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SPECTRUM COMPARISON FOR A CONSERVED REACTION-DIFFUSION SYSTEM WITH A VARIATIONAL PROPERTY*

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Abstract We are dealing with a two-component system of reaction-diffusion equations with conservation of a mass in a bounded domain subject to the Neumann or the periodic boundary conditions. We consider the case that the conserved system is transformed into a phase-field type system. Then the stationary problem is reduced to that of a scalar reaction-diffusion equation with a nonlocal term. We study the linearized eigenvalue problem of an equilibrium solution to the system, and compare the eigenvalues with ones of the linearized problem arising from the scalar nonlocal equation in terms of the Rayleigh quotient. The main theorem tells that the number of negative eigenvalues of those problems coincide. Hence, a stability result for nonconstant solutions of the scalar nonlocal reaction-diffusion equation is applicable to the system.

Keywords Reaction-diffusion system, conservation of a mass, equilibrium solution, linearized eigenvalue problem, Rayleigh-quotient.

MSC(2000) 35B35, 35K57, 49R50.

Dedicated to Professor Yasumasa Nishiura on the occasion of his 60th birthday

1. Introduction

We are concerned with the following 2-component reaction-diffusion system:

$$\begin{cases} u_t = d_1 \Delta u + k_1 \tilde{f}(u,v), \\ v_t = d_2 \Delta v - k_2 \tilde{f}(u,v), \end{cases} \tag{1.1}$$

in a bounded domain, where $d_j, k_j (j = 1, 2)$ are positive parameters and Δ stands for the Laplacian. This class of model equations contains an autocatalytic model, exothermic reaction-diffusion model and a cell polarity model. For instance, in an autocatalytic model $\tilde{f}(u,v) = -u^m v^n$ (m, n are positive integers) is treated where the variables u and v stand for a reactant and an autocatalyst respectively. In an exothermic model the case $\tilde{f}(u,v) = -u\exp(-C/v)$ (C is a positive constant) is studied where u and v correspond to a reactant and temperature respectively. The

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main issues for those models are the existence of travelling waves and the speed of the waves (see, for instance, Chen & Qi [3], Hosono [9], Ikeda etc [10] and the references therein). On the other hand for a cell polarity model an emergence of a pattern under the periodic boundary conditions is studied for $\tilde{f}(u,v) = -g_1(u,v) + g_2(u,v)$ with appropriate functions g_1 and g_2 in the literature Ishihara etc, Otsuji etc [11, 14].

In this paper we are dealing with the case

$$\tilde{f}(u,v) = f(u) + v, \tag{1.2}$$

which corresponds to the case $g_1 = -f(u),\ g_2 = v$ in the cell polarity model. Scaling the variables as

$$k_1 t \to t, \quad \sqrt{k_2/d_2}\,x \to x,$$

and putting

$$d := d_1k_2/d_2k_1, \qquad \tau := k_1/k_2, \tag{1.3}$$

we consider the following normalized system

$$\begin{cases} u_t = d\varDelta u + f(u) + v, \\ \tau v_t = \varDelta v - f(u) - v \end{cases} \qquad (x \in \Omega), \tag{1.4}$$

under the Neumann boundary conditions for a bounded domain Ω in $\mathbb{R}^n$ with smooth boundaries $\partial\Omega$,

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \qquad (x \in \partial\Omega), \tag{1.5}$$

or the periodic boundary conditions for $\Omega = [0, L]$ with

$$\begin{cases} u(0,t) = u(L,t), \quad u_x(0,t) = u_x(L,t), \\ v(0,t) = v(L,t), \quad v_x(0,t) = v_x(L,t). \end{cases} \tag{1.6}$$

A specific case $f(u) = -au/(u^2 + b)\ (a, b > 0)$ is proposed in Ishihara etc [11] and Otsuji etc [14] as a conceptual model for the cell polarity. In those papers the Turing-type instability of a constant solution is verified for appropriate parameter values and the emergence of a localized pattern is shown by numerics. Later, Morita & Ogawa [13] studies the stability of nonconstant solutions to (1.4) rigorously. This article is successive to Morita & Ogawa [13] about the study for the stability analysis of equilibrium solutions to (1.4).

We note that the solution $(u,v) = (u(x,t), v(x,t))$ to (1.4) with the boundary conditions satisfies

$$\frac{d}{dt}\int_\Omega (u + \tau v)dx = \int_\Omega (d\varDelta u + \varDelta v)dx = 0,$$

which implies the conservation of the mass as

$$\int_\Omega (du + v)dx = \text{constant}.$$

In the present paper we use the notation for the spatial average of a function $w = w(x)$ by

$$\langle w \rangle := \frac{1}{|\Omega|}\int_\Omega w(x)dx,$$

where $|\Omega|$ stand for the volume of Ω, and define the average of the mass for the solution

$$s := \langle u(\cdot,t)\rangle + \tau\langle v(\cdot,t)\rangle, \tag{1.7}$$

which is constant for $t \geq 0$, as long as the solution is defined in an appropriate function space.

Throughout the paper, we consider f such that the equations (1.4) with (1.5) or (1.6) allows a unique time global solution for the initial conditions

$$(u(x,0), v(x,0)) = (u_0(x), v_0(x)) \in X$$

in an appropriate function space X, and the solution map

$$(u_0(\cdot), v_0(\cdot)) \mapsto (u(\cdot,t;u_0,v_0), v(\cdot,t;u_0,v_0))$$

for $t \geq 0$ generates a C^1 semiflow in X.

