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PULLBACK EXPONENTIAL ATTRACTORS FOR THE VISCOUS CAHN-HILLIARD EQUATION IN BOUNDED DOMAINS

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Abstract This work is devoted to the construction of a pullback exponential attractor for a viscous Cahn-Hilliard system in bounded domains. Our construction is based on the results obtained by Langa, Miranville and Real in [7].

Keywords Dissipative dynamical system, viscous Cahn-Hilliard equation, pullback attractor, pullback exponential attractor.

MSC(2000) 35D, 35C.

1. Introduction

Attractor's theory is very important to describe the long time behavior of dissipative dynamical systems generated by evolution equations which model physical phenomena. Moreover, there are several kinds of attractors, each one depending on the type of problem studied.

In this article, we will focus on the pullback exponential attractors which are time dependent compact sets, with finite fractal dimension, which are positively invariant and exponentially attract in the pullback sense every bounded set of the phase space.

We will study these attractors for the non-autonomous viscous Cahn-Hilliard equation in $\mathbb{R}^n$, $n = 1, 2, 3$. In this case (non-autonomous case), the solutions strongly depend on two time variables: the final time t and the initial time τ. The (viscous) Cahn-Hilliard equation is very important in materials science: it models the transport of atoms between units cells. It has been proposed and studied in, e.g., [1], [5], [6], [9] and [10].

This equation is written in the following form:

$$\frac{\partial u}{\partial t} = \Delta K(u),$$

$$K(u) = \varepsilon \frac{\partial u}{\partial t} - \Delta u + f(u) + (\Delta)^{-1} m,$$

where $0 \le \varepsilon < 1$. For $\varepsilon = 0$, this equation reduces to the non-autonomous Cahn-Hilliard model introduced in [2]. Thus, the viscous Cahn-Hilliard equation includes certain viscous effects neglected in [2].

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In this article, we construct, in Section 5, pullback exponential attractors for the viscous Cahn-Hilliard system, which necessitates several technical estimates that are obtained in Section 3 and 4.

Indeed, based on the construction of [7], we consider hypotheses similar to those in this reference and then, we construct these attractors for the viscous Cahn-Hilliard problem (see also [4] where a first example of a pullback exponential attractor is given). Under these hypotheses we lose any kind of forward attraction, i.e., we obtain a pullback (and not necessarily forwards) exponential attractor which contains the pullback attractor which is a compact set, is invariant and satisfies a pullback attraction property.

2. Setting of the problem

We consider the following viscous Cahn-Hilliard system:

$$\begin{cases} \dfrac{\partial}{\partial t}\big(u+\varepsilon(-\Delta)u\big)+\Delta^2 u-\Delta f(u)=m(t), \quad x\in\Omega, \\ \dfrac{\partial u}{\partial n}\Big|_{\Gamma}=\dfrac{\partial \Delta u}{\partial n}\Big|_{\Gamma}=0, \\ u|_{t=\tau}=u_0, \end{cases} \tag{2.1}$$

where $\Omega\subset\mathbb{R}^3$ is a bounded domain, u is an unknown function, f is a given function, $m(t)$ is a given external force field, $\varepsilon\geq 0$ is a small parameter, $\tau\in\mathbb{R}$ is a given initial time and u_0 is the initial velocity field.

We assume in this paper that the average value of m is null, i.e., $\displaystyle\int_\Omega m dx=0$. Thus, integrating (2.1) over Ω, we have

$$\frac{d\langle u\rangle}{dt}=0,$$

where $\langle u\rangle=\dfrac{1}{|\Omega|}\displaystyle\int_\Omega u dx$ denotes the spatial average. Thus, we deduce that

$$\langle u(t)\rangle=\langle u_0\rangle,\ \forall\ t\geq 0 \ \ \text{(conservation of mass)}.$$

We rewrite equation (2.1) in the following equivalent form:

$$\frac{\partial}{\partial t}\big((-\Delta)^{-1}+\varepsilon\big)\bar{u}-\Delta u+f(u)=\langle f(u)\rangle+(-\Delta)^{-1}m, \tag{2.2}$$

$$\frac{\partial u}{\partial n}\Big|_{\Gamma}=0, \quad u(\tau)=u_0,$$

where the operator $(-\Delta)^{-1}$ is associated with Neumann boundary conditions and $\bar{u}=u-\langle u\rangle$.

Furthermore, we assume that the nonlinear term $f(u)$ is a polynomial of arbitrary odd degree with strictly positive leading coefficient:

$$f(u)=\sum_{j=1}^{2p-1}a_j u^j,\ p\in\ \mathbb{N},\ p\geq 2,\ \text{where } a_{2p-1}=2pb_{2p}>0.$$

Thus, this fuction satisfies the following conditions (for more details see [12]):

$$\begin{cases} f'(u) \geq -c, \\ \exists c_1 > 0 : \ f(u)u \geq pb_{2p}u^{2p} - c_1, \quad \forall u \in \mathbb{R}, \\ \forall \alpha > 0, \ \exists c_2 = c_2(\alpha) : \ |f(u)| \leq \alpha b_{2p} u^{2p} + c_2(\alpha), \quad \forall u \in \mathbb{R}. \end{cases}$$

We consider the following usual spaces:

$H = L^2(\Omega)$, with inner product $(\cdot, \cdot)$ and associated norm $|\cdot|$,

$V = H^1(\Omega)$, with scalar product $((\cdot, \cdot))$ and associated norm $\|\cdot\|$,

$V_1 = \{v \in H^2(\Omega) : \ \frac{\partial v}{\partial n}|_\Gamma = 0\}$ with norm $\|\cdot\|_2$.

Finally, $\|\cdot\|_{H^{-1}}$ stands for the norm in $V' = H^{-1}(\Omega)$. Note that $\|\cdot\|_{H^{-1}} = \|\cdot\|_{-1} := |(-\Delta)^{-\frac{1}{2}} \cdot|$ for the fuctions with null average.

It follows that $V_1 \subset V \subset H \subset V'$, with dense and compact embeddings. Finally, we introduce the space $E(\varepsilon)$ defined by the norm

$$\|u\|_E := \Big(\|u\|^2_{H^{-1}} + \varepsilon |u|^2\Big)^{\frac{1}{2}}, \quad \text{if } \langle u \rangle = 0,$$

and

$$\|u\|_E := \Big(\|\bar{u}\|_E + (1+\varepsilon)\langle u \rangle^2\Big)^{\frac{1}{2}} = \Big(\|\bar{u}\|^2_{-1} + \langle u \rangle^2 + \varepsilon(|\bar{u}|^2 + \langle u \rangle^2)\Big)^{\frac{1}{2}},$$

if $\langle u \rangle =$ const.

3. Uniform a priori estimates

We have the following existence and uniqueness result:

Theorem 3.1. *If $m \in L^2_{loc}(\mathbb{R}; V')$, then, for any $\tau \in \mathbb{R}$ and all $u_0 \in V$, there exists a unique weak solution $u(t) = u(t, \tau; u_0)$ of (2.1). Moreover, this solution satisfies*

$$u \in L^\infty(\tau, T; V) \cap L^2(\tau, T; V_1) \cap L^{2p}(\tau, T; L^{2p}(\Omega)) \quad \textit{for all } T > \tau.$$

Furthermore, if $u_0 \in V_1$ then, $u \in L^\infty(\tau, T; V_1) \cap L^2(\tau, T; H^3(\Omega))$ and $\partial_t u \in L^2(\tau, T; V)$ for all $T > \tau$, where $p = 2$ when $n = 3$.

Finally, if $u_0 \in H^3(\Omega)$ then, $u \in L^\infty(\tau, T; H^3(\Omega)) \cap L^2(\tau, T; H^4(\Omega))$ for all $T > \tau$, where $p = 2$ when $n = 3$.

