Journal of Applied Analysis and Computation
Volume 2, Number 1, February 2012
Website:http://jaac-online.com/
pp. 91–102

DYNAMIC PROPERTIES OF THE OREGONATOR MODEL WITH DELAY*

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Abstract Delayed feedbacks are quite common in many physical and biological systems and in particular many physiological systems. Delay can cause a stable system to become unstable and vice versa. One of the well-studied non-biological chemical oscillators is the Belousov-Zhabotinsky(BZ) reaction. This paper presents an investigation of stability and Hopf bifurcation of the Oregonator model with delay. We analyze the stability of the equilibrium by using linear stability method. When the eigenvalues of the characteristic equation associated with the linear part are pure imaginary, we obtain the corresponding delay value. We find that stability of the steady state changes when the delay passes through the critical value. Then, we calculate the explicit formulae for determining the direction of the Hopf bifurcation and the stability of these periodic solutions bifurcating from the steady states, by using the normal form theory and the center manifold theorem. Finally, numerical simulations results are given to support the theoretical predictions by using Matlab and DDE-Biftool.

Keywords Delay, Oregonator model, stability, Hopf bifurcation

MSC(2000) 34C23, 37G15.

1. Introduction

Feedback plays a very crucial role in many physical, chemical and biological systems. Dynamics of oscillators with delayed feedback is of considerable practical and theoretical interest [5]. Among all the chemical reactions that show nonlinear dynamics, the BelousovZhabotinsky (BZ) system has received the most extensive attention since its discovery by Belousov in 1951 [1], and has remained a prototype for nonlinear chemical systems. The improved Oregonator model described by the following system (see [2]):

$$\begin{cases} \frac{dX}{dt} = k_3AY - k_2XY + k_5AX - 2k_4X^2, \\ \frac{dY}{dt} = k_3AY - k_2XY + \frac{1}{2}fk_0BZ, \\ \frac{dZ}{dt} = 2k_5AX - k_0BZ, \end{cases} \tag{1}$$

here $x = [HBrO_2]$,$y = [Br]$,[z $=$ [Ce(IV)] and its dimensionless form is

$$\begin{cases} \varepsilon\frac{dx}{dt} = qy - xy + x(1-x), & (2) \\ \delta\frac{dy}{dt} = -qy - xy + fz, & (3) \\ \frac{dz}{dt} = x - z, & (4) \end{cases} \tag{1b}$$

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*This work was supported by the National Natural Science Foundations of China (No 10871056).

where t is dimensionless time $\varepsilon, \delta, q, f$ are constants while f is an adjustable parameter [7].Because δ much less than ε , so the left side of equation (3) can be considered equals 0, equation (3) can be reduced to $y \approx \frac{fz}{q+x}$.

Brought $y \approx \frac{fz}{q+x}$ into equation (2) and equation (4) we can get 2-D simplified model

$$\begin{cases} \varepsilon\frac{dx}{dt} = \frac{qfz}{q+x} - \frac{fxz}{q+x} + x(1-x), \\ \frac{dz}{dt} = x - z. \end{cases} \tag{5}$$

From literature [8], we can know the topological equivalence of 3-D Oregonator to its 2-D simplified model, which testifies that the investigation of 2-D model for the 3-D Oregonator is valid. When electric current is applied, the catalyst $Ce(IV)$ is perturbed and other species are not affected [4]. Consequently, in modeling, the perturbation term is introduced only in equation(4), which now becomes

$$\frac{dz}{dt} = x - z + kz(t-\tau). \tag{6}$$

Then the purpose of this paper is to consider Oregonator model with a delay:

$$\begin{cases} \varepsilon\frac{dx}{dt} = \frac{qfz}{q+x} - \frac{fxz}{q+x} + x(1-x), \\ \frac{dz}{dt} = x - z + kz(t-\tau), \end{cases} \tag{7}$$

where ε=$4 \times 10^{-2}\delta$= $4 \times 10^{-4} q = 8 \times 10^{-4}, f \in (0,1)$ is an adjustable parameter.

The remainder of this paper organized as follows. In the next section, we shall consider the stability and the local Hopf bifurcation. In Section 3, based on the normal form method and the center manifold reduction introduced by Hassard et al.[3], we derive the formulae determining the direction, stability and the period of the bifurcating periodic solution at the critical value of τ, a conclusion is drawn in this section. To verify the theoretic analysis, numerical simulations are given in section 4.