We note that the system (1.4) is transformed into a phase-field type system (see in the next section), hence the system allows a Lyapunov function. Moreover, as for the phase-field system (proposed by Caginalp [2] and Fix [6]) a nice variational property is found in Bates & Fife [1]. In this article, we shall develop the argument in Bates & Fife [1] for the study of spectra of the linearized eigenvalue problem of a nonconstant equilibrium to (1.4).

Corresponding to the system (1.4) under the boundary conditions with (1.7), we consider the scalar reaction-diffusion equation with a nonlocal term

$$u_t = d\Delta u + f(u) - du + \frac{s}{\tau} - \frac{1-\tau d}{\tau}\langle u\rangle \quad (x \in \Omega), \tag{1.8}$$

with the Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = 0 \quad (x \in \partial\Omega), \tag{1.9}$$

or the periodic boundary conditions

$$u(0,t) = u(L,t), \quad u_x(0,t) = u_x(L,t), \tag{1.10}$$

for $\Omega = [0, L]$.

The goal of the present article is to establish a comparison theorem between the spectra for the linearized eigenvalue problem of an equilibrium solution to (1.4) and the ones of (1.8). We see that equilibrium solutions between the system and the scalar equation have one-to-one correspondence in what follows. Let $(u^*(x;s), v^*(x;s))$ be a solution enjoying

$$\begin{cases} d\Delta u + f(u) + v = 0, \\ \Delta v - f(u) - v = 0 \end{cases} \qquad (x \in \Omega), \tag{1.11}$$

and

$$s = \langle u\rangle + \tau\langle v\rangle, \tag{1.12}$$

with the Neumann or the periodic boundary conditions. Then u^* and v^* satisfy

$$d\Delta u + f(u) - du + \frac{s}{\tau} - \frac{1-\tau d}{\tau}\langle u\rangle = 0, \tag{1.13}$$

and

$$v = -du + \frac{s}{\tau} - \frac{1 - \tau d}{\tau}\langle u \rangle, \tag{1.14}$$

respectively. Indeed, from adding the two equations of (1.11) it follows that

$$du^* + v^* = C \tag{1.15}$$

holds for a constant C. Integrating (1.15) and using

$$s = \langle u^* \rangle + \tau \langle v^* \rangle,$$

we obtain (1.13) and (1.14) for u^* and v^*. In sequel, an equilibrium solution u^* of (1.13) together with v^* given by (1.14) provide a solution (u^*, v^*) of (1.11) with (1.12), and vice versa.

Now we write the linearized eigenvalue problem

$$\mathcal{L}\begin{pmatrix} \phi \\ \psi \end{pmatrix} := -\begin{pmatrix} d\Delta\phi + f'(u^*)\phi + \psi \\ \Delta\psi - f'(u^*)\phi - \psi \end{pmatrix} = \lambda \begin{pmatrix} \phi \\ \tau\psi \end{pmatrix} \tag{1.16}$$

with the domain

$$\mathcal{D}_N(\mathcal{L}) := \mathcal{D}_N \times \mathcal{D}_N, \tag{1.17}$$
$$\mathcal{D}_N := \{\varphi \in H^2(\Omega) \ : \ \partial\varphi/\partial\nu = 0 \ \ (x \in \partial\Omega)\}, \tag{1.18}$$

or

$$\mathcal{D}_p(\mathcal{L}) := \mathcal{D}_p \times \mathcal{D}_p, \tag{1.19}$$
$$\mathcal{D}_p := \{\varphi \in H^2(0, L) \ : \ \varphi(0) = \varphi(L), \ \varphi_x(0) = \varphi_x(L) = 0\}, \tag{1.20}$$

under the constraint

$$\langle \phi \rangle + \tau \langle \psi \rangle = 0. \tag{1.21}$$

This condition (1.21) naturally follows when we fix the mass s. On the other hand the corresponding linearized eigenvalue problem to a solution u^* of (1.13) is given by

$$L_0[\varphi] := -\{d\Delta\varphi + f'(u^*)\varphi - d\varphi - \frac{1}{\tau}(1 - \tau d)\langle \varphi \rangle\} = \mu\varphi, \tag{1.22}$$

with the domain $\mathcal{D}(L_0) = \mathcal{D}_N$ or $\mathcal{D}_p$.

We note that for a nonconstant solution $(u^*, v^*) = (u^*(\cdot; s), v^*(\cdot; s))$ to (1.11) with the periodic boundary conditions,

$$\mathcal{L}\begin{pmatrix} u_x^* \\ v_x^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and $L_0[u_x^*] = 0$ hold.

Let us define the eigenvalues and the corresponding eigenfunctions as

$$\mathcal{L}\begin{pmatrix} \phi_j \\ \psi_j \end{pmatrix} = \lambda_j \begin{pmatrix} \phi_j \\ \psi_j \end{pmatrix} \qquad (j = 1, 2, \ldots), \tag{1.23}$$

and

$$L_0[\varphi_j] = \mu_j \varphi_j \qquad (j = 1, 2, \ldots), \tag{1.24}$$

where $\{\lambda_j\}$ and $\{\mu_j\}$ are arranged in an nondecreasing order with counting multiplicity in the way

$$\lambda_1 \le \lambda_2 \le \dots \lambda_j \le \lambda_{j+1} \le \dots, \qquad \mu_1 \le \mu_2 \le \dots \mu_j \le \mu_{j+1} \le \dots \tag{1.25}$$

The main result of the present article is that the number of negative eigenvalues (the Morse index) of the operators $\mathcal{L}$ and L_0 coincides. More precisely we obtain the next theorem.