Using standard techniques we can prove global existence and uniqueness of the solution (see e.g., [11]); we will not develop this classical aspect here and we will just derive the a priori estimates for the solution.

We set, in what follows,

$$U_m(t, \tau)u_0 := u(t; \tau, u_0), \quad \tau \leq t, \quad u_0 \in E(\varepsilon).$$

It is clear that U_m is a process on $E(\varepsilon)$ from Theorem 3.1.

From now on, we assume that m and m', m' denoting the derivative of m with respect to t, satisfie the three following assumptions:

(A1). The functions $m, m' \in L^2_{loc}(\mathbb{R}; E(\varepsilon))$, $\int_t^{t+1} |m|^2 dx = 0$, $\int_t^{t+1} |m'|^2 dx = 0$ and the average value of m' is null, $\int_\Omega m' dx = 0$.

(A2). $M_m(t) := \sup_{r \le t} \int_{r-1}^{r} \|m(s)\|_E^2 ds < \infty$ for all $t \in \mathbb{R}$.

(A3). There exist $t_0 \in \mathbb{R}$ and $q > 2$ such that

$$M_{m,q}(t_0) := \sup_{r \le t_0} \int_{r-1}^{r} \|m(s)\|_E^q ds < \infty.$$

Lemma 3.1. *Let $D \subset E(\varepsilon)$ be a bounded subset. Then,*

$$\|U_m(t,\tau)u_0\|_E \le C_m(t_0), \tag{3.1}$$

for any $t \le t_0$, $\tau \le t - \frac{2}{c} log\big(C_1\|D\|_E\big)$, where $u_0 \in D$ and $C_m(t_0)$ is a positive constant that only depends on m, t_0, Ω and is independent of $\varepsilon \ge 0$.

Proof. Let $u(t) = u(t; \tau, u_0)$. Multiplying equation (2.2) by $\bar{u}$ and integrating over Ω,

$$\frac{1}{2}\frac{d}{dt}\|\bar{u}\|_E^2 + |\nabla u|^2 + (f(u), u) \le \Big|\langle u\rangle \int_\Omega f(u)dx\Big| + \|m\|_{-1}\|\bar{u}\|_{-1}.$$

Using the inequalities

$$f(u)u \ge pb_{2p}u^{2p} - c_1, \quad \|\bar{u}\|_{-1}^2 \le c|\nabla u|^2 \;\text{ and } |\bar{u}|^2 \le c_2|\nabla u|^2,$$

we find

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\bar{u}\|_E^2 + c\|\bar{u}\|_E^2 + \frac{1}{2}pb_{2p}\int_\Omega u^{2p}dx \quad &\le \quad \frac{1}{2}k_1 + \frac{c}{2}\|\bar{u}\|_{-1}^2 + \frac{1}{2c}\|m\|_{-1}^2 \\
&\le \quad \frac{1}{2}k_1 + \frac{c}{2}\|\bar{u}\|_E^2 + \frac{1}{2c}\|m\|_E^2.
\end{aligned}$$

In particular,

$$\frac{d}{dt}\|\bar{u}\|_E^2 + c\|\bar{u}\|_E^2 \le k_1 + \frac{1}{c}\|m\|_E^2.$$

We have $\dfrac{d\langle u\rangle^2}{dt} = 0$. We deduce from the last inequality

$$\frac{d}{dt}\Big(\|\bar{u}\|_E^2 + (1+\varepsilon)\langle u\rangle^2\Big) + c\Big(\|\bar{u}\|_E^2 + (1+\varepsilon)\langle u\rangle^2\Big) \le k_1' + \frac{1}{c}\|m\|_E^2, \tag{3.2}$$

for some strictly positive constants c, k_1', which are independent of $\varepsilon \ge 0$.

Applying now Gronwall's inequality to estimate (3.2), we obtain, for all $t \ge \tau$,

$$\|u(t)\|_E^2 \le e^{-c(t-\tau)}\|u_0\|_E^2 + k_2 + \frac{1}{c}e^{-ct}\int_\tau^t e^{cs}\|m(s)\|_E^2 ds. \tag{3.3}$$

We have

$$\begin{aligned}
e^{-ct}\int_\tau^t e^{cs}\|m(s)\|_E^2 ds &\leq e^{-ct}\int_{-\infty}^t e^{cs}\|m(s)\|_E^2 ds \\
&= e^{-ct}\sum_{n=0}^{\infty} e^{c(t-n)}\int_{t-(n+1)}^{t-n}\|m(s)\|_E^2 ds \\
&\leq (1-e^{-ct})^{-1}M_m(t) \\
&\leq (1+c^{-1})M_m(t).
\end{aligned}$$

We deduce from (3.3) that

$$\|u(t)\|_E^2 \leq e^{-c(t-\tau)}\|u_0\|_E^2 + c^{-1}(1+c^{-1})M_m(t) + k_2 \tag{3.4}$$

for all $t \geq \tau$.

We set, for any bounded subset $D \subset E(\varepsilon)$,

$$\|D\|_E := \max\Big(1, \sup_{v\in D}\|v\|_E\Big).$$

By (3.4), we deduce that

$$\|u(t;\tau,u_0)\|_E^2 \leq 1 + c^{-1}(1+c^{-1})M_m(t_0) \tag{3.5}$$

for all $t \leq t_0,\ \tau \leq t - \frac{2}{c}\log\big(C_1\|D\|_E\big),\ u_0 \in D$.

In the last inequality, τ is obtained by

$$e^{-c(t-\tau)}\|D\|_E^2 + k_2 \leq 1,$$

thus, $\tau \leq t - \frac{2}{c}\log\big(C_1\|D\|_E\big)$, where $C_1 = k_2$, and Lemma 3.1 is proved. □

4. Estimates for the difference of solutions

In this section, we derive several estimates for the difference of solutions of (2.1) that will be essential in Section 4 for the construction of pullback exponential attractors for the proplem (2.1). We start with the following estimate.

Lemma 4.1. *There exists a positive function $L = L(t,\tau)$ which is independent of m and $\varepsilon \geq 0$, and satisfies*

$$\|U_m(t,\tau)u_{01} - U_m(t,\tau)u_{02}\|_E \leq L\|u_{01}-u_{02}\|_E \tag{4.1}$$

for all $\tau \leq t$, $u_{01}, u_{02} \in E(\varepsilon)$.

Proof. We set $u(t) = u_1(t) - u_2(t) = U_{m_1}(t,\tau)u_{01} - U_{m_2}(t,\tau)u_{02}$, $m(t) = m_1(t) - m_2(t)$ and $\langle u_1\rangle = \langle u_2\rangle$. The function $u(t)$ satisfies the problem, noting that $\langle u\rangle = 0$,

$$\begin{cases} \dfrac{\partial}{\partial t}\big((-\Delta)^{-1}u + \varepsilon u\big) - \Delta u + f(u_1) - f(u_2) = (-\Delta)^{-1}m, \\[2mm] u(\tau) = u_{01} - u_{02}. \end{cases} \tag{4.2}$$

Multiplying equation (4.2) by $u(t)$, integrating over Ω, using the interpolation inequality $|u|^2 \leq c|\nabla u|\|u\|_{-1}$ and the assumption $f' \geq -c$, we find

$$\frac{d}{dt}\|u\|_E^2 \leq c\|u\|_E^2 + c_1\|m\|_E^2.$$

Thus, integrating with respect to t, we obtain

$$\|u(t)\|_E^2 \leq c\|u_0\|_E^2 e^{c(t-\tau)} + c_1 \int_\tau^t e^{c(t-s)}\|m(s)\|_E^2 ds. \tag{4.3}$$

Finally, for $m_1 = m_2$, we obtain (4.1), where $L(t,\tau) = e^{\frac{c}{2}(t-\tau)}$, and Lemma 4.1 is proved. □

In order to construct the pullback exponential attractors we need the following lemma.