2. Stability and local Hopf bifurcations

Through out the paper, we assume that $k < 1$.Let (x^*, z^*) be an equilibrium point of system (7). Then there is a unique $x^* > 0$,$z^* > 0$ satisfying

$$\begin{cases} \frac{qfz^*}{q+z^*} - \frac{fx^*z^*}{q+z} + x^*(1-x^*) = 0, \\ x^* - z^* + kz^* = 0, \end{cases}$$

where

$$\begin{cases} x^* = \frac{1-\frac{f}{1-k}-q+\sqrt{(1-\frac{f}{1-k}-q)^2+4q(1+\frac{f}{1-k})}}{2}, \\ z^* = \frac{x^*}{1-k}. \end{cases}$$

Resulting in $\frac{x^*}{z^*} = 1-k > 0$, we have $k < 1$. Let $x = x - x^*, z = z - z^*$. Then we can rewrite (7) as the following equivalent system

$$\begin{cases} \frac{dx}{dt} = \frac{1}{\varepsilon}(\frac{qf(z+z^*)}{q+(x+x^*)} - \frac{f(x+x*)(z+z^*)}{q+(x+x^*)} + (x+x^*)(1-x-x^*)), \\ \frac{dz}{dt} = x - z + kz(t-\tau). \end{cases} \tag{8}$$

The linearization of system (8) at $(0,0)$ is

$$\begin{cases} \frac{dx}{dt} = a_1 x + a_2 z, \\ \frac{dz}{dt} = x - z + kz(t-\tau). \end{cases} \tag{9}$$

Moreover, its corresponding characteristic equation is

$$\begin{vmatrix} \lambda - a_1 & -a_2 \\ -1 & \lambda + 1 - ke^{-\lambda\tau} \end{vmatrix} = \lambda^2 - b_1\lambda - b_2 - k\lambda e^{-\lambda\tau} + ka_1 e^{-\lambda\tau} = 0, \tag{9}$$

where $a_1 = \frac{1}{\varepsilon}(\frac{-2qfz^*}{(q+x^*)^2} + 1 - 2x^*), a_2 = \frac{1}{\varepsilon}\frac{qf - fx^*}{q+x^*}, b_1 = a_1 - 1, b_2 = a_1 + a_2.$

In this section, we first study the distribution of roots of Eq.(10). We first introduce the following important result, which was been proved by Ruan and Wei [6] using Rouche theorem.
For $\tau = 0$, the two roots of (10) have negative real parts if and only $k + b_1 < 0, ka_1 - b_2 > 0$.
We impose the following condition:
(A1)$ka_1 > b_2$, $k < -b_1$.

Lemma 2.1. *Let $\tau = 0$. Then if (A1) is satisfied, all the roots of Eq. (10) have negative real parts, and hence (x^*, z^*) is asymptotically stable.*

Next, we mainly focus on the case of $\tau > 0$. If $\lambda = iw_0 (w_0 > 0)$ is a purely imaginary root of Eq. (10) for $\tau > 0$, then we have

$$-{\omega_0}^2 - b_1 i\omega_0 - b_2 - ki\omega_0 e^{-i\omega_0\tau} + ka_1 e^{-i\omega_0\tau} = 0.$$

Separating the real and imaginary parts, we obtain

$$\begin{cases} -{\omega_0}^2 - b_2 - k\omega_0 \sin\omega_0\tau + ka_1 \cos\omega_0\tau = 0, \\ -b_1\omega_0 - k\omega_0 \cos\omega_0\tau - ka_1 \sin\omega_0\tau = 0, \end{cases} \tag{11}$$

which implies

$$w_0^4 + (2b_2 + b_1^2 - k^2)w_0^2 + b_2^2 - k^2 a_1^2 = 0. \tag{12}$$

Let $z = w^2$ and denote

$$u = 2b_2 + b_1^2 - k^2, r = b_2^2 - k^2 a_1^2.$$

Then, Eq. (12) becomes

$$z^2 + uz + r = 0. \tag{13}$$

In order to seek a positive solution for Eq. (13), we impose the following condition:

$(B1) r < 0$.

Clearly, under the condition $(B1)$, Eq. (13) has a unique positive root $z = \frac{-u + \sqrt{u^2 - 4r}}{2}$.

$(B2) r > 0, u > 0$.

Under the condition $(B2)$, Eq. (13) has no positive root.

$(B3) r > 0, u < 0$.

Under the condition $(B3)$, if there are real positive roots, then $|k|$ is very large, f infinitely close to one, does not match with the actual situation. Summarizing the above discussions, we obtain the following:

Lemma 2.2. *For the polynomial Eq. (13), we have the following result.*
(i)If $r < 0$, then Eq. (13) *has a unique positive root* $z = \frac{-u+\sqrt{u^2-4r}}{2}$.
(ii)If $r < 0$,then Eq. (13) has no positive root.

