^{n}\left(\frac{f(2^k x)}{4^k}-\frac{f(2^{k-1}x)}{4^{k-1}}\right)}\Big(\sum_{k=m+1}^{n} 2^{-k-1}t\Big) \\ \geq &\prod_{k=m+1}^{n} G_{\frac{f(2^k x)}{4^k}-\frac{f(2^{k-1}x)}{4^{k-1}}}(2^{-k-1}t) \ \geq \ F_x(t). \end{aligned} \tag{3.4}$$

Let $c > 0$ and ε be given. Since

$$\lim_{t\to\infty} F_x(t) = 1,$$

there is some $t_0 > 0$ such that $F_x(t_0) \geq 1 - \varepsilon$. Fix some $t > t_0$. The convergence of the series $\sum_{n=1}^{\infty} 2^{-n-1}t$ shows that there exists some $n_0 \geq 0$ such that for each $n > m \geq n_0$, the inequality $\sum_{k=m+1}^{n} 2^{-k-1}t < c$ holds. It follows that,

$$G_{\frac{f(2^n x)}{4^n}-\frac{f(2^m x)}{4^m}}(c) \geq G_{\frac{f(2^n x)}{4^n}-\frac{f(2^m x)}{4^m}}(\textstyle\sum_{k=m+1}^{n} 2^{-k-1}t_0) \geq F_x(t_0) \geq 1-\varepsilon.$$

Hence $\{\frac{f(2^n x)}{4^n}\}$ is a Cauchy sequence in $(Y, G, *)$. Since $(Y, G, *)$ is a Menger PB-space, this sequence converges to some $Q(x) \in Y$. Hence, we can define a mapping $Q : X \to Y$ such that $\lim_{n\to\infty} G_{\frac{f(2^n x)}{4^n}-Q(x)} = 1$. Moreover, if we put $m = 0$ in (3.4) we observe that

$$G_{\frac{f(2^n x)}{4^n}-f(x)}(\textstyle\sum_{k=1}^{n} 2^{-k-1}t) \geq F_x(t).$$

Therefore,

$$G_{\frac{f(2^n x)}{4^n}-f(x)}(t) \geq F_x\left(\frac{t}{\sum_{k=1}^{n} 2^{-k-1}}\right). \tag{3.5}$$

Next we will show that Q is quadratic. Let $x, y \in X$, then we have

$$\begin{aligned}
&G_{Q(x+y)+Q(x-y)-2Q(x)-2Q(y)}(t)\\
\geq &G_{Q(x+y)-\frac{f(2^n(x+y))}{4^n}}\left(\frac{t}{5}\right) * G_{Q(x-y)-\frac{f(2^n(x-y))}{4^n}}\left(\frac{t}{5}\right)\\
&* G_{2\frac{f(2^n x)}{4^n}-2Q(x)}\left(\frac{t}{5}\right) * G_{2\frac{f(2^n y)}{4^n}-2Q(y)}\left(\frac{t}{5}\right)\\
&* G_{\frac{f(2^n(x+y))}{4^n}+\frac{f(2^n(x-y))}{4^n}-2\frac{f(2^n x)}{4^n}-2\frac{f(2^n y)}{4^n}}\left(\frac{t}{5}\right).
\end{aligned}$$

The first four terms on the right hand side of the above inequality tend to 1 as $n \to \infty$, $t \to \infty$ and the fifth term, by (3.1) is greater than or equal to $F_{2^n x}(\frac{4^n t}{10}) * F_{2^n y}(\frac{4^n t}{10}) = F_x(\frac{2^n t}{10}) * F_y(\frac{2^n t}{10})$, which tends to 1 as $n \to \infty$. Therefore $Q(x+y)+Q(x-y) = 2Q(x)+2Q(y)$. Next we approximate the difference between f and Q. For every $x \in X$ and $t > 0$, by (3.5) for large enough n, we have

$$G_{Q(x)-f(x)}(t) \geq G_{Q(x)-\frac{f(2^n x)}{4^n}}\left(\frac{t}{2}\right) * G_{\frac{f(2^n x)}{4^n}-f(x)}\left(\frac{t}{2}\right) \geq F_x(t).$$

Let Q' be another quadratic function from X to Y which satisfies (3.2). We have

$$G_{Q(x)-Q'(x)}(t) \geq G_{Q(x)-f(x)}\left(\frac{t}{2}\right) * G_{f(x)-Q'(x)}\left(\frac{t}{2}\right) \geq F_x(t)$$

for each $t > 0$. Therefore $Q = Q'$. □

4. Stability of pexiderized quadratic equation in Menger PN-spaces

In this section we define the P-approximately pexiderized quadratic functional equation in Menger PN-spaces and then we investigate the Hyers-Ulam-Rassias stability problem for these functional equations.