Lemma 4.2. *Let $u(t) = U_m(t,\tau)u_0$ be a solution of problem* (2.1)*, where $u_0 \in V$. Then, the following estimate is valid:*

$$\begin{aligned}\|U_m(t,\tau)u_0 - u_0\|_E^2 \leq &\Big(k_1'(t-\tau) + c_1\big(\|u_0\|^2 + \|u_0\|^4\big)(t-\tau) \\ &+ c'|u_0|^2(t-\tau) + c^{-1}\int_\tau^t \|m(s)\|_E^2 ds\Big)e^{c(t-\tau)},\end{aligned} \tag{4.4}$$

for all $\tau \leq t$, where the constants are independent of ε.

Proof. We set $w(t) := u(t) - u_0$. This function satisfies the equation, noting that $\langle w \rangle = 0$,

$$\frac{\partial}{\partial t}\big((-\Delta)^{-1}w + \varepsilon w\big) - \Delta u + f(u) = (-\Delta)^{-1}m + \langle f(u)\rangle.$$

Multiplying this equation by $w(t)$ and integrating over Ω, we have

$$\begin{aligned}\frac{d}{dt}\|w\|_E^2 + 2|\nabla w|^2 + 2(f(u),u) \leq\ & 2c|\nabla u_0|^2 + 2|(f(u),u_0)| - 2|(f(u_0),u_0)| \\ &+ 2|((-\Delta)^{-1}m,w)| + 2|(f(u_0),u_0)|.\end{aligned} \tag{4.5}$$

The term $|(f(u),u_0)| - |(f(u_0),u_0)|$ in the right-hand side of (4.5) can be estimated as follows:

$$\begin{aligned}|(f(u),u_0)| - |(f(u_0),u_0)| &\leq |(f(u)-f(u_0),u_0)| \\ &\leq \int_\Omega \big(|u|^{2p-2} + |u_0|^{2p-2} + 1\big)|w||u_0|dx.\end{aligned}$$

In three dimension we have $p = 2$ and we obtain

$$|(f(u),u_0)| - |(f(u_0),u_0)| \leq \int_\Omega \big(|u|^2 + |u_0|^2 + 1\big)|w||u_0|dx. \tag{4.6}$$

First, we estimate the term $\int_\Omega |u|^2|w||u_0|dx$ in the right-hand side of the last inequality, by using Hölder inequality and the embedding $H^1 \subset L^6$, and we obtain

$$\begin{aligned}\int_\Omega |u|^2|w||u_0|dx &\leq \|u\|_{L^6}^2|w|\ \|u_0\|_{L^6} \\ &\leq c\|u\|^4|w|^2 + \|u_0\|^2.\end{aligned}$$

Similarly, we have

$$\int_\Omega |u_0|^2|w||u_0|dx \le c\|u_0\|^4|w|^2 + \|u_0\|^2.$$

From the theorem of existence and uniqueness, we have $\|u\| \le$ const. ($\|u_0\| \le$ const.) and estimate (4.6) yields

$$\begin{aligned} |(f(u),u_0)| - |(f(u_0),u_0)| &\le c|w|^2 + \|u_0\|^2 \\ &\le \frac{1}{2}|\nabla w|^2 + c\|w\|_{-1}^2 + \|u_0\|^2. \end{aligned}$$

We find, in particular, from (4.5) and by the inequality $|(f(u_0),u_0)| \le c|u_0|_{L^{2p}}^{2p} + c'$ (in three dimension, we have $|(f(u_0),u_0)| \le c|u_0|_{L^4}^4 + c'$),

$$\frac{d}{dt}\|w\|_E^2 \le k_1' + c_1\|u_0\|^2 + c'|u_0|^2 + \|u_0\|^4 + c^{-1}\|m\|_E^2 + c\|w\|_E^2.$$

Here, we have used the embedding $H^1 \subset L^4$. Now, applying Gronwall's inequality to the last relation, we obtain estimate (4.4) and Lemma 4.2 is proved. □

The next theorem gives the $E(\varepsilon) \to H^2(\Omega)$-smoothing for the difference of two solutions.

Theorem 4.1. *Let $u_1(t)$ and $u_2(t)$ be two solutions of (2.1) such that $\|u_i(\tau)\|_2 \le R,\ i = 1,2$. Then, the following estimate is valid:*

$$\|u_1(t) - u_2(t)\|_2^2 \le \frac{c_R}{t-\tau} e^{\alpha_R(t-\tau)}\|u_1(\tau) - u_2(\tau)\|_E^2 \quad \textit{for all } \tau < t, \tag{4.7}$$

where the constants c_R and α_R depend on R and are independent of ε.

We divide the proof of this theorem into several lemmata.

Lemma 4.3. *Let the above assumptions hold. Then, the following estimate is valid:*

$$\|u(t)\|_{H^{-1}}^2 + \varepsilon|u(t)|^2 + \int_\tau^t \|u(s)\|^2 ds \le ce^{\alpha(t-\tau)}\|u_1(\tau) - u_2(\tau)\|_E^2, \tag{4.8}$$

for all $\tau \le t$, where the constants c and α are independent of ε.

Proof. The function $u(t) = u_1(t) - u_2(t)$ satisfies the equation

$$\frac{\partial}{\partial t}\Big((-\Delta)^{-1}\bar{u} + \varepsilon\bar{u}\Big) - \Delta u + \ell(t)u = \langle \ell(t)u\rangle, \tag{4.9}$$

where $\ell(t) := \int_0^1 f'\big(su_1(t) + (1-s)u_2(t)\big)ds$. Multiplying equation (4.9) par $\bar{u}$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\|\bar{u}\|_{-1}^2 + \varepsilon|\bar{u}|^2\Big) + |\nabla u|^2 \\ \le\ & -(f(u_1) - f(u_2), u) + \Big|\langle u\rangle \int_\Omega \big(f(u_1) - f(u_2)\big)dx\Big| \qquad (4.10) \\ \le\ & c|u|^2 + \Big|\langle u\rangle \int_\Omega \big(f(u_1) - f(u_2)\big)dx\Big|. \end{aligned}$$

Noting that (see e.g., [8])

$$\begin{aligned}|u|^2 \leq& 2\big(|\bar{u}|^2+\langle u\rangle^2\big) \leq c\big(\|\bar{u}\|_{-1}|\nabla u|+\langle u\rangle^2\big)\\ \leq&\gamma|\nabla u|^2+c\big(\|\bar{u}\|_{-1}^2+\langle u\rangle^2\big), \quad \forall\gamma>0.\end{aligned} \tag{4.11}$$

We recall that p is finite arbitrary if $n=1$ or 2 and $p=2$ if $n=3$. We have for $n=1,2$

$$\begin{aligned}|\langle u\rangle\int_\Omega\big(f(u_1)-f(u_2)\big)dx| &\leq |\langle u\rangle|\,|\int_\Omega\big(\int_0^1 f'(u_1+(1-s)u_2)ds\big)udx|\\ &\leq |\langle u\rangle|\int_\Omega\big(|u_1|^{2p-2}+|u_2|^{2p-2}+1\big)|u|dx\\ &\leq c|\langle u\rangle|\Big(\|u_1\|_{L^{4p-4}}^{2p-2}+|u_2|_{L^{4p-4}}^{2p-2}+1\Big)|u|\\ &\leq c\Big(|u|^2+\big(\|u_1\|^{4p-4}+\|u_2\|^{4p-4}+1\big)\langle u\rangle^2\Big).\end{aligned}$$