Theorem 1.1. *Assume* $0 < \tau d < 1$. *Consider a nonconstant solution* $(u^*(x), v^*(x))$ *to* (1.11) *with the constraint* (1.12) *subject to the Neumann or the periodic boundary conditions. Let* $\{\lambda_j\}_{j=1,2,\dots}$ *and* $\{\mu_j\}_{j=i,2,\dots}$ *be the eigenvalues of* $\mathcal{L}$ *with the condition* (1.21) *and* L_0 *respectively with the ordering* (1.25). *If* $\lambda_k \neq 0$ *or* $\mu_k \neq 0$, *then*

$$\lambda_k \mu_k > 0, \qquad |\lambda_k| < |\mu_k| \tag{1.26}$$

holds. Moreover, if $\mu_k = 0$, *then* $\lambda_k = 0$ *holds, and vice versa.*

We also consider the eigenvalues and the corresponding eigenfunctions of $\mathcal{L}$ without the restriction (1.21), which are denoted as

$$\{\lambda'_j\}_{j=1,2,\dots}, \quad \{(\phi'_j, \psi'_j)\}_{j=1,2,\dots} \tag{1.27}$$

with nondecreasing order and counting the multiplicity. We note that in this case $\mathcal{L}$ always has a zero eigenvalue since

$$\mathcal{L}\begin{pmatrix} \partial u^*/\partial s \\ \partial v^*/\partial s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{1.28}$$

holds, where

$$\langle \partial u^*/\partial s\rangle + \tau\langle \partial v^*/\partial s\rangle = 1,$$

while

$$L_0[\partial u^*/\partial s] = 1/\tau$$

holds. Then we obtain the next corollary.

Corollary 1.1. *Under the same assumption in Theorem 1.1 let* $\{\lambda'_j\}_{j=1,2,\dots}$ *and* $\{\mu_j\}_{j=i,2,\dots}$ *be the eigenvalues of* $\mathcal{L}$ *and* L_0 *respectively with the nondecreasing order and the counting multiplicity. Then the following holds for the Neumann boundary conditions:*

(i) *If* $\mu_1 > 0$, *then* $\lambda'_1 = 0 < \lambda'_2$ *and*

$$\lambda'_{j+1} < \mu_j \quad (j \ge 1) \tag{1.29}$$

hold. On the other hand, if $\lambda'_1 = 0 < \lambda'_2$, *then* (1.29) *holds.*

(ii) *If there is a positive integer* n *such that* $\mu_n < 0 < \mu_{n+1}$, *then* $\lambda'_{n+1} = 0$ *and*

$$\mu_j < \lambda'_j < 0 \;\; (j \le n), \qquad 0 < \lambda'_{j+1} < \mu_j \;\; (j \ge n+1) \tag{1.30}$$

hold. On the other hand, if $\lambda'_{n+1} = 0 < \lambda'_{n+2}$ *for a positive integer* n, *then* (1.30) *holds.*

As for the periodic boundary conditions the next assertions hold:

(i') If $\mu_2 > \mu_1 = 0$, then $\lambda'_1 = \lambda'_2 = 0 < \lambda'_3$ and

$$\lambda'_{j+1} < \mu_j \quad (j \geq 2) \tag{1.31}$$

hold. On the other hand, if $\lambda'_1 = \lambda'_2 = 0 < \lambda'_3$, then (1.31) holds.

(ii') If there is a positive integer n such that $\mu_n < 0 = \mu_{n+1} < \mu_{n+2}$, then $\lambda'_{n+1} = \lambda'_{n+2} = 0$ and

$$\mu_j < \lambda'_j < 0 \quad (j \leq n), \qquad 0 < \lambda'_{j+1} < \mu_j \quad (j \geq n+2) \tag{1.32}$$

hold. On the other hand, if $\lambda'_{n+1} = \lambda'_{n+2} = 0 < \lambda'_{n+3}$ for a positive integer n, then (1.32) holds.

Remark 1.1. As seen in Morita & Ogawa [13], the nonlinear dynamical stability of the equilibrium solution to (1.4) is assured for the case (i) and (i') of Corollary 1.1 (see also Henry [8]).

Remark 1.2. Some partial result of Corollary 1.1 is obtained in Morita & Ogawa [13], where a variational property for the system is not used. In this paper, by virtue of the variational setting found in Bates & Fife [1], we can conclude in the assertions of the theorem and the corollary. Indeed, as mentioned before, our system (1.4) can be transformed into a phase-field type equations. We develop the argument in Bates & Fife [1] to the present problem.

We also give a remark on the result about the spectra in Bates & Fife [1]. Since in Bates & Fife [1] they compare the spectra between the system and scalar equations (without nonlocal term), they obtain a different comparison result. We believe that the comparison to the scalar equation with nonlocal term is more natural. In sequel, the assertion is simple and clear. Moreover, by this comparison, the next corollary follows from Theorem 1.1 and a result in Suzuki & Tasaki [15].

Corollary 1.2. *Consider the system (1.4) in $\Omega = (0, L)$ under the condition $0 < \tau d < 1$. For the Neumann boundary conditions any stable nonconstant equilibrium solution is constant or strictly monotone. For the periodic boundary condition any stable equilibrium solution is constant or it has a single peak. Moreover, in a cylindrical domain*

$$\Omega = \{x = (x_1, x') \in (0, L) \times D\} \subset \mathbb{R}^n,$$

where D is a bounded domain with smooth boundaries in $\mathbb{R}^{n-1}$ $(n \geq 2)$, any stable nonconstant equilibrium solution is constant, or monotone in the x_1 direction.

We prove the theorem together with the corollaries in the rest of this article.

