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EXISTENCE AND ORBITAL STABILITY OF PERIODIC WAVE SOLUTIONS FOR THE NONLINEAR SCHRÖDINGER EQUATION*

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Abstract In this paper, we study the existence and orbital stability of periodic wave solutions for the Schrödinger equation. The existence of periodic wave solution is obtained by using the phase portrait analytical technique. The stability approach is based on the theory developed by Angulo for periodic eigenvalue problems. A crucial condition of orbital stability of periodic wave solutions is proved by using qualitative theory of ordinal differential equations. The results presented in this paper improve the previous approach, because the proving approach does not dependent on complete elliptic integral of first kind and second kind.

Keywords Schrödinger equation, orbital stability, periodic wave solution, Picard-Fuchs equattion.

MSC(2000) 35Q51, 35Q53.

1. Introduction

It is well known that the study of nonlinear wave equations and their solutions are of great importance in many areas of physics. The study of periodic patterns for diverse types of nonlinear evolutionary partial differential equations has considerably increased because of the rich variety of new mathematical problems as well as of the many physical relevancy applications.

In the specific case of dispersive evolution equations such as the Korteweg-de Vries equation [1, 2, 3, 14], the BBM equation [15, 16], the nonlinear Schrödinger equations [4, 5, 9] and the Klein-Gordon type equations [24], there are recent results proving the existence and the nonlinear stability/instability of periodic travelling wave. It has been carried out in a general setting where the periodic waves have a profile given exactly by the classical Jacobi elliptic functions (see [1, 2, 3, 14, 15, 16]). Moreover, the required spectral information in the stability/instability theories has been based on the study of the periodic eigenvalue problem associated to the Jacobi

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form of the Lamé's equation, expressed as

$$\begin{cases} \dfrac{d^2}{d\xi^2}\Psi + [\rho - n(n+1)k^2 sn^2(x;k)]\Psi = 0, \\ \Psi(0) = \Psi(2K(k)), \quad \Psi'(0) = \Psi'(2K(k)), \end{cases} \tag{1.1}$$

with $n \in N, sn(x;k)$ being the Jacobi elliptic function with modulus $k, k \in (0,1)$ and K representing the complete elliptic integral of the first kind defined by

$$K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}. \tag{1.2}$$

In this paper, we consider the existence and stability properties of periodic traveling waves solutions of the nonlinear Schrödinger equation

$$iu_t + u_{xx} + |u|^2 u = 0. \tag{1.3}$$

For obtaining the orbital stability result we use the current techniques developed in Grillakis, Shatah and Strauss [10, 11], Angulo and Natali [2, 5] combined with the recently theory developed by Hakkaev, Iliev and Kirchev in [14, 15]. This last two theories give us sufficient conditions in order to obtain a specific spectral structure of the operator

$$L_{dn} = -\frac{d^2}{dx^2} + \omega - 3\varphi_\omega^2. \tag{1.4}$$

We have the following set of conditions which guarantee the stability of φ_ω in $H^1_{per}([0,T])$(see [1, 5]):

(H1) there is a non-trivial smooth curve of periodic solutions for (1.3) of the form: $\omega \in I \subseteq R \to \varphi_\omega \in H^1_{per}([0,T])$;
(H2) L_{dn} has a unique negative eigenvalue, which is simple;
(H3) the eigenvalue 0 is simple;
(H4) $\frac{d}{d\omega}\int_0^T \varphi_\omega^2(x)dx > 0$.

For the condition (H4), Angulo proved that it holds by using the complete elliptic integral. Here, we give a different proving method based on theory developed by Hakkaev, Iliev and Kirchev in [14, 15].

The paper is organized as follows. In Section 2, we discuss the existence of periodic travelling wave solutions. In Section 3, we prove orbital stability of periodic travelling wave solutions of Eq. (1.3). A short conclusion is given in Section 4.