We obtain from the last inequality and estimate (4.11)

$$\big|\langle u\rangle\int_\Omega\big(f(u_1)-f(u_2)\big)dx\big| \leq \frac{1}{4}|\nabla u|^2+C(\|u_1\|,\|u_2\|)\big(\|\bar{u}\|_{-1}^2+\langle u\rangle^2\big).$$

For $n=3$,

$$\begin{aligned}|\langle u\rangle\int_\Omega\big(f(u_1)-f(u_2)\big)dx| &\leq c|\langle u\rangle|\int_\Omega\big(|u_1|^2+|u_2|^2+1\big)|u|dx\\ &\leq c\Big(|u|^2+\big(\|u_1\|_{L^4}^4+\|u_2\|_{L^4}^4+1\big)\langle u\rangle^2\Big)\\ &\leq \frac{1}{4}|\nabla u|^2+C(\|u_1\|,\|u_2\|)\big(\|\bar{u}\|_{-1}^2+\langle u\rangle^2\big).\end{aligned}$$

Observe that from the theorem of existence and uniqueness one has
$\|u_i\|\leq$ const., $\forall i=1,2$. Hence, we obtain from (4.10)

$$\frac{d}{dt}\Big(\|\bar{u}\|_{-1}^2+\varepsilon|\bar{u}|^2\Big)+|\nabla u|^2 \leq c\big(\|\bar{u}\|_{-1}^2+\langle u\rangle^2\big),$$

$$\frac{d}{dt}\Big(\|\bar{u}\|_{-1}^2+\varepsilon|\bar{u}|^2+(1+\varepsilon)\langle u\rangle^2\Big)+c'\big(|\nabla u|^2+\langle u\rangle^2\big) \leq c\|u\|_E^2.$$

Integrating with respect to $t\in[\tau,t]$ we obtain estimate (4.8). □

Lemma 4.4. *Let the above assumptions hold. Then, the following estimate is valid:*

$$\|u(t)\|^2+\int_\tau^t\big\|\frac{\partial u}{\partial t}(s)\big\|_E^2ds \leq \frac{c}{t-\tau}e^{\alpha(t-\tau)}\|u(\tau)\|_E^2 \quad \textit{for all } \tau<t, \tag{4.12}$$

where the constants c and α depend de R and are independent of ε.

Proof. Multiplying (4.9) by $(t-\tau)\frac{\partial u}{\partial t}$, noting that $\langle\frac{\partial u}{\partial t}>=0$, we obtain

$$\begin{aligned}(t-\tau)\Big(\|\frac{\partial u}{\partial t}\|_{H^{-1}}^2+\varepsilon|\frac{\partial u}{\partial t}|^2\Big)+\frac{t-\tau}{2}\frac{d}{dt}|\nabla u|^2 \leq& -\big(\ell(t)u,(t-\tau)\frac{\partial u}{\partial t}\big)\\ \leq&(t-\tau)|\nabla\ell(t)u|\|\frac{\partial u}{\partial t}\|_{H^{-1}}.\end{aligned}$$

In order to estimate the right-hand side of the last inequality we will prove the following estimate (see [3]):

$$|\nabla \ell(t)u| \leq c_R\|u\|, \quad \forall u \in H^1(\Omega). \tag{4.13}$$

Indeed,

$$|\nabla \ell(t)v| \leq c\Big(\|\ell\|_{L^\infty}|\nabla v| + \|\nabla \ell\|_{L^4}\|v\|_{L^4}\Big).$$

We have $H^1 \subset L^4$, then

$$|\nabla \ell(t)v| \leq c\Big(\|\ell\|_{L^\infty} + \|\nabla \ell\|_{L^4}\Big)\|v\|. \tag{4.14}$$

We prove, for $\|u_i(\tau)\|_2 \leq R$, the estimate $\|u_i(t)\|_2 \leq c_R$. Then, we obtain

$$\|\ell\|_{L^\infty} \leq c(\|u_i\|_2) \leq c(R), \quad \text{for } n = 1, 2, \quad \text{and for } p = 2 \quad \text{when } n = 3.$$

Next, we estiamte $\|\nabla \ell\|_{L^4}$. We have

$$\begin{aligned}
|\nabla \ell| &= \Big|\int_0^1 f''(su_1 + (1-s)u_2)\big(s\nabla u_1 + (1-s)\nabla u_2\big)ds\Big| \\
&\leq c\big(1 + |u_1|^{2p-3} + |u_2|^{2p-3}\big)\Big|\int_0^1 \big(s\nabla u_1 + (1-s)\nabla u_2\big)ds\Big|. \\
\|\nabla \ell\|_{L^4} &\leq c\int_\Omega \Big(1 + |u_1|^{2p-3} + |u_2|^{2p-3}\Big)^4\big(|\nabla u_1|^4 + |\nabla u_2|^4\big)dx \\
&\leq c\Big(1 + \|u_1\|_{L^\infty}^{8p-12} + \|u_2\|_{L^\infty}^{8p-12}\Big)\big(\|\nabla u_1\|_{L^4}^4 + \|\nabla u_2\|_{L^4}^4\big).
\end{aligned}$$

For $n = 1, 2$ and for $n = 3$, we have the inequality

$$\|\nabla \ell\|_{L^4} \leq C\Big(\max_{i=1,2}\|u_i\|_{L^\infty}^r\Big)\Big(\max_{i=1,2}\|\nabla u_i\|_{L^4}^4\Big) \leq c'(R),$$

where $r = 8p-12$ if $n = 1, 2$ and $r = 4$ if $n = 3$. (Here, we have used the embeddings $H^1 \subset L^4$ et $H^2 \subset W^{1,4}$.) Estimate (4.13) follows from the above results.

Hence, we have

$$\begin{aligned}
&\frac{d}{dt}\Big((t-\tau)\big(|\nabla u|^2 + \langle u\rangle^2\big)\Big) + c'(t-\tau)\Big\|\frac{\partial u}{\partial t}\Big\|_E^2 \\
\leq& c(t-\tau)\big(|\nabla u|^2 + \langle u\rangle^2\big) + |\nabla u|^2 + \langle u\rangle^2.
\end{aligned}$$

Applying Gronwall's inequality and taking into account estimate (4.8) we have the result. □

Lemma 4.5. *Under the above assumptions we have the following estimate*

$$\Big\|\frac{\partial u}{\partial t}\Big\|_E^2 + \int_\tau^t \big|\nabla\frac{\partial u}{\partial t}(s)\big|^2 ds \leq \frac{c}{t-\tau}e^{\alpha(t-\tau)}\|u(\tau)\|_E^2, \qquad t > \tau, \tag{4.15}$$

where the constants c and α depend de R and are independent of ε.