2. Variational setting

As shown in Morita & Ogawa [13], putting $w = du + v$ in (1.4) yields a phase-field type system

$$\begin{cases} u_t = \Delta u + f(u) - du + w, \\ \tau w_t + (1 - \tau d) u_t = \Delta w \end{cases} \tag{2.1}$$

which allows a Lyapunov function

$$\mathcal{E}_1(u,w) := \int_\Omega \left\{ \frac{d}{2}|\nabla u|^2 - F(u) + \frac{d}{2}u^2 + \frac{\tau}{2(1-\tau d)}w^2 \right\} dx, \tag{2.2}$$

where

$$F(u) = \int f(u)du.$$

Here, to see a variational property of the system clearly, we transform (1.4) in another way, which is found in Bates & Fife [1]. We put

$$\alpha := \sqrt{\tau(1-\tau d)}, \quad \beta := \sqrt{\frac{1-\tau d}{\tau}}, \tag{2.3}$$

and introduce the new variable as

$$W := \frac{1}{\alpha}(u + \tau v). \tag{2.4}$$

Then the equations (1.4) with (1.7) and the boundary conditions (1.5) or (1.6) are transformed into

$$\begin{cases} u_t = d\Delta u + f(u) - u/\tau + \beta W, \\ \tau W_t = \Delta W - \beta \Delta u \end{cases} \qquad (x \in \Omega), \tag{2.5}$$

$$\langle W(\cdot,t)\rangle = \frac{s}{\alpha} \tag{2.6}$$

with the Neumann or the periodic boundary conditions for both u and W.

The associate Lyapunov function turns to be

$$\mathcal{E}_2(u,W) := \int_\Omega \left\{ \frac{d}{2}|\nabla u|^2 - F(u) + \frac{d}{2}u^2 + \frac{1}{2}(W - \beta u)^2 \right\} dx \tag{2.7}$$

In what follows the stationary equations of (2.5) are obtained as the variational equations of (2.7). We introduce function spaces. Let

$$\overline{L}^2 := \{w \in L^2(\Omega) : \langle w \rangle = 0\}, \qquad \overline{H}^k := \{w \in H^k(\Omega) : \langle w \rangle = 0\} \quad (k \geq 1).$$

and the closed operator A_0 in $\overline{L}^2$ with the domain

$$\mathcal{D}(A_0) := \overline{H}^2_N := \{u \in \overline{H}^2 : \partial u/\partial \nu = 0 \quad (x \in \partial\Omega)\}$$

and the range $\mathcal{R}(A_0) = \overline{L}^2$ such that

$$A_0 u = -\Delta u \quad (u \in \mathcal{D}(A_0)).$$

We use the notation

$$\|u\| := \left(\int_\Omega |u(x)|^2 dx \right)^{1/2}, \quad (u,v)_{L^2} := \int_\Omega u(x)v(x)dx.$$

We note that the fractional power operator A_0^α $(0 < \alpha \leq 1)$ is self-adjoint, i.e.,

$$(A_0^\alpha u, v)_{L^2} = (u, A_0^\alpha v)_{L^2}.$$

Henceforth we fix s so that $\langle W \rangle = s/\tau$. For $\psi' \in \overline{H}^1$ there is a unique function Ψ which solves

$$\psi' = -\Delta\Psi \quad (x \in \Omega), \qquad \frac{\partial \Psi}{\partial \nu} = 0 \quad (x \in \partial\Omega), \quad \langle \Psi \rangle = 0 \tag{2.8}$$

For $(\phi, \psi') \in H^1 \times \overline{H}^1$, with Ψ of (2.8) we can compute

$$\begin{aligned}
&\frac{d}{d\eta}\mathcal{E}_2(u + \eta\phi, W + \eta\psi')_{|\eta=0} \\
&= \int_\Omega \{d\nabla u \cdot \nabla\phi - F'(u)\phi + du\phi + (W - \beta u)(\psi' - \beta\phi)\}dx \\
&= \int_\Omega \{d\nabla u \cdot \nabla\phi - f(u)\phi + du\phi + (W - \beta u)(-\Delta\Psi) - \beta W\phi + \beta^2 u\phi\}dx \\
&= \int_\Omega \{d\nabla u \cdot \nabla\phi - f(u)\phi + \frac{1}{\tau}u\phi - \beta W\phi + \nabla(W - \beta u) \cdot \nabla\Psi\}dx
\end{aligned}$$

By this we obtain the Euler-Lagrange equations as

$$\begin{cases} d\Delta u + f(u) - u/\tau + \beta W = 0, \\ \Delta W - \beta \Delta u = 0 \end{cases} \qquad (x \in \Omega), \tag{2.9}$$

with the boundary conditions. This system is the stationary equations of (2.5).

Next we compute the second variation around the equilibrium solution $(u^*(x), W^*(x))$.

$$\begin{aligned}
&\frac{1}{2}\frac{d^2}{d\eta^2}\mathcal{E}_2(u^* + \eta\phi, W^* + \eta\psi')_{|\eta=0} \\
&= \int_\Omega \{d|\nabla\phi|^2 - f'(u^*)|\phi|^2 + d|\phi|^2 + (\psi' - \beta\phi)^2\}dx \qquad (2.10) \\
&= \int_\Omega \{d|\nabla\phi|^2 - f'(u^*)|\phi|^2 + \frac{1}{\tau}|\phi|^2 - 2\beta\psi'\phi + \psi'^2\}dx
\end{aligned}$$