Suppose that Eq. (13) has positive roots. Without loss of generality, we assume that it has a positive root defined by z.Then, Eq. (12) has a positive root ω_0, moreover ω_0 must satisfies the following equations.

$$\left(\frac{w_0^2 + a_1 b_2}{k(w^2 + a_1^2)}\right)^2 + \left(\frac{w_0^3 + w_0(a_1^2 + a_2)}{kw_0^2 + ka_1^2}\right)^2 = 1.$$

By (11), we have

$$\cos(w_0\tau) = \frac{w_0^2 + a_1 b_2}{k(w^2 + a_1^2)}, \sin(w_0\tau) = -\frac{w_0^3 + w_0(a_1^2 + a_2)}{kw_0^2 + ka_1^2}.$$

Thus, denoting

$$a = -\frac{w_0^3 + w_0(a_1^2 + a_2)}{kw_0^2 + ka_1^2}, b = \frac{w_0^2 + a_1 b_2}{k(w^2 + a_1^2)},$$

$$\tau_j = \begin{cases} \frac{1}{w_0}(\arccos b + 2j\pi), & a \geq 0, \\ \frac{1}{w_0}(2\pi - \arccos b + 2j\pi), & a < 0, \end{cases}$$

where $j = 0, 1, 2, \cdots$,then $\pm iw_0$ is a pair of purely imaginary roots of Eq. (10) with $\tau = \tau_j$.
Note that when $\tau = 0$, Eq. (10) becomes

$$\lambda^2 - (k + b_1)\lambda + ka_1 - b_2 = 0. \tag{14}$$

Using Lemmas 2.1 and 2.2, we have the following results.

Lemma 2.3. *For the exponential polynomial equation Eq.* (10)*, we have*
(i)If $r > 0$ and the condition (A1) is satisfied, then all roots with positive real parts of Eq. (10) *has the same sum as those of the polynomial Eq. (14) for all* $r > 0$.
(ii)If $r < 0$ then all roots with positive real parts of Eq. (10) *has the same sum as those of the polynomial Eq.* (14) *for* $\tau \in [0, \tau_0)$.

Let $\lambda(\tau) = a(\tau) + iw(\tau)$ be the root of Eq.(7) near $\tau = \tau_j$ satisfying $a(\tau_j) = 0, w(\tau_j) = w_0$.Then, the following transversality condition holds.

Lemma 2.4. *Suppose that $r < 0$, then* $\frac{\mathrm{Re}\{\lambda(\tau_j)\}}{d\tau} > 0$.

Proof. Substituting $\lambda(\tau)$ into Eq.(10) and differentiating the resulting equation in τ, we obtain

$$(2\lambda - b_1\lambda + \tau k\lambda e^{-\lambda\tau} - ke^{-\lambda\tau} - \tau a_1\lambda e^{-\lambda\tau})\frac{d\lambda}{d\tau} = \lambda k a_1 e^{-\lambda\tau} - k\lambda^2 e^{-\lambda\tau}.$$

Thus,

$$\begin{aligned}\left(\frac{d\lambda}{d\tau}\right)^{-1} &= \frac{2\lambda - b_1\lambda + \tau k\lambda e^{-\lambda\tau} - ke^{-\lambda\tau} - \tau a\lambda e^{-\lambda\tau}}{\lambda k a e^{-\lambda\tau} - k\lambda^2 e^{-\lambda\tau}} \\ &= \frac{\tau(k - a_1)}{k(a_1 - \lambda)} - \frac{1}{\lambda(a_1 - \lambda)} + \frac{(2 - b_1)e^{\lambda\tau}}{k(a_1 - \lambda)},\end{aligned}$$

we can easily obtain

$$\left(\frac{d(\text{Re}\lambda(\tau_j))}{d\tau}\right)^{-1}_{\tau=\tau_j} = \text{Re}\left\{\frac{\tau(k-a_1)}{k(a_1-\lambda)} - \frac{1}{\lambda(a_1-\lambda)} + \frac{(2-b_1)e^{\lambda\tau}}{k(a_1-\lambda)}\right\}_{\tau=\tau_j}$$

$$= \frac{1}{w_0^2+a_1^2} + \frac{3w_0^2+2b_2-a_1w_0^2+\tau k^2a_1-\tau ka_1^2}{k^2(a_1^2+w_0^2)}.$$

For $a_1 < 0$, we have $k < 0$.In the previous part of this paper we know $|k|$ can't be very large .As mentioned above, it can be obtained that

$$sign\left[\frac{1}{w_0^2+a_1^2} + \frac{3w_0^2+2b_2-a_1w_0^2+\tau k^2a_1-\tau ka_1^2}{k^2(a_1^2+w_0^2)}\right] > 0.$$

For $a_1 < 0$, we have $0 < k < 1-2f$ and $a_1 < 2$. It can be obtained that

$$sign\left[\frac{1}{w_0^2+a_1^2} + \frac{3w_0^2+2b_2-a_1w_0^2+\tau k^2a_1-\tau ka_1^2}{k^2(a_1^2+w_0^2)}\right] > 0.$$