2. Existence of periodic travelling wave solutions

In this section, we establish the existence of some smooth curves of periodic travelling wave solutions to the Schrödinger equation (1.3) of the form

$$u(x,t) = e^{i\omega t}\varphi_\omega(x). \tag{2.1}$$

In order to obtain the existence results of periodic wave solutions, we apply the phase portrait analytical technique [17, 18, 26, 25, 12, 13, 8, 7] to Eq. (1.3). Substituting (2.1) into Eq. (1.3) and setting $\varphi = \varphi_\omega$, we obtain

$$\varphi'' + \varphi^3 - \omega\varphi = 0. \tag{2.2}$$

Eq. (2.2) is equivalent to the Hamiltonian system

$$\begin{cases} \dfrac{d\varphi}{dx} = y, \\ \dfrac{d\varphi}{dx} = \omega\varphi - \varphi^3, \end{cases} \tag{2.3}$$

with the Hamiltonian

$$H(\varphi, y) = \frac{y^2}{2} - \frac{\omega}{2}\varphi^2 + \frac{1}{4}\varphi^4 = h, \tag{2.4}$$

where h is is an integral constant.

Clearly, system (2.3) has three singular points under the condition $\omega > 0$. Denote them by $O(0,0), P_1(\sqrt{\omega}, 0))$ and $P_2(-\sqrt{\omega}, 0))$, respectively. By the qualitative theory of ordinary differential equations, we know that $O(0,0)$ is a saddle point and $P_1(\sqrt{\omega}, 0))$ and $P_2(-\sqrt{\omega}, 0))$ are two centers.

For a fixed $h \in R$, the curve

$$\Gamma_h = \{(\varphi, y) \in R \times R : H(\varphi, y) = h\} \tag{2.5}$$

is called a level curve with the energy level h. Obviously, each orbit of Eq. (2.3) is a branch of certain energy curve. For convenience, we call the orbit as the orbit with the energy level h. To facilitate further analysis, we investigate the relation between the bounded orbit of Eq. (2.3) and the energy level h. Put

$$U_h(\varphi) = 2h + \omega\varphi^2 - \frac{1}{2}\varphi^4. \tag{2.6}$$

It is easy to obtain the three extreme points of $U_h(\varphi)$ as follows:

$$\varphi_0 = 0, \qquad \varphi_\pm = \pm\sqrt{\omega}. \tag{2.7}$$

Let $h_c = H(\sqrt{\omega}, 0) = -\frac{\omega^2}{4}$, then we can easily draw the graphics of the function $U_h(\varphi)$ in Fig. 1(1-1). We can see from Fig. 1(1-1) that for each given h:

(i) If $h_c < h < 0$, then $U_h(\varphi) = 0$ has four different real roots. Denote them by $\pm\varphi_1, \pm\varphi_2$ with $0 < \varphi_2 < \varphi_1$;
(ii) If $h = 0$, then $U_h(\varphi) = 0$ has a zero root and two different nonzero real roots which are denoted by $\pm\varphi_3$ with $\varphi_3 > 0$;
(iii) If $h > 0$, then $U_h(\varphi) = 0$ has two different real roots which are denoted by $\pm\varphi_4$ with $\varphi_4 > 0$.

The energy curves Γ_h are equivalent to the curves defined by $y^2 = U_h(\varphi)$. According to the above arguments, we obtain the following results (see Fig. 1(1-2)).

(1) System (2.3) does not have any bounded orbit with energy level h satisfying $h \le h_c$;
(2) System (2.3) has two families of periodic orbits $\Gamma_h = \{(\varphi, y) : H(\varphi, y) = h, h_c < h < 0\}$ which lie in the inside of the different bounded regions determined by two homoclinic orbits ;
(3) System (2.3) has two homoclinic orbits with energy level 0;

(4) System (2.3) has a family of periodic orbits $\Gamma_h = \{(\varphi, y) : H(\varphi, y) = h, h > 0\}$ which lie in the outside of the bounded region determined by two homoclinic orbits.

Corresponding to periodic orbits, we have the Newton's equation

$$y^2 = U(\varphi) = 2h + \omega\varphi^2 - \frac{1}{2}\varphi^4. \tag{2.8}$$

If $h \in (h_c, 0)$, then $U(\varphi) = \frac{1}{2}(\varphi_1^2 - \varphi^2)(\varphi^2 - \varphi_2^2), \varphi_1 > \varphi_2 > 0$, we obtain the smooth periodic travelling wave solutions

$$\varphi(x) = \varphi_1 dn(\nu x, k), \tag{2.9}$$

where

$$\nu = \frac{\sqrt{2}}{2}\varphi_1, \quad k^2 = \frac{\varphi_1^2 - \varphi_2^2}{\varphi_1^2}.$$