Putting

$$\psi' = A_0^{1/2}\Psi, \tag{2.11}$$

(not as in (2.8)) and using

$$\phi^Q := \phi - \langle \phi \rangle,$$

we can write (2.10) as

$$\begin{aligned}
\mathcal{K}[\phi, \Psi] := &\int_\Omega \{d|\nabla\phi|^2 - f'(u^*)|\phi|^2 + \frac{1}{\tau}|\phi|^2\}dx \\
&- 2\beta(A_0^{1/2}\Psi, \phi^Q)_{L^2} + \|A_0^{1/2}\Psi\|^2, \qquad (2.12)
\end{aligned}$$

where we used

$$(A_0^{1/2}\Psi, \phi)_{L^2} = (A_0^{1/2}\Psi, \phi^Q)_{L^2}.$$

Define the Rayleigh quotient

$$\mathcal{R}[\phi, \Psi] := \frac{\mathcal{K}[\phi, \Psi]}{\|\phi\|^2 + \tau\|\Psi\|^2}. \tag{2.13}$$

Any critical point of the minimizing problem of this Rayleigh quotient in $(\phi,\Psi)\in H^1\times\overline{H}^1$ $((\phi,\Psi)\neq(0,0))$ leads to the linearized eigenvalue problem (1.16) except for the zero corresponding to the eigenfunction $(\partial u^*/\partial s,\partial v^*/\partial s)$. To prove it, we compute the variation

$$\begin{aligned}
&\frac{1}{2}\frac{d^2}{d\eta^2}\mathcal{K}[\phi+\eta\phi_1,\Psi+\eta\Psi_1]_{|\eta=0}\\
&=\int_\Omega\{d\nabla\phi\cdot\nabla\phi_1-f'(u^*)\phi\phi_1+\frac{1}{\tau}\phi\phi_1\}dx\\
&\quad-\beta(A_0^{1/2}\Psi,\phi_1^Q)_{L^2}-\beta(A_0^{1/2}\Psi_1,\phi^Q)_{L^2}+(A_0^{1/2}\Psi,A_0^{1/2}\Psi_1)_{L^2}\\
&=\int_\Omega\{d\nabla\phi\cdot\nabla\phi_1-f'(u^*)\phi\phi_1+\frac{1}{\tau}\phi\phi_1\}dx\\
&\quad-\beta(A_0^{1/2}\Psi,\phi_1)_{L^2}+(A_0^{1/2}(A_0^{1/2}\Psi-\beta\phi^Q),\Psi_1)_{L^2},
\end{aligned}$$

while

$$\frac{1}{2}\frac{d}{d\eta}(\|\phi+\eta\phi_1\|^2+\tau\|\Psi+\eta\Psi_1\|^2)_{|\eta=0}=(\phi,\phi_1)_{L^2}+\tau(\Psi,\Psi_1)_{L^2}$$

Thus the Euler-Lagrange equations of the minimization problem is given as

$$\begin{cases}-(d\Delta\phi+f'(u^*)\phi-\frac{1}{\tau}\phi+\beta A_0^{1/2}\Psi)=\lambda\phi\\ A_0\Psi-\beta A_0^{1/2}\phi^Q=\tau\lambda\Psi\end{cases}\tag{2.14}$$

with the boundary conditions

$$\frac{\partial\phi}{\partial\nu}=0\quad(x\in\partial\Omega)$$

for ϕ. For the solutions of (2.14) we see $\Psi\in\mathcal{D}(A_0^{3/2})$ from the bootstrap argument. Acting $A_0^{1/2}$ on the second equation and putting $\psi'=A_0^{1/2}\Psi$ yield

$$\begin{cases}-(d\Delta\phi+f'(u^*)\phi-\frac{1}{\tau}\phi+\beta\psi')=\lambda\phi,\\ -\Delta\psi'+\beta\Delta\phi=\tau\lambda\psi',\end{cases}\tag{2.15}$$

with the boundary conditions and the constraint

$$\langle\psi'\rangle=0.\tag{2.16}$$

This eigenvalue problem (2.15) with (2.16) is nothing but the one around the solution (u^*,W^*) of (2.9), and it is equivalent to (1.16) with (1.21) through the relation

$$\psi'=\frac{1}{\alpha}(\phi+\tau\psi).$$

Note that defining

$$\tilde{\mathcal{L}}\begin{pmatrix}\phi\\ \Psi\end{pmatrix}:=\begin{pmatrix}-(d\Delta\phi+f'(u^*)\phi-\frac{1}{\tau}\phi+\beta A_0^{1/2}\Psi)\\ A_0\Psi-\beta A_0^{1/2}\phi^Q\end{pmatrix},$$

and

$$\left\{\begin{pmatrix}\phi_1\\ \Psi_1\end{pmatrix},\begin{pmatrix}\phi_2\\ \Psi_2\end{pmatrix}\right\}_{L^2\times L^2}:=(\phi_1,\Psi_1)_{L^2}+(\phi_2,\Psi_2)_{L^2}$$

we can check $\tilde{\mathcal{L}}$ satisfy

$$\left\{\tilde{\mathcal{L}}\begin{pmatrix}\phi_1\\ \Psi_1\end{pmatrix},\begin{pmatrix}\phi_2\\ \Psi_2\end{pmatrix}\right\}_{L^2\times L^2}=\left\{\begin{pmatrix}\phi_1\\ \Psi_1\end{pmatrix},\tilde{\mathcal{L}}\begin{pmatrix}\phi_2\\ \Psi_2\end{pmatrix}\right\}_{L^2\times L^2}$$

in $H_N^2\times\mathcal{D}(A_0)$.

3. Proof of the main Theorem

Define

$$K_0[\varphi]:=\int_\Omega\{d|\nabla\varphi|^2-(f'(u^*)-d)|\varphi|^2\}dx+\frac{1-\tau d}{\tau}|\Omega|\langle\varphi\rangle^2. \tag{3.1}$$

Lemma 3.1. *For $\mathcal{K}[\cdot,\cdot]$ and $K_0[\cdot]$*

$$\mathcal{K}[\phi,\Psi]=K_0[\phi]+\|A_0^{1/2}\Psi-\beta\phi^Q\|^2 \tag{3.2}$$

holds.