Definition 4.1. Let $(X, F, *)$ be a Menger PN-space and $(Y, G, *)$ a Menger PB-space. A mapping $f : X \to Y$ is said to be P-approximately pexiderized quadratic mapping in Menger PN-space if

$$G_{f(x+y)+f(x-y)-2g(x)-2h(y)}(t+s) \geq F_x(t) * F_y(s), \quad \forall x, y \in X,\ t, s \in [0, \infty). \tag{4.1}$$

The following result gives a Hyers-Ulam-Rassias stability of the P-approximately pexiderized quadratic functional equation.

Proposition 4.1. *Let $f : X \to Y$ be a P-approximately pexiderized quadratic mapping in Menger PN-space, such that f, g and h are odd functions from X to Y. Also suppose that $F_{2x} = F_x(\frac{t}{|\alpha|})$ for some real number α with $0 < |\alpha| < 2$. Then there exists a unique additive mapping $T : X \to Y$ such that*

$$G_{T(x)-f(x)}(t) \geq F_x\left(\frac{(2-\alpha)t}{48}\right). \tag{4.2}$$

Proof. By changing the roles of x and y in (4.1) and putting $t = s$, we get

$$G_{f(x+y)-f(x-y)-2g(y)-2h(x)}(2t) \geq F_y(t) * F_x(t). \tag{4.3}$$

It follows from (4.1) and (4.3) that

$$\begin{aligned} &G_{f(x+y)-g(x)-h(y)-g(y)-h(x)}(4t) \\ \geq &G_{f(x+y)+f(x-y)-2g(x)-2h(y)}(2t) * F_{f(x+y)-f(x-y)-2g(y)-2h(x)}(2t) \\ \geq &F_x(t) * F_y(t). \end{aligned} \tag{4.4}$$

If we put $y = 0$ in (4.4), we obtain

$$G_{f(x)-g(x)-h(x)}(4t) \geq F_x(t). \tag{4.5}$$

Similarly by putting $x = 0$ in (4.4), we have

$$G_{f(y)-g(y)-h(y)}(4t) \geq F_y(t). \tag{4.6}$$

From (4.4), (4.5) and (4.6) we conclude that

$$\begin{aligned} &G_{f(x+y)-f(x)-f(y)}(12t) \\ \geq &G_{f(x+y)-g(x)-h(y)-g(y)-h(x)}(4t) \\ &* G_{f(x)-g(x)-h(x)}(4t) * G_{f(y)-h(y)-g(y)}(4t) \\ \geq &F_x(t) * F_y(t). \end{aligned} \tag{4.7}$$

If we put $x = y$ in (4.7), we get

$$G_{f(2x)-2f(x)}(t) \geq F_x\left(\frac{t}{12}\right). \tag{4.8}$$

Replacing x by $2^n x$ in (4.8) and by assumption we have

$$G_{f(2^{n+1}x)-2f(2^n x)}(t) \geq F_{2^n x}\left(\frac{t}{12}\right) = F_x\left(\frac{t}{12\alpha^n}\right).$$

Thus

$$G_{\frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^n x)}{2^n}}(t) = G_{f(2^{n+1}x)-f(2^n x)}(2^n t) \geq F_x\left(\frac{(\frac{2}{\alpha})^n t}{12}\right).$$

Hence

$$G_{\frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^n x)}{2^n}}\left((\frac{\alpha}{2})^n t\right) \geq F_x\left(\frac{t}{12}\right).$$

Therefore for each $n > m \geq 0$,

$$\begin{aligned}
&G_{\frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m}} \left(\sum_{k=m+1}^{n} (\frac{\alpha}{2})^{k-1} t \right) \\
=&G_{\sum_{k=m+1}^{n} \frac{f(2^k x)}{2^k} - \frac{f(2^{k-1} x)}{2^{k-1}}} \left(\sum_{k=m+1}^{n} (\frac{\alpha}{2})^{k-1} t \right) \\
\geq &\prod_{k=m}^{n-1} G_{\frac{f(2^k x)}{2^k} - \frac{f(2^{k-1} x)}{2^{k-1}}} \left((\frac{\alpha}{2})^{k-1} t \right) \\
\geq &F_x \left(\frac{t}{12} \right)
\end{aligned} \tag{4.9}$$