Now, since elliptic function dn has fundamental period $2K$, where $K = K(k)$ represents the complete elliptic integral of first kind, it follows that the dnoidal wave solution φ_ω in (2.2) has fundamental period T, given by

$$T = \frac{2\sqrt{2}}{\varphi_1}K(k). \tag{2.10}$$

The expression for the dimensionless wavelength (2.10) can be manipulated into an easy form, and so we can obtain basic information about the behavior of the fundamental period T. In fact, from (2.5) we get that given a fixed $\omega > 0$, then $0 < \varphi_2 < \sqrt{\omega} < \varphi_2 < \sqrt{2\omega}$ and we can see (2.10) as a function of a unique variable φ_2, namely,

$$T = \frac{2\sqrt{2}}{\sqrt{2\omega - \varphi_2^2}}K(k(\varphi_2)) \ \ with \ \ k^2(\varphi_2) = \frac{2\omega - 2\varphi_2^2}{2\omega - \varphi_2^2}. \tag{2.11}$$

If $\varphi_2 \to 0$ then $k(\varphi_2) \to 1^-$, and so $K(k(\varphi_2)) \to +\infty$. Therefore, $T(\varphi_2) \to +\infty$ as $\varphi_2 \to 0$. Now, if $\varphi_2 \to \sqrt{\omega}$ then $k(\varphi_2) \to 0$, and so $K(k(\varphi_2)) \to \frac{\pi}{2}$. Therefore, $T(\varphi_2) \to \frac{\sqrt{2}\pi}{\sqrt{\omega}}$ as $\varphi_2 \to \sqrt{\omega}$. Finally, since $\varphi_2 \to T(\varphi_2)$ is a strictly decreasing function (see [21]), we have $T > \frac{\sqrt{2}\pi}{\sqrt{\omega}}$. Considering φ_2 tends to zero, $\varphi_1^2 \to 2\omega$ and $k \to 1^-$, then the dnoidal wave loses its periodicity in this limit and we obtain a solitary wave with "infinity period" of the form

$$\varphi(x) = \sqrt{2\omega} sech(\sqrt{\omega}x), \tag{2.12}$$

which corresponds the homoclinic orbit in $(\varphi, y)-$phase plane.

Next we will construct, for a fixed period T, a smooth curve of dnoidal wave solutions for Eq. (2.2). We start by showing the existence of a family of dnoidal waves with a fixed period. In fact, it considers $T > 0$ arbitrary but fixed, and $\omega > 0$ such that $\sqrt{\omega} > \frac{\sqrt{2}\pi}{T}$. We know that the mapping $\varphi_2 \in (0, \sqrt{\omega}) \to T(\varphi_2)$ is strictly decreasing, so by (2.11) there is a unique $\varphi_2 = \varphi_2(\omega) \in (0, \sqrt{\omega})$ such that the fundamental period of the dnoidal wave will be $T(\varphi_2(\omega)) = T$. Note that in this case the modulus k can be seen as a function of ω, since by (2.11) we have

$$k^2(\omega) = \frac{2\omega - 2\varphi_2^2}{2\omega - \varphi_2^2}, \tag{2.13}$$

with $\varphi_2 = \varphi_2(\omega)$. By the analysis made above we have the following theorem.

Theorem 2.1. *[1] Let $T > 0$ arbitrary but fixed. Consider $\omega_0 > \frac{2\pi^2}{T^2}$ and the unique $\varphi = \varphi(\omega_0) \in (0, \sqrt{\omega_0})$ such that $T_{\varphi_{\omega_0}} = T$. Then*

(1) there exist an interval $I(\omega_0)$ around ω_0, an interval $B(\varphi_2)$ around $\varphi_2(\omega_0)$, and a unique smooth function $f : I(\omega_0) \to B((\varphi_2)$ such that $f(\omega_0) = \varphi_2$ and

$$T = \frac{2\sqrt{2}}{2\omega - \varphi_2^2} K(k), \tag{2.14}$$

where $\omega \in I(\omega_0), \varphi_2 = f(\omega)$, and $k^2 = k^2(\omega) \in (0,1)$ is defined by (2.13);

(2) the dnoidal wave solution in (2.9) has fundamental period T and satisfies (2.2). Moreover, the mapping

$$\omega \in I(\omega_0) \to \varphi_\omega \in H^1_{per}([0,T]) \tag{2.15}$$

is a smooth function;

(3) $I(\omega_0)$ can be chosen as $(\frac{2\pi^2}{T^2}, +\infty)$.

