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CAUCHY PROBLEM FOR THE ZAKHAROV SYSTEM ARISING FROM ION-ACOUSTIC MODES WITH LOW REGULARITY DATA*

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Abstract We prove local well-posedness results for the Zakharov System Arising from Ion-Acoustic Modes in two spacial dimension with large initial data in low regularity Sobolev space $(\dot{H}^1 \cup H^{\frac{1}{2}}) \times L^2 \times H^{-1}$. Using "derivative sharing", the local well-posedness results in $(\dot{H}^1 \cup H^{\frac{1}{2}-\delta}) \times H^{\delta} \times H^{-1+\delta}$ are also obtained, for any $0 \le \delta \le \frac{1}{2}$.

Keywords Zakharov System, Bourgain space, Cauchy problem, Local well-posedness.

MSC(2000) 35Q55, 35L70.

1. Introduction

In this paper, we consider the Cauchy problem for the Zakharov System Arising from Ion-Acoustic Modes (IZS) in two spacial dimension:

$$i\triangle\varphi_t + \triangle^2\varphi + i\alpha\triangle\varphi + i\beta\varphi + \frac{1}{i}\nabla\varphi \cdot \overline{\nabla}n = 0, \tag{1.1}$$

$$n_{tt} - \triangle n + \gamma n_t + \frac{\omega}{i}\nabla\varphi^* \cdot \overline{\nabla}\varphi = 0, \tag{1.2}$$

$$\varphi(0,x) = \varphi_0(x), n(0,x) = n_0(x), n_t(0,x) = n_1(x), \tag{1.3}$$

where $\varphi(t,x)$ is a complex valued function of $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^2$, $\mathbb{R}_+ := [0,+\infty]$, φ^* is the complex conjugation of φ. $n(t,x)$ is a real valued function of $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^2$, α, β, γ and ω are real constants $\beta < 0, \omega > 0, i = \sqrt{-1}$,

$$\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}), \overline{\nabla} = (\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_1}), \triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}.$$

The system of (1.1) and (1.2) arises in the study of plasma physics, which describes the modulation instability and collapse of wave in the dynamics of strong Langmuir

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turbulence. In physics, it can be used to discuss the modulation instability of lower-hybrid waves in the auroral region of the Earth's ionosphere. We can refer to [15] for the physical background of this mode.

The local well-posedness of classical Zakharov system

$$\begin{cases} i\mathcal{E}_t + \Delta\mathcal{E} - n\mathcal{E} = 0, \\ \alpha^{-2} n_{tt} - \Delta n = \Delta|\mathcal{E}|^2 \end{cases} \tag{1.4}$$

in 2 dimension space were wildly studied by many authors. Most of them used the Fourier restriction norm method. Bourgain-Colliander in [4] proved the local well-posedness for (1.4) in spaces which comprise the energy space and they also obtained global well-posedness in the energy space under some assumption. Later, the local result was improved by Ginibre-Tsutsumi-Velo in [6] to $H^{1/2} \times L^2 \times H^{-1}$. Recently, applying angular frequency decomposition, Bejenaru-Herr-Holmer-Tataru in [2] improved the local result to $L^2 \times H^{-1/2} \times H^{-3/2}$, one-half derivative better than [6]. They also show it is the space of optimal regularity in the sense that the data-to-solution map fails to be smooth at the origin for any rough pair of spaces in the L^2-based Sobolev scale. And soon Bejenaru-Herr in [3] extend their previous work to higher dimension case.

For (IZS) equation (1.1)–(1.3), using compactness method, Guo and Yuan [7] studied the well-posedness of smooth solutions and showed that (1.1)–(1.3) has a unique global solution, when the initial data belongs to $H^{m+2} \times H^{m+1} \times H^m$, $m \in \mathbb{N}$. However, there is no result about (1.1)–(1.3) with low regularity data. In this paper, we will consider the local well-posedness for system (1.1) –(1.3). We get the following results:

Theorem 1.1. *When the initial data* (φ_0, n_0, n_1) *belong to* $(\dot{H}^1 \cup H^{\frac{1}{2}}) \times L^2 \times H^{-1}$, *then there exists a constant* T *and a unique solution* $(\varphi, n) \in X_T \cap H_T^{\frac{1}{2},0}$ *to the Cauchy problem of* (1.1)–(1.3). *The space* $H_T^{k,\ l}$ *is defined as the Banach space of all pairs of space-time distributions* (φ, n)

$$\begin{aligned} &\varphi \in C([0,T]; \dot{H}^1 \cup H^k(\mathbb{R}^2; C)), \\ &n \in C([0,T]; H^l(\mathbb{R}^2; \mathbb{R})) \cap C^1([0,T]; H^{l-1}(\mathbb{R}^2; \mathbb{R})), \end{aligned} \tag{1.5}$$

endowed with the standard norm defined as

$$\|(\varphi, n)\|^2_{H_T^{k,l}} = \|\varphi\|^2_{L^\infty([0,T]; \dot{H}^1 \cup H^k)} + \|n\|^2_{L^\infty([0,T]; H^l)} + \|n_t\|^2_{L^\infty([0,T]; H^{l-1})}. \tag{1.6}$$