Proof. We differentiate equation (4.9) with respect to t and set $\theta(t) := \dfrac{\partial u}{\partial t}$. This function satisfies the equation, noting that $\langle\theta\rangle = 0$,

$$\frac{\partial}{\partial t}\big((-\Delta)^{-1}\theta + \varepsilon\theta\big) - \Delta\theta = -\ell(t)\theta - \ell'(t)u + \langle\frac{\partial}{\partial t}\big(\ell(t)u\big)\rangle.$$

Multiplying this equation by $(t-\tau)\theta$, we have

$$\frac{t-\tau}{2}\frac{d}{dt}\big(\|\theta\|^2_{H^{-1}} + \varepsilon|\theta|^2\big) + (t-\tau)|\nabla\theta|^2 \leq c(t-\tau)|\theta|^2 + c'(t-\tau)|\theta|\ |\ell'(t)u|. \quad (4.16)$$

We estimate the last term in the right-hand side of this inequality as follows:

$$\begin{aligned} |\ell'(t)| &= \Big|\int_0^1 f''(su_1 + (1-s)u_2)\big(s\frac{\partial u_1}{\partial t} + (1-s)\frac{\partial u_2}{\partial t}\big)ds\Big| \\ &\leq c\big(1 + |u_1|^{2p-3} + |u_2|^{2p-3}\big)\Big(\Big|\frac{\partial u_1}{\partial t}\Big| + \Big|\frac{\partial u_2}{\partial t}\Big|\Big). \end{aligned}$$

For $n = 1, 2$

$$\begin{aligned} |\ell'(t)u|^2 &\leq c\int_\Omega \big(1 + |u_1|^{4p-6} + |u_2|^{4p-6}\big)\Big(\Big|\frac{\partial u_1}{\partial t}\Big|^2 + \Big|\frac{\partial u_2}{\partial t}\Big|^2\Big)|u|^2 dx \\ &\leq c\big(\max_{i=1,2}\|u_i(t)\|^{4p-6}_{L^\infty}\big)\Big(\Big\|\frac{\partial u_1}{\partial t}\Big\|^2_{L^4} + \Big\|\frac{\partial u_2}{\partial t}\Big\|^2_{L^4}\Big)\|u\|^2_{L^4} \\ &\leq C(R)\Big(\Big\|\frac{\partial u_1}{\partial t}\Big\|^2 + \Big\|\frac{\partial u_2}{\partial t}\Big\|^2\Big)\|u\|^2. \end{aligned}$$

Similarly, if $n = 3$ and $p = 2$

$$\begin{aligned} |\ell'(t)u|^2 &\leq c\int_\Omega \big(1 + |u_1|^2 + |u_2|^2\big)\Big(\Big|\frac{\partial u_1}{\partial t}\Big|^2 + \Big|\frac{\partial u_2}{\partial t}\Big|^2\Big)|u|^2 dx \\ &\leq c\big(\max_{i=1,2}\|u_i(t)\|^2_{L^\infty}\big)\Big(\Big\|\frac{\partial u_1}{\partial t}\Big\|^2_{L^4} + \Big\|\frac{\partial u_2}{\partial t}\Big\|^2_{L^4}\Big)\|u\|^2_{L^4} \\ &\leq C(R)\Big(\Big\|\frac{\partial u_1}{\partial t}\Big\|^2 + \Big\|\frac{\partial u_2}{\partial t}\Big\|^2\Big)\|u\|^2. \end{aligned}$$

In order to complete the proof of the lemma, we need the following result.

Lemma 4.6. *Let the assumptions of Theorem 4.1 hold. Then, the following estimate is valid:*

$$\int_t^{t+1}\Big|\nabla\frac{\partial u_i}{\partial t}(s)\Big|ds \leq C(R), \quad i = 1, 2, \quad (4.17)$$

where the constant $C(R)$ is independent of ε.

Proof. The function u_i, for $i = 1, 2$, satisfies the equation

$$\frac{\partial}{\partial t}\big((-\Delta)^{-1}\bar{u}_i + \varepsilon\bar{u}_i\big) - \Delta u_i + f(u_i) = \langle f(u_i)\rangle + (-\Delta)^{-1}m + c''. \quad (4.18)$$

Multiplying this equation by $\bar{u}_i$ and integrating over Ω, we obtain the inequality

$$\frac{d}{dt}\|\bar{u}_i\|_E^2 + 2|\nabla u_i|^2 + c'\int_\Omega u_i^{2p}dx \le c\|\bar{u}_i\|_E^2 + \|m\|_E^2.$$

Applying Gronwall's inequality and using assumption (A1), we have, in particular,

$$\|\bar{u}_i(t)\|_E^2 + 2\int_\tau^t |\nabla u_i(s)|^2 ds \le e^{c(t-\tau)}\|\bar{u}_i(\tau)\|_E^2 + c_1 \le C_T, \quad \tau \le t \le T,$$

where C_T is independent of ε.

We recall that $\|u_i\|_E^2 \le \|\bar{u}_i(t)\|_E^2 + \langle u_i\rangle^2$, so that we find

$$\|u_i(t)\|_E^2 + 2\int_\tau^t |\nabla u_i(s)|^2 ds \le C_T. \tag{4.19}$$

We have also, by multiplying (4.18) by $\dfrac{\partial u_i}{\partial t}$, integrating over Ω and applying Gronwall's inequality (after simple transformations), the inequality

$$\begin{aligned}&|\nabla u_i(t)|^2 + 2\int_\tau^t \Big(\big\|\frac{\partial u_i}{\partial t}\big\|_{-1}^2 + \varepsilon\big|\frac{\partial u_i}{\partial t}\big|^2\Big)ds + \int_\Omega g(u_i(t))dx\\ \le&|\nabla u_i(\tau)|^2 + \int_\Omega g(u_i(\tau))dx + C_m \le \text{ const.}\end{aligned} \tag{4.20}$$

(Here we have used assumption (A1)).

Now, we multiply the equation

$$\frac{\partial \bar{u}_i}{\partial t} + \varepsilon(-\Delta)\frac{\partial \bar{u}_i}{\partial t} + \Delta^2 u_i - \Delta f(u_i) = m, \tag{4.21}$$

by $(t-\tau)\bar{u}_i$ and we have

$$\begin{aligned}&\frac{t-\tau}{2}\frac{d}{dt}\big(|\bar{u}_i|^2 + \varepsilon|\nabla u_i|^2\big) + (t-\tau)|\Delta u_i|^2\\ \le\ & -(t-\tau)(f'(u_i)\nabla u_i, \nabla u_i) + (t-\tau)\|m\|_{-1}|\nabla u_i|\\ \le\ & c(t-\tau)|\nabla u_i|^2 + c'(t-\tau)\|m\|_E^2.\end{aligned}$$

Then,

$$\begin{aligned}&\frac{d}{dt}\Big((t-\tau)\big(|\bar{u}_i|^2 + \varepsilon|\nabla u_i|^2\big)\Big) + 2(t-\tau)|\Delta u_i|^2\\ \le\ & \big(|\bar{u}_i|^2 + \varepsilon|\nabla u_i|^2\big) + c(t-\tau)\big(|\bar{u}_i|^2 + \varepsilon|\nabla u_i|^2\big) + c'(t-\tau)\|m\|_E^2.\end{aligned}$$

Applying Gronwall's inequality and using estimate (4.19) and the interpolation inequality $|\bar{u}_i|^2 \le c\|\bar{u}_i\|_{-1}|\nabla u_i|$, we obtain

$$|u_i(t)|^2 + \varepsilon|\nabla u_i(t)|^2 + c'\int_\tau^t |\Delta u_i(s)|^2 ds \le C_T, \tag{4.22}$$

where C_T is independent of ε.

Let us now multiply equation (4.21) by $(t-\tau)\dfrac{\partial u_i}{\partial t}$, note that $\langle\dfrac{\partial u_i}{\partial t}\rangle = 0$, and integrate over Ω to obtain

$$\begin{aligned}&(t-\tau)\Big(\Big|\frac{\partial u_i}{\partial t}\Big|^2+\varepsilon\Big|\nabla\frac{\partial u_i}{\partial t}\Big|^2\Big)+\frac{t-\tau}{2}\frac{d}{dt}|\Delta u_i|^2\\ \le\ & c(t-\tau)|\Delta f(u_i)|^2+(t-\tau)|m|\ \Big|\frac{\partial u_i}{\partial t}\Big|.\end{aligned}$$

One can easily obtain, from the definition of the function f, the estimate $|\Delta f(u_i)|^2 \le c(R)|\Delta u_i|^2 + c'(R)$. We thus find

$$\begin{aligned}&\frac{d}{dt}\big((t-\tau)|\Delta u_i|^2\big)+c'(t-\tau)\Big(\Big|\frac{\partial u_i}{\partial t}\Big|^2+\varepsilon\Big|\nabla\frac{\partial u_i}{\partial t}\Big|^2\Big)\\ \le\ & c(t-\tau)|\Delta u_i|^2+|\Delta u_i|^2+c_T|m|^2+c'(R).\end{aligned}$$

Applying Gronwall's inequality and using estimate (4.22) and assumption (A1), we have the following estimate:

$$|\Delta u_i(t)|^2+\int_\tau^t\Big(\Big|\frac{\partial u_i}{\partial t}\Big|^2+\varepsilon\Big|\nabla\frac{\partial u_i}{\partial t}\Big|^2\Big)ds\le C_T,\quad \tau<t\le T, \tag{4.23}$$

where the constant C_T is independent of ε.