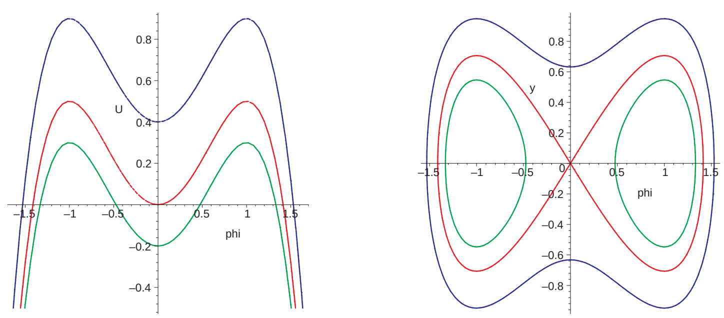

Fig. 1. The graphs of function $U_h(\varphi)$ and phase portrait of system (2.3).

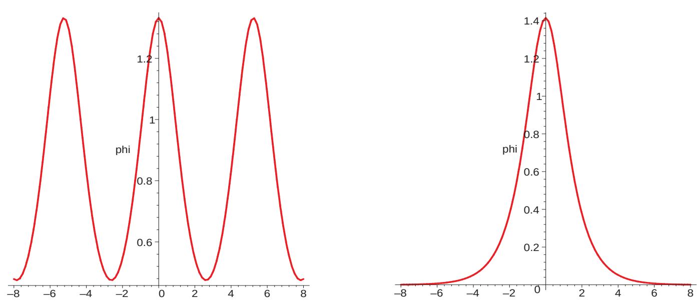

Fig. 2. The profiles of waves.(2-1) periodic wave, (2-2) solitary wave

3. Orbital stability of periodic wave solutions

We now turn to our main stability problem. Take $\omega \in R$ and denote by $e^{i\theta}\varphi_\omega(x)$ the periodic wave solution of Eq. (1.3).

Definition 3.1. The periodic wave solution $u(x,t) = e^{i\theta}\varphi_\omega(x)$ is said to be orbital stable, if for every $\epsilon > 0$ there is $\delta > 0$ such that if

$$\|u_0 - e^{i\theta}\varphi_\omega(\cdot + r)\|_1 < \delta,$$

then the solution $u(x,t)$ of (1.3) with initial data u_0 satisfies that

$$\|u(t) - e^{i\theta}\varphi_\omega(\cdot + r)\|_1 < \epsilon$$

for all $t \in R$.

We now discuss the spectral properties of the linear operator L_{dn} defined by (1.4).

Theorem 3.1. *The linear operator L_{dn} defined by (1.4) has the following spectral properties for $\omega > 0$:*
(i) L_{dn} has a unique negative simple eigenvalue;
(ii) The second eigenvalue of L_{dn} is $\lambda_2 = 0$, which is simple.

Proof. Let $\xi = \nu x$, then using (2.9) we get

$$\begin{aligned} L_{dn} &= -\frac{d^2}{dx^2} + \omega - 3\varphi_1^2 dn^2(\nu x, k) \\ &= -\nu^2\frac{d^2}{d\xi^2} + \omega - 3\varphi_1^2 dn^2(\xi, k) \\ &= \nu^2[-\frac{d^2}{d\xi^2} - 6dn^2(\xi,k) + 2 - k^2]. \end{aligned}$$

Let $\Lambda = -\frac{d^2}{d\xi^2} - 6dn^2(\xi,k)$, then the first three simple eigenvalues of operator Λ in the interval $[0, 2K(k)]$ are (see [19])

$$\mu_1 = 2k^2 - 4 - 2\sqrt{1 - k^2 + k^4}, \tag{3.1}$$
$$\mu_2 = k^2 - 2, \tag{3.2}$$
$$\mu_3 = 2k^2 - 4 + 2\sqrt{1 - k^2 + k^4}. \tag{3.3}$$

Thus the first three simple eigenvalues of operator L_{dn} in the interval $[0, 2K(k)]$ are

$$\lambda_1 = \nu^2(k^2 - 2 - 2\sqrt{1 - k^2 + k^4}), \tag{3.4}$$
$$\lambda_2 = 0, \tag{3.5}$$
$$\lambda_3 = \nu^2(k^2 - 2 + 2\sqrt{1 - k^2 + k^4}). \tag{3.6}$$

It is easy to see that $\lambda_1 < 0, \lambda_2 = 0$ and $\lambda_3 > 0$. □

We next consider the condition (H4).