for all $x \in X$ and $t > 0$ where $\prod_{j=1}^{n} a_j = a_1 * a_2 * ... * a_n$. Let $\varepsilon > 0$ and $t_0 > 0$ be given. Thanks to the fact that $\lim_{t \to \infty} F_x(t) = 1$, we can find some $t_1 > t_0$ such that $F_x(t_1) > 1 - \varepsilon$. The convergence of the series $\sum_{n=1}^{\infty} (\frac{\alpha}{2})^n t_1$ gives some $n_0 \in \mathbb{N}$ such that for each $n > m \geq n_0$,

$$\sum_{k=m+1}^{n} (\tfrac{\alpha}{2})^{k-1} t_1 < t_0.$$

Therefore

$$G_{\frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m}}(t_0) \geq G_{\frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m}} (\textstyle\sum_{k=m+1}^{n} (\frac{\alpha}{2})^{k-1} t_1) \geq F_x(t_1) > 1 - \varepsilon.$$

So $\{\frac{f(2^n x)}{2^n}\}$ is a Cauchy sequence in the Menger PB-space $(Y, G, *)$. Hence $\{\frac{f(2^n x)}{2^n}\}$ converges to some point $T(x) \in Y$. Define $T : X \to Y$ such that $lim_{n \to \infty} G_{\frac{f(2^n x)}{2^n} - T(x)}(t) = 1$. Fix $x, y \in X$ and $t > 0$. It follows from (4.7) that

$$\begin{aligned}
&G_{\frac{f(2^n(x+y))}{2^n} - \frac{f(2^n x)}{2^n} - \frac{f(2^n y)}{2^n}} \left(\frac{t}{4} \right) \\
\geq &F_x \left(\frac{2^n t}{48} \right) * F_y \left(\frac{2^n t}{48} \right)
\end{aligned} \tag{4.10}$$

for all n. Moreover,

$$\begin{aligned}
&G_{T(x+y) - T(x) - T(y)}(t) \\
\geq &G_{T(x+y) - \frac{f(2^n(x+y))}{2^n}} \left(\frac{t}{4} \right) * G_{T(x) - \frac{f(2^n x)}{2^n}} \left(\frac{t}{4} \right) \\
&* G_{T(y) - \frac{f(2^n y)}{2^n}} \left(\frac{t}{4} \right) * G_{\frac{f(2^n(x+y))}{2^n} - \frac{f(2^n x)}{2^n} - \frac{f(2^n y)}{2^n}} \left(\frac{t}{4} \right)
\end{aligned} \tag{4.11}$$

for all n. Letting $n \to \infty$ and $t \to \infty$ in (4.10) and (4.11), we obtain $T(x + y) = T(x) + T(y)$.

Furthermore, using (4.9) with $m = 0$, we see that for large n,

$$\begin{aligned}
G_{T(x) - f(x)}(t) &\geq G_{T(x) - \frac{f(2^n x)}{2^n}} \left(\frac{t}{2} \right) * G_{\frac{f(2^n x)}{2^n} - f(x)} \left(\frac{t}{2} \right) \\
&\geq F_x \left(\frac{(2 - \alpha)t}{48} \right)
\end{aligned} \tag{4.12}$$

For the uniqueness of T, let T' be another additive mapping which satisfies (4.12). We have

$$G_{T(x)-T'(x)}(t) \geq G_{T(x)-\frac{f(2^n x)}{2^n}}\left(\tfrac{t}{2}\right) * G_{\frac{f(2^n x)}{2^n}-T'(x)}\left(\tfrac{t}{2}\right).$$

Therefore $T = T'$. □

Proposition 4.2. *Let $f : X \to Y$ be a P-approximately pexiderized quadratic mapping in Menger PN- space such that f, g and h are even. Also suppose that $F_{2x} = F_x(\frac{t}{|\alpha|})$ for some real number α with $0 < |\alpha| < 4$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$G_{f(x)-Q(x)}(t) \geq F_x\left(\frac{(4-\alpha)t}{96}\right), \qquad \forall x \in X, t > 0. \tag{4.13}$$

Proof. Change the roles of x and y and putting $s = t$ in (4.1) to get

$$G_{f(x+y)+f(x-y)-2g(y)-2h(x)}(2t) \geq F_y(t) * F_x(t). \tag{4.14}$$

Put $y = x$ and $t = s$ in (4.14) to obtain

$$G_{f(2x)-2g(x)-2h(x)}(2t) \geq F_x(t). \tag{4.15}$$

Put $x = 0$ and $t = s$ in (4.1), we get

$$G_{2f(y)-2h(y)}(2t) \geq F_y(t). \tag{4.16}$$

Similarly, putting $y = 0$ and $t = s$ in (4.1) we get

$$G_{2f(x)-2g(x)}(2t) \geq F_x(t). \tag{4.17}$$

Combining (4.14), (4.16) and (4.17) we get

$$\begin{aligned} &G_{f(x+y)+f(x-y)-2f(x)-2f(y)}(6t) \\ \geq &G_{f(x+y)+f(x-y)-2g(y)-2h(x)}(2t) * G_{2f(y)-2h(y)}(2t) \\ &* G_{2f(x)-2g(x)}(2t) \\ \geq &F_x(t) * F_y(t). \end{aligned} \tag{4.18}$$