Proof. The assertion follows from the computation

$$\begin{aligned}&\|A_0^{1/2}\Psi\|^2-2\beta(A_0^{1/2}\Psi,\phi^Q)_{L^2}\\ =&\|A_0^{1/2}\Psi-\beta\phi^Q\|^2-\beta^2\|\phi^Q\|^2\\ =&\|A_0^{1/2}\Psi-\beta\phi^Q\|^2-\frac{1-\tau d}{\tau}\left\{\|\phi\|^2-\frac{1}{|\Omega|}\left(\int_\Omega\phi dx\right)^2\right\}\end{aligned}$$

and the definitions of $\mathcal{K}[\cdot,\cdot]$ and $K_0[\cdot]$.

□

We let

$$R_0[\varphi]:=\frac{K_0[\varphi]}{\|\varphi\|^2}. \tag{3.3}$$

The associate eigenvalue problem is given by (1.22).

We denote

$$\begin{aligned}&\mathcal{M}_n:\text{a set of all } n-\text{dimensional subspaces in } L^2\times\overline{L}^2,\\ &\mathcal{N}_n:\text{a set of all } n-\text{dimensional subspaces in } L^2.\end{aligned}$$

We use the notation $\varphi\perp\tilde{\varphi}$ which implies $(\varphi,\tilde{\varphi})_{L^2}=0$. Similarly,

$$(\phi,\Psi)\perp(\tilde{\phi},\tilde{\Psi})$$

implies

$$(\phi,\tilde{\phi})_{L^2}+(\Psi,\tilde{\Psi})_{L^2}=0.$$

We also denote by $Z^\perp$ the orthogonal space of Z of L^2 or $L^2\times\overline{L}^2$.

Recall $\{\lambda_j\}$ and $\{\mu_j\}$ are the sets of the eigenvalues of $\mathcal{L}$ with (1.21) and L_0 respectively, which are arranged in the nondecreasing order and $\{(\phi_j,\psi_j)\}$ and $\{\varphi_j\}$

are the corresponding eigenfunctions. Then the eigenfucntions to (2.14) are given by

$$(\phi_j, \Psi_j) = (\phi_j, \frac{1}{\alpha} A_0^{-1/2}(\phi_j + \tau\psi_j)) \quad (j = 1, 2, \ldots).$$

Those eigenvalues are variationally achieved as

$$\begin{aligned} \lambda_n &= \inf\{\mathcal{R}[\phi, \Psi] : (\phi, \Psi) \in H^1 \times \overline{H}^1, (\phi, \Psi) \neq (0,0), \\ &\qquad\qquad (\phi, \Psi) \perp (\phi_j, \Psi_j) \quad (j \leq n-1)\} \\ \mu_n &= \inf\{R_0[\varphi] : \varphi \in H^1, \varphi \neq 0, \varphi \perp \varphi_j \quad (j \leq n-1)\} \end{aligned}$$

and

$$\begin{aligned} \lambda_n &= \sup_{X_{n-1} \in \mathcal{M}_{n-1}} \inf\{\mathcal{R}[\phi, \Psi] : (\phi, \Psi) \in H^1 \times \overline{H}^1, (\phi, \Psi) \neq (0,0), \\ &\qquad\qquad (\phi, \Psi) \in X_{n-1}^{\perp}\} \\ \mu_n &= \sup_{Y_{n-1} \in \mathcal{N}_{n-1}} \inf\{R_0[\varphi] : \varphi \in H^1, \varphi \neq 0, \varphi \in Y_{n-1}^{\perp}\} \end{aligned}$$

We also recall the Min-Max principle such as

$$\begin{aligned} \lambda_n &= \inf_{X_n \in \mathcal{M}_n} \sup\{\mathcal{R}[\phi, \Psi] : (\phi, \Psi) \in X_n, (\phi, \Psi) \neq (0,0)\}, \\ \mu_n &= \inf_{Y_n \in \mathcal{N}_n} \sup\{R_0[\varphi] : \varphi \in Y_n, \varphi \neq 0\} \end{aligned}$$

(see Davies [4]).

Now we are ready to prove the theorem.

Proof of Theorem 1.1. First we observe

$$\lambda_1 = \mathcal{R}[\phi_1, \Psi_1] \geq \frac{K_0[\phi_1]}{\|\phi_1\|^2 + \tau\|\Psi_1\|^2} \geq \frac{\mu_1\|\phi_1\|^2}{\|\phi_1\|^2 + \tau\|\Psi_1\|^2}. \tag{3.4}$$

This yields

$$\mu_1 > 0 \text{ (resp. } \mu_1 = 0) \implies \lambda_1 > 0 \text{ (resp. } \lambda_1 \geq 0), \tag{3.5}$$

$$\lambda_1 < 0 \text{ (resp. } \lambda_1 = 0) \implies \mu_1 < \lambda_1 < 0 \text{ (resp. } \mu_1 \leq 0). \tag{3.6}$$

We can exclude the possibility $\mu_1 = \lambda_1$ in (3.6) if $\lambda_1 < 0$. Indeed, if this equality holds, then $\Psi = 0$ and $\phi^Q = 0$, that is, ϕ must be constant, which is not allowed by the nonconstant u^*.