Theorem 3.2. *Let $d(\omega) = \int_0^T \varphi_\omega^2(x)dx$, then $d'(\omega) > 0$ for $\omega > 0$.*

To prove theorem 3.2, we first give some useful lemmas based on qualitative theory of ordinal differential equations [20, 21, 22, 23, 27, 28].

Lemma 3.1. *Denote $I_n(h) = \oint_{H=h} \varphi^n y dx$, then*

$$d'(\omega) = \frac{1}{2I_0''}[I_4'' I_0'' - (I_2'')^2]. \tag{3.7}$$

Proof. For $T \in I$ and $\varphi_2 \in B$, denote

$$G(\omega, \varphi_2) = \overline{G}(\omega, h) = T(\varphi_2) - T = \oint_{H=h} \frac{dx}{y} - T.$$

Then

$$\frac{\partial \overline{G}}{\partial h}\dot{h} + \frac{\partial \overline{G}}{\partial \omega} = 0. \tag{3.8}$$

From $H(\varphi, y) = h$, we obtain

$$y\frac{dy}{dh} = 1, \qquad y\frac{dy}{d\omega} = \frac{1}{2}\varphi^2. \tag{3.9}$$

Eq. (3.8) gives

$$I_0''\dot{h} + \frac{1}{2}I_2'' = 0, \tag{3.10}$$

where we use $I_n'(h) = \oint_{H=h} \frac{\varphi^n}{y} dx$. The function $d(\omega)$ can been write in the form

$$d(\omega) = \int_0^T \varphi_\omega^2(x)dx = \oint_{H=h} \frac{\varphi^2 d\varphi}{y}. \tag{3.11}$$

Thus we have

$$\begin{aligned} d'(\omega) &= \dot{h}\frac{d}{dh}\oint_{H=h}\frac{\varphi^2 d\varphi}{y} + \frac{d}{d\omega}\oint_{H=h}\frac{\varphi^2 d\varphi}{y} \\ &= \dot{h}\frac{d}{dh}\oint_{H=h}\frac{\varphi^2 d\varphi}{y} + \frac{d}{dh}\oint_{H=h}\frac{\varphi^4 d\varphi}{2y} \\ &= \frac{1}{2I_0''}[I_4''I_0'' - (I_2'')^2]. \end{aligned}$$

□

Lemma 3.2. *If we set $J(h) = column(I_0(h), I_2(h))$, then the integrals I_0 and I_2 satisfy the Picard-Fuchs system*

$$\Delta(h)J'(h) = \begin{pmatrix} a_{11}(h) & a_{12}(h) \\ a_{21}(h) & a_{22}(h) \end{pmatrix} J(h), \tag{3.12}$$

where

$$\Delta(h) = 4h(4h + \omega^2)$$

and

$$a_{11}(h) = 4(3h + \omega^2), \quad a_{12}(h) = -5\omega, \quad a_{21}(h) = -4h\omega, \quad a_{22}(h) = 20h.$$

Proof. Along the curve Γ_h we have $y^2 = 2h + \omega\varphi^2 - \frac{1}{2}\varphi^4$. Therefore

$$\begin{aligned} I_n(h) &= \oint \varphi^n y d\varphi && (3.13)\\ &= \oint \frac{\varphi^n}{y}(2h + \omega\varphi^2 - \frac{1}{2}\varphi^4)d\varphi && (3.14)\\ &= 2hI_n' + \omega I_{n+2}' - \frac{1}{2}I_{n+4}'. && (3.15) \end{aligned}$$

On the other hand, by integrating by parts we have

$$\begin{aligned} I_n(h) &= \oint \varphi^n y d\varphi = \frac{1}{n+1}\oint y d\varphi^{n+1} = -\frac{1}{n+1}\oint \varphi^{n+1} dy && (3.16)\\ &= -\frac{1}{n+1}\oint \frac{1}{y}(\omega\varphi^{n+2} - \varphi^{n+4}) d\varphi && (3.17)\\ &= -\frac{\omega}{n+1} I'_{n+2} + \frac{1}{n+1} I'_{n+4}. && (3.18)\end{aligned}$$