Finally, we differentiate equation (4.18) with respect de t and set $\theta_i := \dfrac{\partial u_i}{\partial t}$, for $i = 1, 2$, to find

$$(-\Delta)^{-1}\frac{\partial\theta_i}{\partial t}+\varepsilon\frac{\partial\theta_i}{\partial t}-\Delta\theta_i=-f'(u_i)\theta_i+\langle f'(u_i)\theta_i\rangle+(-\Delta)^{-1}m'.$$

Multiplying this equation by $(t-\tau)\theta_i$, noting that $\langle\theta_i\rangle = 0$, and using the interpolation inequality $|\theta_i|^2 \le c\|\theta_i\|_{-1}|\nabla\theta_i|$, we have

$$\begin{aligned}&\frac{t-\tau}{2}\frac{d}{dt}\big(\|\theta_i\|_{-1}^2+\varepsilon|\theta_i|^2\big)+(t-\tau)|\nabla\theta_i|^2\\ =\ & -(t-\tau)\big(f'(u_i)\theta_i,\theta_i\big)+(t-\tau)\big((-\Delta)^{-1}m',\theta_i\big)\\ \le\ & c(t-\tau)|\theta_i|^2+c'(t-\tau)\|m'\|_{-1}^2+c(t-\tau)\|\theta_i\|_{-1}^2\\ \le\ & c(t-\tau)\big(\|\theta_i\|_{-1}^2+\varepsilon|\theta_i|^2\big)+\frac{t-\tau}{2}|\nabla\theta_i|^2+c'(t-\tau)\|m'\|_{-1}^2.\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{d}{dt}\Big((t-\tau)\big(\|\theta_i\|_{-1}^2+\varepsilon|\theta_i|^2\big)\Big)+(t-\tau)|\nabla\theta_i|^2\\ \le\ & c(t-\tau)\big(\|\theta_i\|_{-1}^2+\varepsilon|\theta_i|^2\big)+\|\theta_i\|_{-1}^2+\varepsilon|\theta_i|^2+c'(t-\tau)\|m'\|_{-1}^2.\end{aligned}$$

Applying Gronwall's inequality, and using assumption (A1) and estimate (4.20), we obtain

$$\|\theta_i(t)\|_{-1}^2+\varepsilon|\theta_i(t)|^2+\int_\tau^t|\nabla\theta_i(s)|^2ds\le C_T,\qquad \tau<t\le T, \tag{4.24}$$

where the constant C_T is independent of ε. Lemma 4.6 is proved. □

Now, we obtain from (4.16)

$$\begin{aligned}
&\frac{t-\tau}{2}\frac{d}{dt}\big(\|\theta\|^2_{H^{-1}}+\varepsilon|\theta|^2\big)+(t-\tau)|\nabla\theta|^2\\
\leq\ & c(t-\tau)|\theta|^2+C(R)(t-\tau)\Big(\big\|\frac{\partial u_1}{\partial t}\big\|^2+\big\|\frac{\partial u_2}{\partial t}\big\|^2\Big)\|u\|^2\\
\leq\ & c(t-\tau)|\nabla\theta|\ \|\theta\|_{H^{-1}}+C(R)(t-\tau)\Big(\big\|\frac{\partial u_1}{\partial t}\big\|^2+\big\|\frac{\partial u_2}{\partial t}\big\|^2\Big)\|u\|^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{d}{dt}\Big((t-\tau)\big(\|\theta\|^2_{H^{-1}}+\varepsilon|\theta|^2\big)\Big)+(t-\tau)|\nabla\theta|^2\\
\leq\ & c(t-\tau)\big(\|\theta\|^2_{H^{-1}}+\varepsilon|\theta|^2\big)+\|\theta\|^2_{H^{-1}}+\varepsilon|\theta|^2\\
&+C(R)(t-\tau)\Big(\big\|\frac{\partial u_1}{\partial t}\big\|^2+\big\|\frac{\partial u_2}{\partial t}\big\|^2\Big)\|u\|^2.
\end{aligned}$$

Applying Gronwall's inequality and using estimates (4.12) and (4.17) we find (4.15). This finishes the proof of Lemma 4.5. □

Now, having estimate (4.15) and interpreting equation (4.9) as an elliptic equation

$$\Delta u-\ell(t)u+\langle\ell(t)u\rangle=\frac{\partial}{\partial t}\big((-\Delta)^{-1}\bar{u}+\varepsilon\bar{u}\big):=h(t), \tag{4.25}$$

we obtain from the above estimates

$$\begin{aligned}
|h(t)|^2=\big|(-\Delta)^{-1}\frac{\partial u}{\partial t}+\varepsilon\frac{\partial u}{\partial t}\big|^2\ &\leq\ 2\Big(\big\|\frac{\partial u}{\partial t}\big\|^2_{H^{-1}}+\varepsilon\big|\frac{\partial u}{\partial t}\big|^2\Big)\\
&\leq\ \frac{c}{t-\tau}e^{\alpha(t-\tau)}\|u(\tau)\|^2_E,\qquad t>\tau.
\end{aligned}$$

Multiplying now equation (4.25) by Δu (t is fixed), we obtain, from estimate (4.13),

$$\begin{aligned}
|\Delta u(t)|^2\leq&|h(t)|\ |\Delta u(t)|+|\nabla\ell(t)u|\ |\nabla u|\\
\leq&c\big(|h(t)|^2+|\nabla u(t)|^2+\|u(t)\|^2\big).
\end{aligned}$$

Hence

$$\begin{aligned}
\|u(t)\|_2^2\leq&c'\big(|h(t)|^2+\|u(t)\|^2\big)\\
\leq&\frac{c}{t-\tau}e^{\alpha(t-\tau)}\|u(\tau)\|^2_E,\qquad\text{for all } t>\tau.
\end{aligned}$$

Thus, we have estimate (4.7) and we finish the proof of Theorem 4.1.

5. Construction of pullback exponential attractors

We consider the family of mappings

$$U_{m,t_0}:=\big\{U_m(t,\tau):\tau\leq t\leq t_0\big\},$$

and, for any $\delta > 0$, $K > 0$ and $B \subset E(\varepsilon)$, we denote by $S_{\delta,K}(B)$ the set of mappings $S : E(\varepsilon) \to E(\varepsilon)$ such that

$$S\big(\mathcal{O}_\delta(B)\big) \subset B$$

and

$$\|Su_1 - Su_2\|_2 \leq K\|u_1 - u_2\|_E \qquad \text{for all } u_1, u_2 \in \mathcal{O}_\delta(B),$$

where $\mathcal{O}_\delta(B) := \big\{u \in E(\varepsilon) : \inf_{v\in B} \|u - v\|_E < \delta\big\}$.