Setting $y = x$ in (4.18), we have

$$G_{f(2x)-4f(x)}(t) \geq F_x\left(\frac{t}{6}\right). \tag{4.19}$$

It follows from (4.19) that

$$G_{f(2^{n+1}x)-4f(2^n x)}(t) \geq F_{2^n x}\left(\frac{t}{6}\right) = F_x\left(\frac{t}{6\alpha^n}\right). \tag{4.20}$$

By (4.20),

$$G_{\frac{f(2^{n+1}x)}{4^{n+1}}-\frac{f(2^n x)}{4^n}}\left(\frac{\alpha^n t}{4^{n+1}}\right) \geq F_x\left(\frac{t}{6}\right).$$

Therefore for each $n > m \geq 0$,

$$\begin{aligned}
&G_{\frac{f(2^n x)}{4^n} - \frac{f(2^m x)}{4^m}} \left(\sum_{k=m+1}^{n} \frac{\alpha^{k-1} t}{4^k} \right) \\
=&G_{\sum_{k=m+1}^{n} \frac{f(2^k x)}{4^k} - \frac{f(2^{k-1} x)}{4^{k-1}}} \left(\sum_{k=m+1}^{n} \frac{\alpha^{k-1} t}{4^k} \right) \\
\geq &\prod_{k=m+1}^{n} G_{\frac{f(2^k x)}{4^k} - \frac{f(2^{k-1} x)}{4^{k-1}}} \left(\frac{\alpha^{k-1} t}{4^k} \right) \\
\geq &F_x \left(\frac{t}{6} \right).
\end{aligned} \tag{4.21}$$

Let $\varepsilon > 0$ and $t_0 > 0$ be given. Since $\lim_{t\to\infty} F_x(t) = 1$, there is some $t_1 > t_0$ such that $F_x(t_1) > 1 - \varepsilon$. The convergence of the series $\sum_{k=1}^{\infty} \frac{\alpha^{k-1} t_1}{4^k}$ gives some n_0 such that $\sum_{k=m+1}^{n} \frac{\alpha^{k-1} t_1}{4^k} < t_0$ for each $n > m \geq n_0$. It follows that for each $n > m > n_0$,

$$G_{\frac{f(2^n x)}{4^n} - \frac{f(2^m x)}{4^m}}(t_0) \geq G_{\frac{f(2^n x)}{4^n} - \frac{f(2^m x)}{4^m}}(\textstyle\sum_{k=m+1}^{n} \frac{\alpha^{k-1} t_1}{4^k}) \geq F_x(t_1) > 1 - \varepsilon.$$

This shows that $\{\frac{f(2^n x)}{4^n}\}$ is a Cauchy sequence in the Menger PB-space $(Y, G, *)$, therefore it is convergence to some $Q(x)$. So we can define a mapping $Q : X \to Y$ by $\lim_{n\to\infty} G_{\frac{f(2^n x)}{4^n} - Q(x)}(t) = 1$. Fix $x, y \in X$ and $t > 0$. It follows from (4.18) that

$$\begin{aligned}
&G_{\frac{f(2^n(x+y))}{4^n} + \frac{f(2^n(x-y))}{4^n} - 2\frac{f(2^n x)}{4^n} - 2\frac{f(2^n y)}{4^n}} \left(\frac{t}{5} \right) \\
=&G_{f(2^n(x+y)) + f(2^n(x-y)) - 2f(2^n x) - 2f(2^n y)} \left(\frac{4^n t}{5} \right) \\
\geq &F_x \left(\frac{4^n t}{30\alpha^n} \right) * F_y \left(\frac{4^n t}{30\alpha^n} \right)
\end{aligned} \tag{4.22}$$