The definition of X_T *can be found in Definition 2.1 and Remark 2.1.*

Moreover, the map $(\varphi_0, n_0, n_1) \mapsto (\varphi, n)$ *is locally Lipschitz-continuous.*

Corollary 1.1. *In fact, we can also prove that, when the initial data* (φ_0, n_0, n_1) *belong to* $(\dot{H}^1 \cup H^{\frac{1}{2}-\delta}) \times H^\delta \times H^{-1+\delta}$, *for all* $0 \le \delta \le \frac{1}{2}$. *then there exists a unique solution* $(\varphi, n) \in X_T^\delta \cap H_T^{\frac{1}{2}-\delta,\ \delta}$ *to the Cauchy problem of* (1.1)–(1.3). *The definition of* X_T^δ *can also be found in Definition 2.1 and Remark 2.1.*

Remark. Using "derivative sharing", we can obtain this Corollary from similar proof as Theorem 1.1(See case 1 in Proposition 3.3 for detail).

Our main tools in this paper are angular frequency decomposition and dyadic Bourgain space. It seems that if we apply the method in [2] directly to solve equation (1.1)–(1.3), we can only solve the problem with initial data in $\dot{H}^1 \times H^{1/2} \times H^{-1/2}$, which is worse than our Theorem 1.1 and 1.2. We will treat the low frequency part specially in this paper.

Now I will give a sketch explanation on our proof. First we apply the standard procedure to factor the wave operator in order to derive a first order system (see also [6]). Suppose that (φ, n) is a sufficiently regular solution to (IZS), we define $v = n + \mathrm{i}\langle\nabla\rangle^{-1}\partial_t n$, where $\langle\nabla\rangle = (1-\triangle)^{1/2}$, then we can write Eq. (1.1)–(1.3) as following:

$$\mathrm{i}\varphi_t + \triangle\varphi + \mathrm{i}\beta\triangle^{-1}\varphi = -\mathrm{i}\alpha\varphi + \mathrm{i}\triangle^{-1}[\nabla\varphi\cdot\overline{\nabla}(\mathrm{Re}v)], \tag{1.7}$$

$$-\mathrm{i}v_t + \langle\nabla\rangle v = -\mathrm{i}\omega\langle\nabla\rangle^{-1}[\nabla\varphi^*\cdot\overline{\nabla}\varphi] + \langle\nabla\rangle^{-1}\mathrm{Re}v + \mathrm{i}\gamma(v-\mathrm{Re}v) \tag{1.8}$$

In this way, if (φ, v) is a solution to Eq. (1.7)–(1.8) with the initial data (φ_0, v_0), we can obtain a solution to the original system (IZS) by setting $n = \mathrm{Re}v$. So it is convenient for us to study the system (1.7)–(1.8) instead of the original system (1.1)–(1.3).

Then we write Eq. (1.7)–(1.8) into integral equation (3.1) and (3.2). Then we show the linear estimates for the semigroup $S_\beta := e^{\mathrm{i}t\triangle}e^{\beta t\triangle^{-1}}u_0$ and integral operator $I^{S_\beta}(f) := \int_0^t S_\beta(t-\tau)f(\tau)d\tau$. (see Proposition 3.1 and 3.2)

Next we show multilinear estimates for the nonlinear term. Noticing that, in Eq. (1.7), there is one order derivative in each of φ and v, and there is also two order negative derivative before them. Using Bourgain space method we can see the so called "high× high → low interactions" will occur, which is the worst case. Our idea is dividing the nonlinear term into high frequency part and low frequency part, then estimate them in different space, respectively. So we will estimate $\|P_{\geq 2}\triangle^{-1}[\nabla\varphi\cdot\overline{\nabla}(\mathrm{Re}v)]\|_{X^S_{0,\frac{5}{12},1}}$ and $\|P_{\lesssim 1}\triangle^{-1}[\nabla\varphi\cdot\overline{\nabla}(\mathrm{Re}v)]\|_{Y_i}, i = 1,2$, respectively, where Y_i is used to control low frequency term, while $X^S_{0,\frac{5}{12},1}$ is used to control high frequency term (see Proposition 3.3 and 3.4 for detail).

2. Notations and function spaces

In the sequel C will denote a universal positive constant which can be different at each appearance. $x \lesssim y$ (for x, $y > 0$) means that $x \leq Cy$, and $x \sim y$ stands for $x \lesssim y$ and $y \lesssim x$. Throughout this work we will denote dyadic numbers $2^n, n \in \mathbb{Z}$ by capital letters, this means, we write $N = 2^n$, $L = 2^l$ and so on.

For any $s \in \mathbb{R}$, define homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^2)$ as $(-\triangle)^{-s/2}L^2$, and define inhomogeneous Sobolev space $H^s(\mathbb{R}^2)$ as $(1-\triangle)^{-s/2}L^2$, respectively.