Eliminating I'_{n+4} from Eqs. (3.13)-(3.18) we find that

$$(n+3)I_n(h) = 4hI'_n + \omega I'_{n+2}. \tag{3.19}$$

Taking $n = 0, 2$ in (3.19), we have

$$3I_0 = 4hI'_0 + \omega I'_2, \tag{3.20}$$

$$5I_2 = 4hI'_2 + \omega I'_4. \tag{3.21}$$

Taking $n = 0$ in (3.15), we have

$$I_0 = 2hI'_0 + \omega I'_2 - \frac{1}{2} I'_4. \tag{3.22}$$

Eliminating I'_4 from Eqs. (3.20)-(3.22) we obtain the Picard-Fuchs system (3.12). □

By differentiating Eq. (3.12) with respect to h, we have the following lemma.

Lemma 3.3. *The integrals I_0 and I_2 satisfy the following relations*

$$\Delta(h)J''(h) = \begin{pmatrix} b_{11}(h) & b_{12}(h) \\ b_{21}(h) & b_{22}(h) \end{pmatrix} J'(h), \tag{3.23}$$

where

$$b_{11}(h) = -4h, \quad b_{12}(h) = -\omega, \quad b_{21}(h) = -4h\omega, \quad b_{22}(h) = 4h.$$

Lemma 3.4. *The ratio $R(h) = \frac{I'_2}{I'_0}$ satisfy the Riccati equation*

$$4h(4h + \omega^2)R' = \omega R^2 + 8hR - 4h\omega. \tag{3.24}$$

Proof. Using (3.12) and (3.23), we have

$$\begin{aligned} R'(h) &= \frac{I''_2 I'_0 - I''_0 I'_2}{(I'_0)^2} && (3.25)\\ &= \frac{I''_2}{I'_0} - \frac{I''_0}{I'_0} R && (3.26)\\ &= \frac{\omega}{4h(4h+\omega^2)} R^2 + \frac{2}{4h+\omega^2} R - \frac{\omega}{4h+\omega^2}. && (3.27)\end{aligned}$$

□

Proof of theorem 3.2. Using (3.7) and (3.23), we have

$$\begin{aligned} d'(\omega) &= \frac{1}{2I_0''}[I_4''I_0'' - (I_2'')^2] && (3.28)\\ &= \frac{1}{6I_0}[4h(I_0')^2 + (I_2')^2] && (3.29)\\ &= \frac{(I_0')^2}{6I_0}F(R,h), && (3.30)\end{aligned}$$

where

$$F(R,h) = R^2 + 4h. \tag{3.31}$$

For the Riccati equation (3.24), we denote it in the form

$$\begin{cases} \dfrac{dh}{ds} = 4h(4h+\omega^2), \\ \dfrac{dR}{ds} = \omega R^2 + 8hR - 4h\omega. \end{cases} \tag{3.32}$$

Since $I_0 > 0$, the sign $d'(\omega)$ is determined by $F(R,h)$. The curve $L : F(R,h) = 0$ divides the $(h,R)-$plane into two parts $F_+(F>0)$ and $F_-(F<0)$ according to the sign of $F(R,h)$. We have to determine location of the curve Γ_R with respect to F_+ and F_-.

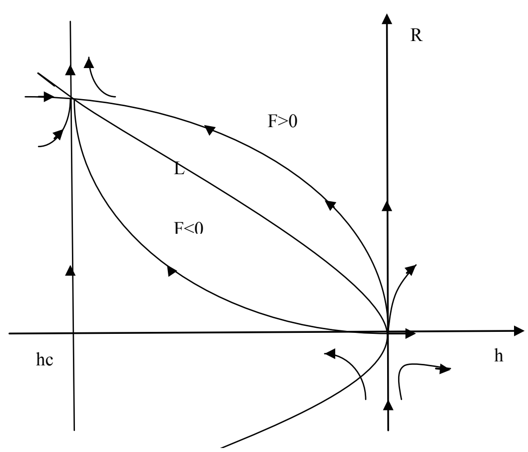

Fig. 3. Phase portrait of system (3.32) and parabola L.