Therefore, the transversality condition holds and Hopf-bifurcation occurs at $\tau = \tau_j$. □

From Lemmas 2.4 and 2.5, we have the following:

Theorem 2.1. (1)*If* $r > 0$ *and the condition* $(A1)$ *is satisfied, then the zero solution of system* (7) *is asymptotically stable for all* $\tau > 0$.
(2)*If* $r < 0$, *then the zero solution of system* (7) *is asymptotically stable for* $\tau \in (0, \tau_0)$, *unstable for* $\tau > \tau_0$. *The system* (7) *undergoes a Hopf bifurcation at the zero solution when* $\tau = \tau_j (j = 0, 1, 2 \cdots)$.

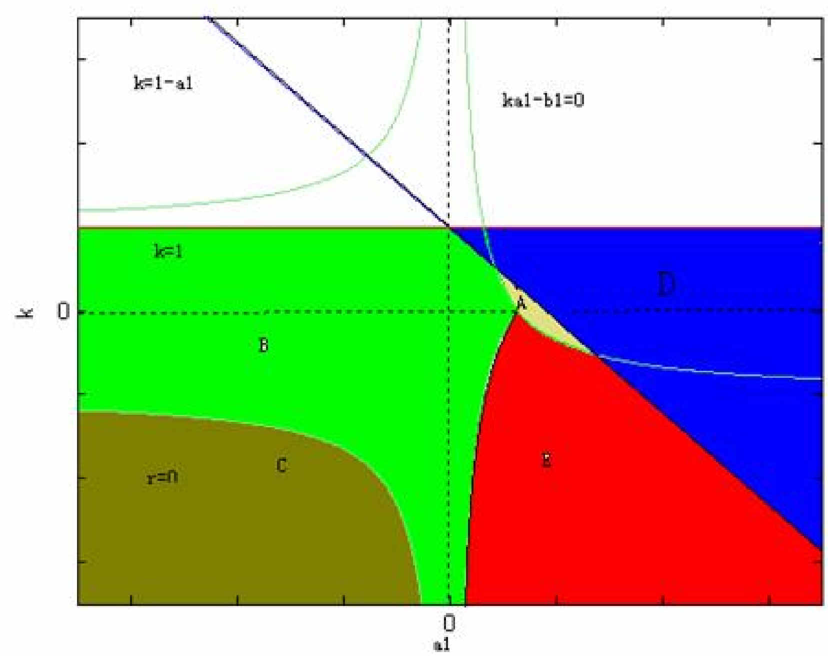


Fig.1.The line $k = 1, k + b_1 = 0, ka_1 - b_2 = 0$ and $r = 0$ divide the a_1, k plane

into five regions. D and E is the unstable region. A and C are the stable region. B is the conditional stable region.

3. Direction and stability of the Hopf bifurcation

In the previous section, we have obtained some conditions to ensure that the system (7) undergoes a single Hopf bifurcation at the origin (x^*, z^*) when $\tau = \tau_0$ passes through certain critical values. In this section, we shall study the direction, stability, and the period of the bifurcating periodic solutions. The method we used is based on the normal form method and the center manifold theory introduced by Hassard et al. [3].

We first re-scale the time by $t \mapsto t/\tau$, to normalize the delay so that system (8) can be written as

$$\begin{cases} \frac{dx}{dt} = \frac{\tau}{\varepsilon}\left(\frac{qf(z+z^*)}{q+(x+x^*)} - \frac{f(x+x*)(z+z^*)}{q+(x+x^*)} + (x+x^*)(1-x-x^*)\right), \\ \frac{dz}{dt} = \tau x - \tau z + \tau k z(t-1). \end{cases} \tag{15}$$

Letting $\tau{=}\tau_0 + a(a \in R)$, then $a = 0$ is Hopf bifurcation value of Eq. (15), Eq. (15) can be rewritten as:

$$\begin{cases} \frac{dx}{dt} = (\tau_0 + a)(a_1 x + a_2 z + M), \\ \frac{dz}{dt} = (\tau_0 + a)(x - z + kz(t-1)), \end{cases}$$

where $M = \frac{1}{\varepsilon}\left[\frac{fz^*(q-x^*+1)}{(q+x^*)^2}\right]x^2 + \frac{1}{\varepsilon}\frac{-2qf}{(q+x^*)}xz.$