In order to obtain a family of pullback exponential attractors we need to prove the following conditions, which are similar to those of (H1), (H2), (H3) and (H4) in [7]:

(H0) Let $\tau_0 > 0$ be fixed. Then, for all $B \subset E(\varepsilon)$ bounded and closed in $E(\varepsilon)$,

$$U_m(t, t - \tau_0) \in S_{\delta,K}(B) \qquad \text{for all } t \leq t_0.$$

(H1) There exist $C_0 > 0$, $0 < \varepsilon_0 \leq \tau_0$ and $\gamma > 0$ such that, for all $t \leq t_0$, $\tau_0 \leq r \leq 2\tau_0, 0 \leq s \leq \varepsilon_0$ and $v \in \mathcal{O}_\delta(B)$,

$$\|U_m(t, t - r)v - U_m(t - s, t - r - s)v\|_E \leq C_0|s|^\gamma.$$

(H2) There exists a constant $C_B > 0$ such that

$$\|U_m(t, t - s)v - U_m(t, t - s)w\|_E \leq C_B\|v - w\|_E,$$

for all $v, w \in B$, for any $t \leq t_0$, $0 \leq s \leq 2\tau_0$.

(H3) There exist $C_0' > 0$ and $\gamma' > 0$ such that

$$\|U_m(t, t - r)v - U_m(t - s, t - r)v\|_E \leq C_0'|s|^{\gamma'},$$

for all $t \leq t_0$, $\tau_0 \leq r \leq 2\tau_0$, $0 \leq s \leq \varepsilon_0$ and $v \in B$.

(H4) For any $t > t_0$ and D_1, D_2 bounded subsets of $E(\varepsilon)$, there exists a constant $L(t, D_1, D_2) > 0$ such that

$$\|U_m(t, t_0)v - U_m(t, t_0)w\|_E \leq L(t, D_1, D_2)\|v - w\|_E,$$

for all $v \in D_1$, $w \in D_2$.

U_m satisfies (H0):

We consider the ball

$$B := \big\{u \in E(\varepsilon) : \|u\|_E \leq C_m(t_0)\big\}$$

and set

$$\tau_0 := 1 + 2c^{-1}\log\Big(C_1 \max\{1, 1 + C_m(t_0)\}\Big).$$

From Lemma 3.1, we have

$$\|U_m(t, t - \tau_0)u_0\|_E \leq C_m(t_0)$$

for all $t \leq t_0$, $t - \tau_0 \leq t - c^{-1}\log\Big(C_1\|\mathcal{O}_1(B)\|_E\Big)$, $u_0 \in \mathcal{O}_1(B)$.

By Theorem 4.1, we have

$$\|U_m(t, t - \tau_0)u_{01} - U_m(t, t - \tau_0)u_{02}\|_2^2 \leq K\|u_{01} - u_{02}\|_E^2,$$

where $K = \frac{c_R}{\tau_0} e^{\alpha_R \tau_0}$. Thus,

$$U_m(t, t-\tau_0) \in S_{1,K}(B) \quad \text{for all } t \leq t_0.$$

We have also, from estimate (4.1), that the mapping $U_m(t,s) : E(\varepsilon) \to E(\varepsilon)$ is continous for any $s \leq t$.
Thus, U_m satisfies (H0).

U_m satisfies (H2) and (H4):
We have by Lemma 4.1

$$\|U_m(t,t-s)u_{01} - U_m(t,t-s)u_{02}\|_E \leq e^{\frac{c}{2}s}\|u_{01} - u_{02}\|_E,$$

for all $t \leq t_0$, $s \in [0, 2\tau_0]$, u_{01}, $u_{02} \in B$.
Thus, U_m satisfies (H2), with $C_B = e^{c\tau_0}$.
Also, from estimate (4.1), we deduce that U_m satisfies (H4) with $L(t,, D_1, D_2) = e^{\frac{c}{2}(t-t_0)}$.

U_m satisfies (H1) and (H3):
Set $u(t) = U_m(t,\tau)u_0$. For any $s \geq 0$, $t - s \geq \tau$, we have

$$u(t) - u(t-s) = \int_{t-s}^{t} \frac{\partial u}{\partial t}(\theta)d\theta,$$

$$\begin{aligned} \|U_m(t,\tau)u_0 - U_m(t-s,\tau)u_0\|_E &\leq \int_{t-s}^{t} \|\frac{\partial u}{\partial t}(\theta)\|_E d\theta \\ &\leq s^{\frac{1}{2}}\Big(\int_{t-s}^{t} \|\frac{\partial u}{\partial t}(\theta)\|_E^2 d\theta\Big)^{\frac{1}{2}}. \end{aligned} \tag{5.1}$$

We now estimate the right-hand side of (5.1).
Multiplying the equation

$$\frac{\partial}{\partial t}\big((-\Delta)^{-1}\bar{u} + \varepsilon\bar{u}\big) - \Delta\bar{u} + f(u) = \langle f(u)\rangle + (-\Delta)^{-1}m$$

by $\frac{\partial u}{\partial t}$ and noting that $\langle \frac{\partial u}{\partial t}\rangle = 0$, we have

$$\begin{aligned} \|\frac{\partial u}{\partial t}\|_E^2 + \frac{1}{2}\frac{d}{dt}\big(|\nabla u|^2 + 2\int_\Omega g(u)dx\big) &\leq c\|m\|_{-1}^2 + \frac{1}{2}\|\frac{\partial u}{\partial t}\|_{-1}^2 \\ &\leq c\|m\|_E^2 + \frac{1}{2}\|\frac{\partial u}{\partial t}\|_E^2, \end{aligned}$$

where g denotes an antiderivative of f. Integrating the last inequality over $[t-s,t]$ (and over $[0, t-s]$) we have

$$\begin{aligned} &\int_{t-s}^{t} \|\frac{\partial u}{\partial t}\|_E^2 + |\nabla u(t)|^2 + 2\int_\Omega g(u(t))dx \\ \leq\ & c\int_0^t \|m(\theta)\|_E^2 d\theta + |\nabla u_0|^2 + 2\int_\Omega g(u_0)dx \\ \leq\ & \text{const.}, \end{aligned}$$

We deduce from (5.1) that

$$\|U_m(t,\tau)u_0 - U_m(t-s,\tau)u_0\|_E \le \hat{c}_1 s^{\frac{1}{2}}, \qquad \text{for all } t \le t_0, \tag{5.2}$$

$0 \le s \le 1$, $\tau \le t - 2c^{-1}\log\big(C_1\|\mathcal{O}_1(B)\|_E\big)$, $u_0 \in \mathcal{O}_1(B)$.
Let $r \ge \tau_0$. Then,

$$t - r \le t - 1 - 2c^{-1}\log\big(C_1\|\mathcal{O}_1(B)\|_E\big). \tag{5.3}$$

By (5.2) and (5.3)

$$\|U_m(t,t-r)u_0 - U_m(t-s,t-r)u_0\|_E \le \hat{c}_1 s^{\frac{1}{2}}, \qquad \text{for all } t \le t_0,$$

$0 \le s \le 1$, $r \ge \tau_0$, $u_0 \in \mathcal{O}_1(B)$.
Thus, U_m satisfies (H3) with $\gamma' = \frac{1}{2}$.