It is easy to see that system (3.32) has two singular points $A(-\frac{\omega^2}{4},\omega)$ and $O(0,0)$. For singular point $O(0,0)$, system (3.32) has the following Jacobian matrix

$$M = \begin{pmatrix} 4\omega^2 & 0 \\ -4\omega & 0 \end{pmatrix}. \tag{3.33}$$

Because $D = detM = 0$ and $E = trM = 4\omega^2 \neq 0$, $O(0,0)$ is degenerate singular point. From $\omega R^2 + 8hR - 4h\omega = 0$, we obtain

$$h = \frac{\omega R^2}{4\omega - 8R}. \tag{3.34}$$

Using Eq. (3.34), we get

$$Q(R, h(R)) = 4h(4h + \omega^2) = \frac{\omega^2 R^2 (R - \omega)^2}{(2R - \omega)^2}. \tag{3.35}$$

Therefore $O(0,0)$ is a saddle-node(see [6]). Similarly, we can know that $A(-\frac{\omega^2}{4}, \omega)$ is also saddle-node. The two singular points are connected by separatrix trajectory Γ_R and Γ_R^d. The phase portrait of system (3.32) is shown in Fig. 3. As we know, the ratio $R(h)$ is analytic in a neighbourhood of $h = h_c$ and, by the mean value theorem

$$\lim_{h \to h_c} R(h) = \lim_{h \to h_c} \frac{\xi^2 \int_{\varphi_1(h)}^{\varphi_2(h)} \frac{d\varphi}{y}}{\int_{\varphi_1(h)}^{\varphi_2(h)} \frac{d\varphi}{y}} = \omega, \tag{3.36}$$

where we use the fact that $\varphi_1 < \xi < \varphi_2$ with $\varphi_i \to \sqrt{\omega}(i = 1,2)$ as $h \to h_c$. For $h \in (h_c, 0)$, then $4h + \omega^2 > 0$ and $\Delta^* = 16h(4h + \omega^2) < 0$, thus $\omega R^2 + 8hR - 4h\omega > 0$. We further get

$$\frac{dR}{dh} = \frac{\omega R^2 + 8hR - 4h\omega}{4h(4h + \omega^2)} < 0, \tag{3.37}$$

therefore Γ_R is decreasing in $(h_c, 0)$. It is easy to see that $R'(h_c) = 0$ for Γ_R and $R'(h_c) = -\frac{2}{\omega}$ for L. This yields that for h close to h_c, Γ_R is placed above L.

Below, we proceed to determine the number of contact points that $L : R = r(h)$ has with the vector field (3.32). Because of the type of critical points, in our case the number of intersections is less than or equal to the number of contact points. As is well known, the equation of the contact points is given by

$$\frac{d}{ds}(R - r(h)) \mid_{R=r(h)} = \dot{R} - \dot{h} r'(h) \mid_{R=r(h)} = -8h\omega - 4\omega^2 \sqrt{-h} = 0. \tag{3.38}$$

Eq. (3.38) holds if and only if $h = -\frac{\omega^2}{4}$. As a result, Γ_R does not intersect L in $h \in (h_c, 0)$ therefor $\Gamma_R \in F_+$. Theorem 3.2 is completely proved. □

Using Theorem 2.1, Theorem 3.1 and Theorem 3.2, we obtain the main result.

Theorem 3.3. *The equation (1.3) has two families of periodic wave solutions for $\omega > 0$ and $h \in (h_c, 0)$, which are orbital stable.*

4. Conclusion and discussion

In this paper,we study the existence and orbital stability of periodic wave solutions for the Schrödinger equation. The existence of periodic wave solution is obtained by using the phase portrait analytical technique. The stability approach is based on the theory developed by Angulo for periodic eigenvalue problems. A crucial condition of orbital stability of periodic wave solutions is proved by using qualitative theory of ordinal differential equations.

In this paper, we only consider the orbital stability of periodic waves of Eq. (1.3) for $h \in (h_c, 0)$. In this case, relative periodic orbits lie in the inside of two homoclinic orbits. Now, a natural problem is how we study orbital stability of periodic waves of Eq. (1.3) for $h \in (0, +\infty)$. Obviously, this is really a question which may make us think deeply.

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