Select the phase space $C = C([-1,0], R^2)$. For any $\phi{=}\ (\phi_1, \varphi_2)^{\mathrm{T}} \in C$, setting

$$L_a(\phi) = (\tau_0 + a)\begin{pmatrix} a_1 & a_2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} + (\tau_0 + a)\begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix}\begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \end{pmatrix}$$

$$= (\tau_0 + a)A\phi(0) + (\tau_0 + a)B\phi(-1),$$

and

$$f(a, \phi)\underline{\underline{def}}(\tau_0 + a)\begin{pmatrix} M \\ 0 \end{pmatrix},$$

By the Riesz representation theorem, there exists a function $\eta(\theta, \mu)(0 \le \theta \le 1)$,whose elements are of bounded variation such that

$$L_a\phi = \int_{-1}^{0} d\eta(\theta, a)\phi(\theta), \qquad \phi \in C.$$

In fact, we choose

$$\eta(\theta, \alpha) = (\tau + \alpha)A\delta(\theta) + (\tau + \alpha)B\delta(\theta + 1),$$

where δ is dened by

$$\delta(\theta){=}\begin{cases} 1, \theta{=}0, \\ 0, \theta \ne 0. \end{cases}$$

For $\phi \in C^1([-1,0], R^2)$, define

$$A(a)\phi = \begin{cases} d\phi(\theta)/d\theta, & \theta \in [-1,0), \\ \int_{-1}^{0} d\eta(t,a)\phi(t), & \theta = 0, \end{cases}$$

and

$$R(a)\phi = \begin{cases} 0, \theta \in [-1, 0), \\ f(a, \phi), \theta = 0. \end{cases}$$

Then, system (11) is equivalent to the following operator equation

$$\dot{u_t} = A(\alpha)u_t + R(\alpha)u_t, \tag{16}$$

where $u_t = u(t + \theta)(\theta \in ([-1, 0])$.

For $\psi \in C^1([0, 1], (R^2)^*)$, define

$$A^*\psi(s) = \begin{cases} -d\psi(s)/ds, & s \in (0, 1], \\ \int_{-1}^{0} d\eta^T(s, a)\phi(-s), & s = 0, \end{cases}$$

and a bilinear form

$$< \psi(s), \phi(\theta) >= \overline{\psi}(0)\phi(0) - \int_{\theta=1}^{0} \int_{\xi=0}^{\theta} \overline{\psi}(\xi - \theta)d\eta(\theta)\phi(\xi)d\xi,$$

where $\eta(\theta) = \eta(\theta, 0)$, then $A(0)$ and A^* are adjoint operators. Setting $q(\theta)$ and $q^*(s)$ is the eigenvector of $A(0)$ and A^* corresponding to $i\tau_0 w_0$ and $-i\tau_0 w_0$.By direct calculation we have

$$q(\theta) = (\tfrac{a_2}{iw_0+a_1}, 1)^T e^{iw_0\tau_0\theta},$$
$$q^*(s) = D(\tfrac{1}{iw_0+a_1}, 1)e^{iw_0\tau_0 s}.$$

where

$$D = -\frac{{w_0}^2 + {a_1}^2}{a_2(2ke^{-iw_0\tau_0} - 2ke^{iw_0\tau_0})}.$$

Then $\langle q^*, q\rangle = 1$ and $\langle q^*, \overline{q}\rangle = 0$.

In the following, we follow the ideas in Hassard et al.[14] and by using the same notations as there to compute the coordinates describing the center manifold C_0 at $a = 0$. Let u_t be the solution of (16) when $a = 0$. Dene $z(t) =< q^*, u_t >$, $W(t, \theta) = u_t(\theta) - 2\mathrm{Re}\{z(t)q(\theta)\}$.

On the center manifold C_0 we have $W(t, \theta) = W(z(t), \bar{z}(t), \theta)$, where

$$W(z, \bar{z}, \theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{z^2}{2} + W_{30}\frac{z^3}{6} + \cdots,$$

z and $\overline{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\overline{q^*}$. Note that W is real if u_t is real .We only consider real solutions. For solution $u_t \in C_0$ of (16), since $a = 0$, we have

$$\begin{aligned} z'(t) &= \mathrm{i}w_0 z + \langle q^*(\theta),\ \tilde{F}(W + 2\mathrm{Re}\{z(t)q(\theta)\})\rangle \\ &\overset{\mathrm{def}}{=\!=} \mathrm{i}w_0 + \bar{q}^*(0)\tilde{F}_0(z,\ \bar{z}). \end{aligned}$$

We rewrite this equation as

$$z'(t) = \mathrm{i}w_0 z(t) + g(z,\ \bar{z}), \tag{17}$$

with

$$g(z, \bar{z}) = \bar{q}^*(0)\tilde{F}(W(z, \bar{z}, 0) + 2\mathrm{Re}\{z(t)q(0)\})$$

$$= g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + ... \tag{18}$$

It follows from (16) and (17) that

$$W' = u_t' - z'q - \bar{z}'\bar{q} = \begin{cases} AW - 2\text{Re}\{\bar{q}*(0)\tilde{F}_0 q(\theta)\}, & \theta \in [-1,0), \\ AW - 2\text{Re}\{\bar{q}*(0)\tilde{F}_0 q(\theta)\} + \tilde{\text{F}}_0\,, & \theta = 0. \end{cases}$$