Next, for a positive integer k define a k-dimensional subspace of $H^1 \times \overline{H}^1$ as

$$X'_k := L.H.\{(\varphi_1, \beta A_0^{-1/2}\varphi_1^Q), (\varphi_2, \beta A_0^{-1/2}\varphi_2^Q), \ldots, (\varphi_k, \beta A_0^{-1/2}\varphi_k^Q)\}. \tag{3.7}$$

For any $(\varphi, \Psi) \in X'_k$ there is a family of constants $\{c_j\}$ such that

$$(\varphi, \Psi) = \sum_{j=1}^{k} c_j(\varphi_j, \beta A_0^{-1/2}\varphi_j^Q).$$

With the aid of this expression we compute

$$\begin{aligned}\mathcal{R}[\varphi,\Psi] &= \frac{K_0[\sum_{j=1}^k c_j\varphi_j]}{\|\sum_{j=1}^k c_j\varphi_j\|^2+\tau\|\sum_{j=1}^k c_j\beta A_0^{-1/2}\varphi_j^Q\|^2}\\ &= \frac{\sum_{j=1}^k c_j^2\mu_j\|\varphi_j\|^2}{\sum_{j=1}^k c_j^2\|\varphi_j\|^2+\tau\beta^2\|A_0^{-1/2}\sum_{j=1}^k c_j\varphi_j^Q\|^2}\\ &\le \frac{\mu_k\sum_{j=1}^k c_j^2\|\varphi_j\|^2}{\sum_{j=1}^k c_j^2\|\varphi_j\|^2+\tau\beta^2\|A_0^{-1/2}\sum_{j=1}^k c_j\varphi_j^Q\|^2} \qquad (3.8)\end{aligned}$$

We take $(c_1,c_2,\dots,c_k)\neq(0,0,\dots,0)$ so that $\tilde\varphi:=\sum_{j=1}^k c_j\varphi_j$ achieves the maximum of $\mathcal{R}[\varphi,\Psi]$ in X_k', that is,

$$\sup\{\mathcal{R}[\varphi,\Psi]:(\varphi,\Psi)\in X_k',(\varphi,\Psi)\neq(0,0)\}=\mathcal{R}[\tilde\varphi,\beta A_0^{-1/2}\tilde\varphi^Q]. \qquad (3.9)$$

If $\lambda_k>0$, then by the Mini-Max principle and (3.8) we obtain

$$0<\lambda_k\le\mathcal{R}[\tilde\varphi,\beta A_0^{-1/2}\tilde\varphi^Q]<\mu_k, \qquad (3.10)$$

where $\lambda_k=\mu_k$ is not allowed since

$$\lambda_k=\mathcal{R}[\tilde\varphi,0]=\mu_k$$

implies that $\tilde\varphi$ is constant. On the other hand, if $\mu_k<0$, then $\lambda_k<0$ by (3.8) and (3.9). In addition $\lambda_k=0$ yields $\mu_k\ge 0$ while $\lambda\le 0$ if $\mu_k=0$. Summarizing these properties, we obtain

$$\lambda_k>0\ \ (\text{resp. } \lambda_k=0)\Longrightarrow\ 0<\lambda_k<\mu_k\ \ (\text{resp. } \mu_k\ge 0) \qquad (3.11)$$
$$\mu_k<0\ \ (\text{resp. } \mu_k=0)\Longrightarrow\ \lambda_k<0\ \ (\text{resp. } \lambda_k\le 0). \qquad (3.12)$$

Hence, for $k=1$, combining (3.5), (3.6), (3.11) and (3.12) together, we obtain that for $\lambda_1\neq 0$ or $\mu_1\neq 0$,

$$\lambda_1\mu_1>0,\quad |\lambda_1|<|\mu_1| \qquad (3.13)$$

holds. In the case $\mu_1=0$ we use (3.5) and (3.11) to obtain $\lambda_1=0$ while $\lambda_1=0$ implies $\mu_1=0$ by (3.6) and (3.12). Thus we conclude the proof for the case $k=1$.

In the case $k\ge 2$ we need another inequality. We put

$$\begin{aligned}&\tilde Y_{k-1}:=L.H.\{\varphi_1,\varphi_2,\dots,\varphi_{k-1}\}\subset H^1,\\ &\tilde X_{k-1}:=L.H.\{(\varphi_1,0),(\varphi_2,0),\dots,(\varphi_{k-1},0)\}\subset\overline{H}^1.\end{aligned}$$

For any $(\phi,\Psi)\in\tilde X_{k-1}^\perp$ with $(\phi,\Psi)\neq(0,0)$, since $\phi\in\tilde Y_{k-1}^\perp$,

$$\begin{aligned}\mathcal{R}[\phi,\Psi] &= \frac{K_0[\phi]+\|A_0^{1/2}\Psi-\beta\phi^Q\|^2}{\|\phi\|^2+\tau\|\Psi\|^2}\\ &\ge \frac{K_0[\phi]}{\|\phi\|^2+\tau\|\Psi\|^2}\ge\frac{\mu_k\|\phi\|^2}{\|\phi\|^2+\tau\|\Psi\|^2} \qquad (3.14)\end{aligned}$$

holds, while letting $(\tilde\phi,\tilde\Psi)$ be a minimizer of $\mathcal{R}[\phi,\Psi]$ over the space $\tilde X_{k-1}^\perp$, we have

$$\mathcal{R}[\tilde\phi,\tilde\Psi]=\inf\{\mathcal{R}[\phi,\Psi]:(\phi,\Psi)\in\tilde X_{k-1}^\perp,(\phi,\Psi)\neq(0,0)\}\le\lambda_k \qquad (3.15)$$

By virtue of (3.14) and (3.15), for $k \geq 2$ we obtain

$$\mu_k > 0 \ \ (\text{resp. } \mu_k = 0) \Longrightarrow \ \ \lambda_k > 0 \ \ (\text{resp. } \lambda_k \geq 0), \tag{3.16}$$

$$\lambda_k < 0 \ \ (\text{resp. } \lambda_k = 0) \Longrightarrow \ \ \mu_k < \lambda_k < 0 \ \ (\text{resp. } \mu_k \leq 0). \tag{3.17}$$