Furthermore,

$$\begin{aligned} & \|U_m(t,t-r)u_0 - U_m(t-s,t-s-r)u_0\|_E \\ \le\ & \|U_m(t,t-r)u_0 - U_m(t-s,t-r)u_0\|_E \\ & + \|U_m(t-s,t-r)u_0 - U_m(t-s,t-s-r)u_0\|_E. \end{aligned} \tag{5.4}$$

By (4.3), we have

$$\begin{aligned} & \|U_m(t-s,t-r)u_0 - U_m(t-s,t-s-r)u_0\|_E \\ =\ & \|U_m(t-s,t-r)u_0 - U_m(t-s,t-r)U_m(t-r,t-r-s)u_0\|_E \\ \le\ & e^{\frac{c}{2}(r-s)}\|u_0 - U_m(t-r,t-r-s)u_0\|_E \\ \le\ & e^{c\tau_0}\|u_0 - U_m(t-r,t-r-s)u_0\|_E,\ \tau_0 \le r \le 2\tau_0, 0 \le s \le 1. \end{aligned}$$

From estimate (4.4), we deduce that

$$\|u_0 - U_m(t-r,t-r-s)u_0\|_E \le \tilde{C}_1 s^{\frac{1}{2}} + c^{-\frac{1}{2}}\Big(\int_{t-r-s}^{t-r}\|m(\theta)\|_E^2 d\theta\Big)^{\frac{1}{2}}, \tag{5.5}$$

where $\tilde{C}_1 = \big(k_1' + c_1(\|u_0\|^2 + \|u_0\|^4) + c'|u_0|^2\big)^{\frac{1}{2}}$.
Observe that, as $0 \le s \le 1$ and $t \le t_0$,

$$\begin{aligned} \int_{t-r-s}^{t-r}\|m(\theta)\|_E d\theta &\le \Big(\int_{t-r-s}^{t-r}\|m(\theta)\|_E^q d\theta\Big)^{\frac{2}{q}}\Big(\int_{t-r-s}^{t-r} 1 d\theta\Big)^{\frac{q-2}{q}} \\ &\le \Big(M_{m,q}(t_0)\Big)^{\frac{2}{q}} s^{\frac{q-2}{q}}, \end{aligned}$$

and then, by (5.5) and the fact that $s^{\frac{1}{2}} \le s^{\frac{q-2}{2q}}$ for all $0 \le s \le 1$, we obtain

$$\|u_0 - U_m(t-r,t-r-s)u_0\|_E \le \Big(\tilde{C}_1 + c^{-\frac{1}{2}}\big(M_{m,q}(t_0)\big)^{\frac{1}{q}}\Big)\, s^{\frac{q-2}{2q}}, \tag{5.6}$$

for all $t \le t_0$, $0 \le s \le 1$, $\tau_0 \le r \le 2\tau_0$, $u_0 \in E(\varepsilon)$.

From estimates (5.2), (5.4) and (5.6), we have

$$\|U_m(t,t-r)u_0 - U_m(t-s,t-s-r)u_0\|_E \le \hat{C}_2\, s^{\frac{q-2}{2q}},$$

for all $t \le t_0$, $0 \le s \le 1$, $\tau_0 \le r \le 2\tau_0$, $u_0 \in \mathcal{O}_1(B)$.
Thus, U_m satisfies (H1) with $\delta = 1$, $\varepsilon = 1$, $C_0 = \hat{C}_2$.

Now, we can apply the results obtained in [7] and we find the following result.

Theorem 5.1. *We assume that* m *appearing in* (2.1) *satisfies (A1), (A2) and (A3). Then, there exists a family* $\widetilde{\mathcal{M}}_{U_m} := \{\widetilde{\mathcal{M}}_{U_m}(t) : t \in \mathbb{R}\}$ *of nonempty subsets of* $E(\varepsilon)$ *which satisfies:*

1. $U_m(t,\tau)\widetilde{\mathcal{M}}_{U_m}(\tau) \subset \widetilde{\mathcal{M}}_{U_m}(t)$, *for all* $\tau \le t$,
2. $\widetilde{\mathcal{M}}_{T_{-\tau}U_m}(t) = \widetilde{\mathcal{M}}_{U_m}(t-\tau)$, *for all* $\tau \ge 0$ *and any* $t \le t_0$ *and* $\widetilde{\mathcal{M}}_{T_{-\tau}U_m}(t) \subset \widetilde{\mathcal{M}}_{U_m}(t-\tau)$, *for all* $\tau \ge 0$ *and any* $t > t_0$, *where* $T_{-\tau}U_m(t,s) := U_m(t-\tau, s-\tau)$,
3. *For any* $D \subset E(\varepsilon)$ *bounded,*
$$dist_{E(\varepsilon)}(U_m(t,t-\tau)D, \widetilde{\mathcal{M}}_{U_m}(t)) \le \tilde{C}_1 e^{\tilde{\alpha} s_D} e^{-\tilde{\alpha}\tau},$$
$$\text{for all } \tau \ge s_D \quad \text{and any } t \le t_0$$
and
$$\begin{aligned} &dist_{E(\varepsilon)}(U_m(t,t-\tau)D, \widetilde{\mathcal{M}}_{U_m}(t)) \\ \le\ & L_m(t, D, \mathcal{M}_{U_m}(t_0))\tilde{C}_1 e^{\tilde{\alpha}(s_D+t-t_0)} e^{-\tilde{\alpha}\tau}, \end{aligned}$$
for all $t > t_0$ *and any* $\tau \ge s_D + t - t_0$, *where* $\tilde{C}_1$ *and* $\tilde{\alpha}$ *are positive constants only depending on* Ω *and* $M_m(t_0)$,
4. *for all* $t \in \mathbb{R}$, $\widetilde{\mathcal{M}}_{U_m}(t)$ *is a compact subset of* $E(\varepsilon)$, *with finite fractal dimension and, more precisely,*
$$dim_F(\widetilde{\mathcal{M}}_{U_m}(t), E(\varepsilon)) \le \begin{cases} \check{C}_1, & \text{if } t \le t_0, \\ \dfrac{\check{C}_1}{L_m(t, \widetilde{\mathcal{M}}_{U_m}(t_0), \widetilde{\mathcal{M}}_{U_m}(t_0))}, & \text{if } t > t_0, \end{cases}$$
where $\check{C}_1 > 0$ *is a positive constant depending on* Ω *and* $M_m(t_0)$,
5. *there exists the pullback attractor for* U_m, $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$, *which satisfies*
$$\mathcal{A}(t) \subset \widetilde{\mathcal{M}}_{U_m}(t), \quad \text{for all } t \in \mathbb{R}.$$
Furthermore
$$dim_F(\mathcal{A}(t), E(\varepsilon)) \le \begin{cases} \check{C}_1, & \text{if } t \le t_0, \\ \dfrac{\check{C}_1}{L_m(t, \widetilde{\mathcal{M}}_{U_m}(t_0), \widetilde{\mathcal{M}}_{U_m}(t_0))}, & \text{if } t > t_0, \end{cases}$$
6. *for all* $0 \le r \le 1$ *and any* $t \le t_0$,
$$dist^{symm}_{E(\varepsilon)}\left(\widetilde{\mathcal{M}}_{U_m}(t), \widetilde{\mathcal{M}}_{U_m}(t-r)\right) \le \check{C}_2 |r|^{(k+1)\gamma},$$
where $\gamma = \dfrac{q-2}{2q}$ *and* $\check{C}_2$ *and* k *are positive constants only depending on* $\Omega, q, M_m(t_0)$ *and* $M_{m,q}(t_0)$.

All the constants in this theorem are independent of the parameter ε.

Remark 5.1. It would be interesting to study the limit $\varepsilon \to 0$ to obtain the continuity of exponential attractors under perturbations and this will be done in a forthcoming. More precisely, we will study the continuity of exponential attractors for a viscous Cahn-Hilliard system to an exponential attractor for the limit Cahn-Hilliard system and will obtain an estimate for the symmetric distance between the perturbed and non-perturbed exponential attractors.

6. Conclusion

We have constructed a pullback exponential attractor for the non-autonomous viscous Cahn-Hilliard system in a bounded domain of $\mathbb{R}^n$, $n = 1, 2, 3$. This construction gives a set containing the associated pullback attractor, so that it yields, by this construction, that the pullback attractors have finite fractal dimension.

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