Comparing of coefficients we have:

$$\begin{aligned}
g_{20} &= 2q_1{}^*(0)\cdot \tfrac{1}{\varepsilon}\tfrac{fz^*(q-x^*+1)}{(q+x^*)^2},\\
g_{11} &= 2q_1{}^*(0)\cdot \tfrac{1}{\varepsilon}\tfrac{-2qf}{(q+x^*)},\\
g_{02} &= 0,\\
g_{21} &= 2q_1{}^*(0)*(\tfrac{1}{\varepsilon}\tfrac{fz^*(q-x^*+1)}{(q+x^*)^2}(\tfrac{2a_1a_2}{a_1{}^2+w_0^2}w_{11}^1(0) + \tfrac{2a_1a_2}{a_1{}^2+w_0^2}w_{20}^1(0))\\
&\quad + \tfrac{1}{\varepsilon}\tfrac{-2qf}{(q+x^*)}(\tfrac{1}{2}w_{20}^1 + \tfrac{1}{2}\tfrac{2a_1a_2}{a_1{}^2+w_0^2}w_{20}^2(0) + \tfrac{2a_1a_2}{a_1{}^2+w_0^2}w_{11}^2(0)),
\end{aligned}$$

where

$$\begin{aligned}
q_1{}^*(0) &= \tfrac{1}{a_1+w_0},\\
W_{20}(\theta) &= -\tfrac{g_{20}}{i\omega_0}q(0)e^{i\omega_0\theta} - \tfrac{\bar{g}_{20}}{3i\omega_0}\bar{q}(0)e^{-i\omega_0\theta} + E_1e^{2i\omega_0\theta},\\
W_{11}(\theta) &= \tfrac{g_{11}}{i\omega_0}q(0)e^{i\omega_0\theta} - \tfrac{\bar{g}_{11}}{i\omega_0}\bar{q}(0)e^{-i\omega_0\theta} + E_2.
\end{aligned}$$

Moreover E_1 and E_2 satisfies the following equations, respectively

$$\begin{pmatrix} 2iw_0 - a_1 & -a_2 \\ -1 & 2iw_0 + 1 - ke^{-2iw_0} \end{pmatrix} E_1 = \begin{pmatrix} \frac{1}{\varepsilon}\frac{fz^*(q-x^*+1)}{(q+x^*)^2} \\ 0 \end{pmatrix},$$
$$\begin{pmatrix} -a_1 & -a_2 \\ -1 & 1-k \end{pmatrix} E_2 = \begin{pmatrix} \frac{1}{\varepsilon}\frac{-2qf}{(q+x^*)} \\ 0 \end{pmatrix}.$$

Then we can compute the following quantities:

$$\begin{aligned}
c_1(0) &= \tfrac{i}{2\omega_0}(g_{20}g_{11} - 2|g_{11}|^2 - \tfrac{1}{3}|g_{02}|^2) + \tfrac{g_{21}}{2},\\
v_2 &= -\tfrac{\text{Re}c_1(0)}{\text{Re}\lambda'(\tau_0)},\\
T_2 &= -\tfrac{Im\{c_1(0)\}+\mu_2 Im\{\lambda'(\tau_0)\}}{\omega_0},\\
\beta_2 &= 2\text{Re}c_1(0).
\end{aligned} \tag{18}$$

Hence we have the following theorem by the result of Hassard et al.[3].

Theorem 3.1. *In* (19)*, the sign of* v_2 *determined the direction of Hopf bifurcation: if* $v_2 > 0 (v_2 < 0)$*, then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solution exist for* $\tau > \tau_0 (\tau < \tau_0)$*.* β_2 *determined the stability of the bifurcating periodic solution: the bifurcating periodic solution is stable (unstable) if* $\beta_2 < 0 (\beta_2 > 0)$*, and* T_2 *determines the period of the bifurcating periodic solution: the period increase (decrease) if* $T_2 > 0 (T_2 < 0)$*.*

4. Numerical simulations

In order to check our computation for Theorem 3.1, we perform some numerical simulations. We choose parameters as follows: $q = 8 \times 10^{-4}, f = 2/3, k = -2.5$, for the above parameters, we can calculate $\tau_0 \approx 1.67$.