We can exclude the case $\mu_k = \lambda_k = 0$ in (3.17) if $\lambda_k < 0$ by the same reason in the previous inequalities (3.6) and (3.11). It is not difficult to see from (3.11), (3.12), (3.16) and (3.17) yields that if $\lambda_k \neq 0$ or $\mu_k \neq 0$, then

$$\lambda_k \mu_k > 0, \quad |\lambda_k| < |\mu_k| \quad (k \geq 2), \tag{3.18}$$

and

$$\mu_k = 0 \Longleftrightarrow \lambda_k = 0.$$

Consequently, we get to the desired conclusion for the proof of Theorem 1.1. □

Proof of Corollary 1.1. We consider the Neumann case. In the absence of the constraint (1.21) we cannot use the variational formulation as in the proof of Theorem 1.1. We need to consider (2.15) under the condition $\langle \psi' \rangle \neq 0$. Then the problem is reduced to the case $\lambda = 0$ in (2.15) since

$$0 = \int_\Omega \Delta \psi' dx - \beta \int_\Omega \Delta \phi dx = \tau \lambda \int_\Omega \psi' dx.$$

If L_0 has no zero eigenvalues, we apply Lemma 3.1 in Morita & Ogawa [13] to obtain the simplicity of the zero eigenvalue of $\mathcal{L}$. In the remaining part we prove the assertion that L_0 has no zero eigenvalue if $\mathcal{L}$ has the simple zero eigenvalue. By (2.15) we solve

$$d\Delta\phi + f'(u^*)\phi - \frac{1}{\tau}\phi + \beta\psi' = 0, \tag{3.19}$$

$$\Delta\psi' - \beta\Delta\phi = 0 \tag{3.20}$$

with

$$\langle \psi' \rangle = 1$$

From (3.20)

$$\psi' - \beta\phi = \langle \psi' \rangle - \beta\langle \phi \rangle = 1 - \beta\langle \phi \rangle.$$

Thus (3.19) turns to be

$$-L_0[\phi] = d\Delta\phi + f'(u^*)\phi - d\phi - \frac{1 - \tau d}{\tau}\langle \phi \rangle = \beta \tag{3.21}$$

(recall $\beta = \sqrt{(1 - \tau d)/\tau}$). Let $\tilde{\phi}$ be a solution to (3.21). Then $(\phi, \psi') = (\tilde{\phi}, \beta\tilde{\phi} - \beta\langle \tilde{\phi} \rangle + 1)$ gives an eigenfunction of $\mathcal{L}$ corresponding to the simple zero eigenvalue. If L_0 has a zero eigenvalue $\mu_k = 0$ with the corresponding eigenfunction φ_k, then $\langle \varphi_k \rangle = 0$ by the solvability condition for (3.21). Thus

$$(\phi, \psi') = (\tilde{\phi}, \beta\tilde{\phi} - \beta\langle \tilde{\phi} \rangle + 1) + (\varphi_k, \beta\varphi_k)$$

also gives an eigenfunction. This contradicts the simplicity of the zero eigenvalue. Combining this observation with the assertion of Theorem 1.1, we conclude the proof of the corollary for the Neumann case.

As for the periodic boundary conditions we use Lemma 3.1 in Morita & Ogawa [13] again, that is, if L_0 has a simple zero eigenvalue with the eigenfunction $\phi = u_x^*$, then the zero eigenvalues of $\mathcal{L}$ has the geometric multiplicity 2. Then, to complete the proof, we use the above argument for the Neumann case with a slight modification. Since the argument is simple, we leave the detail to the readers. □

Proof of Corollary 1.2. Let $(u^*(x), v^*(x))$ be a nonconstant solution of (1.11) with (1.12). If the first eigenvalue μ_1 of the linearized operator L_0 for u^* is negative, then Theorem 1.1 tells the nonconstant solution is unstable. Thus, it suffices to consider the eigenvalue problem for L_0. In the Neumann case the proof for the desired result for L_0 follows from Theorem 3 of Suzuki & Tasaki [15] or the argument in Gurtin & Matano [7].

Here, we give a proof for the periodic case. For the nonconstant solution u^*, assume u_x^* has two zeros in $(0, L)$. L_0 has a zero eigenvalue and the corresponding eigenfunction u_x^*. We also consider the Rayleigh quotient

$$R_{00}[\zeta] := \frac{K_{00}[\zeta]}{\|\zeta\|^2}, \quad K_{00}[\zeta] = \int_0^L \{d|\zeta_x|^2 - (f'(u^*) - d)\zeta^2\}dx$$

and the associate eigenvalue problem

$$L_{00}[\zeta] = -(\zeta_{xx} + f'(u^*)\zeta - d\zeta) = \sigma\zeta,$$

with the periodic boundary conditions. Then $L_0[u_x^*] = L_{00}[u_x^*] = 0$. Since u_x^* has two zeros in $(0, L)$, the first and the second eigenvalues σ_1, σ_2 satisfy

$$\sigma_1 < \sigma_2 < 0.$$

We let ζ_1 and ζ_2 be the corresponding eigenfunctions. Then we can take c so that $\langle c\zeta_1 + \zeta_2 \rangle = 0$. Thus

$$K_0[c\zeta_1 + \zeta_2] = c^2 K_{00}[\zeta_1] + K_{00}[\zeta_2] = c^2\sigma_1\|\zeta_1\|^2 + \sigma_2\|\zeta_2\|^2 < 0$$

This implies the instability of the solution. □

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