for all n. Moreover,

$$\begin{aligned}
&G_{Q(x+y)+Q(x-y)-2Q(x)-2Q(y)}(t) \\
\geq &G_{Q(x+y) - \frac{f(2^n(x+y))}{4^n}} \left(\frac{t}{5} \right) * G_{Q(x-y) - \frac{f(2^n(x-y))}{4^n}} \left(\frac{t}{5} \right) \\
&* G_{2Q(x) - 2\frac{f(2^n x)}{4^n}} \left(\frac{t}{5} \right) * G_{2Q(y) - 2\frac{f(2^n y)}{4^n}} \left(\frac{t}{5} \right) \\
&* G_{\frac{f(2^n(x+y))}{4^n} + \frac{f(2^n(x-y))}{4^n} - 2\frac{f(2^n x)}{4^n} - 2\frac{f(2^n y)}{4^n}} \left(\frac{t}{5} \right)
\end{aligned} \tag{4.23}$$

for all n. Since each factor in the right hand side of (4.22) and (4.23) tends to 1 as $n \to \infty$ and $t \to \infty$, one can easily see that

$$Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y).$$

Furthermore, using (4.21) with $m = 0$, we see that for large n,

$$\begin{aligned} G_{Q(x)-f(x)}(t) \geq & G_{Q(x)-\frac{f(2^n x)}{4^n}}\left(\frac{t}{2}\right) * G_{\frac{f(2^n x)}{4^n}-f(x)}\left(\frac{t}{2}\right) \\ \geq & G_{Q(x)-\frac{f(2^n x)}{4^n}}\left(\frac{t}{2}\right) * F_x\left(\frac{(4-\alpha)t}{96}\right) \\ \geq & F_x\left(\frac{(4-\alpha)t}{96}\right). \end{aligned} \tag{4.24}$$

The uniqueness assertion can be proved by a known strategy as in Proposition (4.1). □

Theorem 4.1. *Let $f : X \to Y$ be a P-approximately quadratic mapping. Then there are unique mappings T and Q from X to Y such that T is additive, Q is quadratic and*

$$G_{f(x)-T(x)-Q(x)}(t) \geq F_x\left(\frac{(2-\alpha)t}{96}\right), \quad \forall x \in X,\ t > 0. \tag{4.25}$$

Proof. Passing to the odd part f^o and even part f^e of f we deduce from (4.1) that

$$G_{f^o(x+y)+f^o(x-y)-2f^o(x)-2f^o(y)}(t) \geq F_x(t) * F_y(t),$$

and

$$G_{f^e(x+y)+f^e(x-y)-2f^e(x)-2f^e(y)}(t) \geq F_x(t) * F_y(t).$$

Using the proofs of Propositions (4.1) and (4.2) we get unique additive mapping T and unique quadratic mapping Q satisfying

$$G_{f^o(x)-T(x)}(t) \geq F_x\left(\frac{(2-\alpha)t}{48}\right), \quad \forall x \in X,\ t > 0,$$

also

$$G_{f^e(x)-Q(x)}(t) \geq F_x\left(\frac{(4-\alpha)t}{96}\right), \quad \forall x \in X,\ t > 0.$$

Therefore

$$G_{f(x)-T(x)-Q(x)}(t) \geq G_{f^o(x)-T(x)}\left(\frac{t}{2}\right) * G_{f^e(x)-Q(x)}\left(\frac{t}{2}\right) \geq F_x\left(\frac{(2-\alpha)t}{96}\right).$$

□

5. Conclusion

The study of Menger probabilistic normed spaces was initiated by Alsina, Schweizer and Sklar [1] and continued by others, especially in connection with an important notion in physics, called fractal spacetime theory which was pioneered by Richard Feynman and Garnet Ord as well as Laurent Nottale. The first comprehensive paper published in an international journal with the title Fractal Spacetime was by the English-Canadian Garnet Ord who discussed this subject with Nobel Laureate Richard Feynman and was strongly influenced by Feynmans views on the subject

[17]. A little later and seemingly independently, a young but well known French astrophysicist Laurent Nottale[15]published in 1989 his paper that was a sequel and generalization of his paper in 1984 [16].

A mathematical foundation for E-infinity was given by El Naschie including infinity categories are united in a mathematical theory called Highly structured ring spectrum which deals with multiplicative processes similar to E-infinity and is usually designated in the mathematical literature by E-infinity rings, E-infinity loop algebra and probabilistic normed spaces [14].

Menger spaces were found to be relevant to quantum entanglement: El Naschie in [13] gave two fundamentally different derivations. The first derivation is purely logical and uses a probability theory which combines the discrete with the continuum. The second derivation is purely geometrical and topological using the fundamental equations of a Cantorian spacetime theory.

Penrose fractal tiling is one of the simplest generic examples for a noncommutative space. A quasicrystal, is a structure that is ordered but not periodic. A quasicrystalline pattern can continuously fill all available space. Menger spaces are relevant to quasicristals and Penrose tiling [4].

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