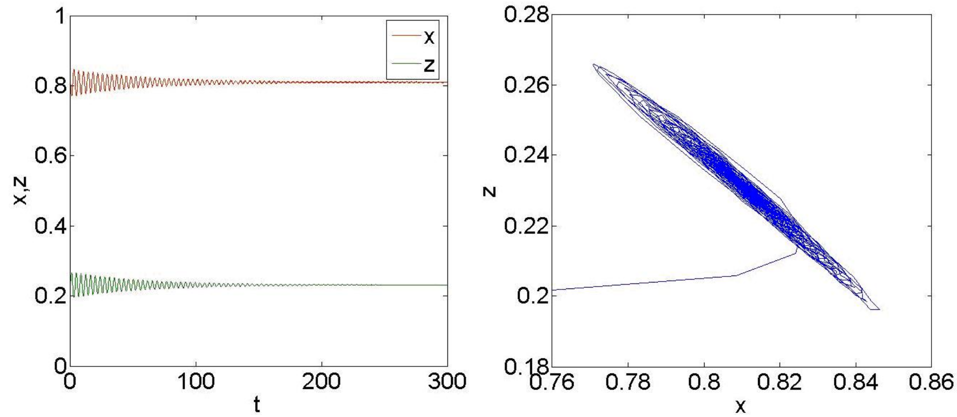


Fig.2. For the following parameter values,$q = 8 \times 10^{-4}, f = 2/3, k = -2.5$, and $\tau = 1.5 < \tau_0$, system solution curve and X-Y Phase diagram.
Figure 2 shows that the equilibrium point of the system is asymptotically stable.

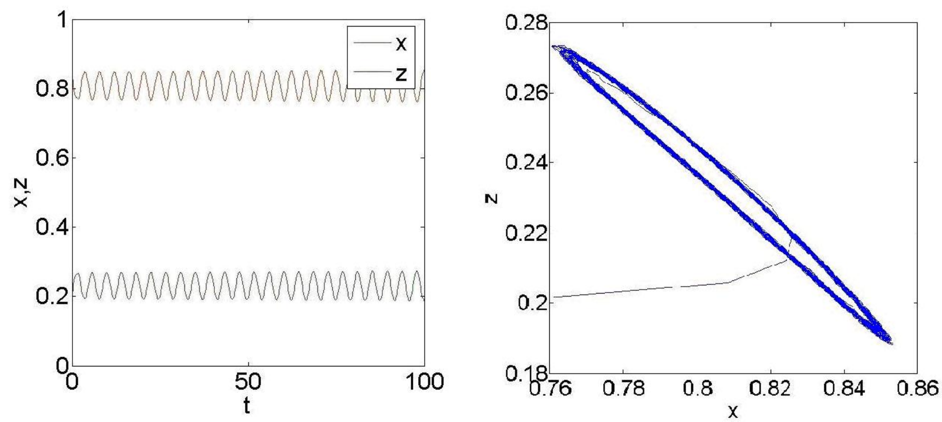


Fig.3. For the following parameter values, $q = 8 \times 10^{-4}, f = 2/3, k = -2.5$, and $\tau = \tau = 1.7 > \tau_0$, system solution curve and X-Y Phase diagram.
Figure 3 shows there exists a periodic orbit which is orbitally asymptotically stable when $\tau > \tau_0$ and is close to τ_0.

5. Conclusions

For an Oregonator model with delay, an important issue is how delays change the stability of Oregonator model states, steady or oscillatory, causing further oscillations or significantly altering existing ones and hence inducing delay-controlled periodic behavior. In this paper, experimental and numerical investigations on the effect of electrical feedback in the oscillating BelousovZhabotinsky reaction are s-tudied. By studying the distribution of the roots in the characteristic equations associated with their corresponding linearizations, we have provided the bifurcation sets in the appropriate parameter plane. From the bifurcation sets, the conclusion can be drawn that in some regions (region of asymptotic stability and unstable region). However, in the region of conditional stability, the stability changes when the sum of delays reaches a certain critical value. In addition, using the normal form theory and center manifold reduction, the stability and bifurcating direction of periodic solutions are determined.

From a chemical viewpoint, both means that time delay could cause a stable equilibrium to become unstable and cause the properties in an Oregonator model to fluctuate: if $\tau < \tau_0$, the density of various elements reach an equilibrium. If τ increases and crosses the value τ_0, then this equilibrium becomes unstable: the density of various elements oscillates around the unstable equilibrium.

Appendix

In this Appendix, we employ the algorithm of DDE-Biftool to analysis the model. Obtained directly from the solution of Eq. (12). We have a positive root

$$\omega_0=\sqrt{\frac{-(2b_2+b_1^2-k^2)+\sqrt{(2b_2+b_1^2-k^2)^2-4(b_2^2-k^2a_1^2)}}{2}}.$$

We choose a set of parameters as follows: $q = 8 \times 10^{-4}, f = 2/3, k = -2$, we have $\tau_0 \approx 2.36$ Select the initial amount of delay $\tau = 3$.

stst.parameter= [0.0008 2/3 -2 0.04 3]

stst =

kind: 'stst'

parameter: $[8.0000e - 0040.6667 - 20.04003]$

x: $[2x1double]$

stability: [ ]

First, draw the picture of the root of the characteristic equation in the equilibrium point. In figure 4, we can see that near the imaginary axis there are at least a pair of characteristic roots is real.

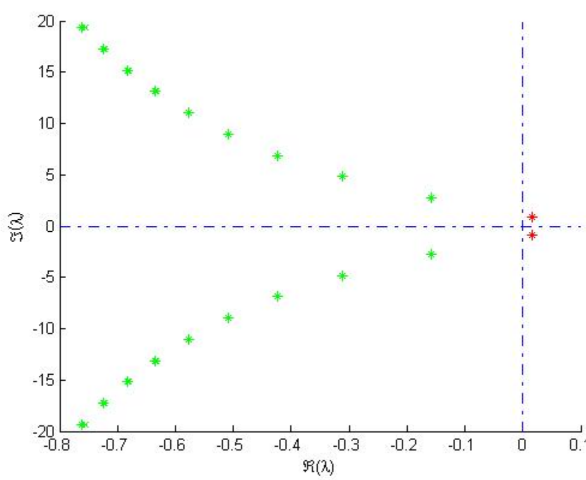
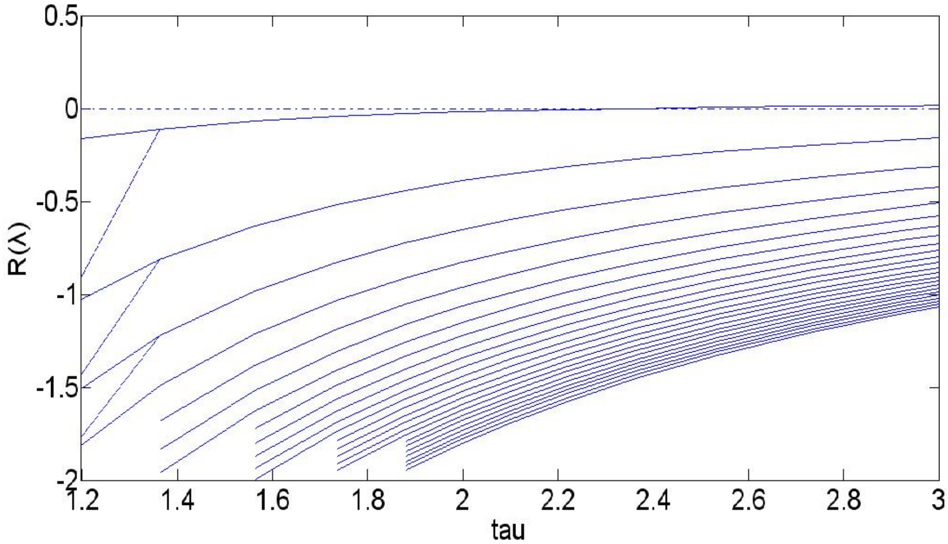


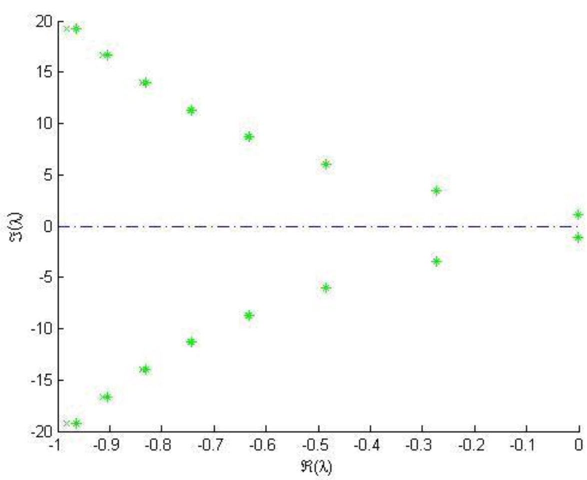
*Fig*4

By adjusting the parameter values, correcting the characteristic roots to find the Hopf bifurcation values.



$Fig5$

By correction, we find the Hopf bifurcation poin $\tau = 2.3640$
branch1.point(9)
stst =
kind: 'stst'
parameter: $[8.0000e - 0040.6667 - 20.04003]$
x: $[2x1double]$
stability: [ ]
When $\tau = 2.3640$ there is a pair of pure imaginary roots appear in Figure 5.



$Fig6$

We can see $\tau \approx \tau_0$, so we verify the previous theoretical.

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