Next we introduce the Littlewood-Paley decomposition. Let $\eta \in C_0^\infty$ be an even, non-negative function with the property $\eta(\xi) = 1$ for $|\xi| \leq 1$ and $\mathrm{supp}\eta \subset [-2,2]$. Then write $\eta_1 = \eta$ and $\eta_N(\xi) = \eta(\frac{\xi}{N}) - \eta(\frac{2\xi}{N})$ for $N = 2^n \geq 2$. In this way we have $1 = \sum_{N\geq 1}\eta_N$. Define dyadic frequency localization operators P_N by

$$P_N f(x) = \mathscr{F}_x^{-1}[\eta_N(|\xi|)\mathscr{F}_x f(\xi)](x).$$

Define operator $P_{\leq M}, P_{\geq M}$ as

$$P_{\leq M} = \sum_{N\leq M} P_N, \quad P_{\geq M} = \sum_{N\geq M} P_N. \tag{2.1}$$

Then we can define the inhomogeneous(homogeneous) Besov space $\dot{B}^s_{2,1}(\mathbb{R})$ $(B^s_{2,1}(\mathbb{R}))$(see [16]) as the completion of $\mathscr{S}(\mathbb{R})$ with respect to the semi-norm

$$\|g\|_{B^s_{2,1}} = \sum_{L\geq 1} L^s \|P_L g\|_{L^2},$$
$$\|g\|_{\dot{B}^s_{2,1}} = \sum_{L=-\infty}^{+\infty} L^s \|P_L g\|_{L^2}, \quad L = 2^l, \quad l \in \mathbb{Z}. \tag{2.2}$$

We follow the notation in [2] and denote the Fourier support of P_N by the corresponding letter:

$$\Gamma_1 = \{(\xi,\tau) \in \mathbb{R}^2 \times \mathbb{R} \mid |\xi| \leq 2\},$$
$$\Gamma_N = \{(\xi,\tau) \in \mathbb{R}^2 \times \mathbb{R} \mid N/2 \leq |\xi| \leq 2N\}.$$

Moreover, for dyadic $L \geq 1$ we define the modulation localization operators as following

$$\mathscr{F}(S_L u)(\tau,\xi) = \eta_L(\tau + |\xi|^2)\mathscr{F}u(\tau,\xi) \tag{2.3}$$
$$\mathscr{F}(W_L u)(\tau,\xi) = \eta_L(\tau + |\xi|)\mathscr{F}u(\tau,\xi) \tag{2.4}$$

and the corresponding Fourier supports

$$\Lambda_1 = \{(\xi,\tau) \in \mathbb{R}^2 \times \mathbb{R} \mid |\tau + |\xi|^2| \leq 2\},$$
$$\Lambda_L = \{(\xi,\tau) \in \mathbb{R}^2 \times \mathbb{R} \mid L/2 \leq |\tau + |\xi|^2| \leq 2L\}, \quad L \geq 2,$$

and respectively

$$\Upsilon_1 = \{(\xi,\tau) \in \mathbb{R}^2 \times \mathbb{R} \mid |\tau + |\xi|| \leq 2\},$$
$$\Upsilon_L = \{(\xi,\tau) \in \mathbb{R}^2 \times \mathbb{R} \mid L/2 \leq |\tau + |\xi|| \leq 2L\}, \quad L \geq 2.$$

We also define an equidistant partition of unity in $\mathbb{R}$,

$$1 = \sum_{j\in\mathbb{Z}} \beta_j, \quad \beta_j(s) = \eta(s-j)\Big(\sum_{k\in\mathbb{Z}} \eta(s-k)\Big)^{-1}. \tag{2.5}$$

Finally, for $A \in \mathbb{N}$ we define an equidistant partition of unity on the unit circle,

$$1 = \sum_{j=0}^{A-1} \beta_j^A, \quad \beta_j^A(\theta) = \beta_j(\frac{A\theta}{\pi}) + \beta_{j-A}(\frac{A\theta}{\pi}).$$

Define

$$\Theta_j^A := [\frac{\pi}{A}(j-2), \frac{\pi}{A}(j+2)] \cap [-\pi + \frac{\pi}{A}(j-2), -\pi + \frac{\pi}{A}(j+2)].$$

We observe that $(\beta_j^A) \in \Theta_j^A$. We introduce the angular frequency localization operators Q_j^A,

$$\mathscr{F}_x(Q_j^A f)(\xi) = \beta_j^A(\theta)\mathscr{F}_x f(\xi), \quad \xi = |\xi|(\cos\theta, \sin\theta).$$

The operators $(Q_j^A u)(t,x)$ localize functions in frequency to the sets

$$\Omega_j^A = \{(|\xi|\cos\theta, |\xi|\sin\theta, \tau) \in \mathbb{R}^2 \times \mathbb{R} \mid \theta \in \Theta_j^A\}.$$

For $A \in \mathbb{N}$ we can now decompose $u : \mathbb{R}^2 \times \mathbb{R} \to C$ as

$$u = \sum_{j=0}^{A-1} Q_j^A u.$$

Now we define our resolution space. For $\delta, b \in \mathbb{R}$, $1 \le p < \infty$,

Definition 2.1.

$$\begin{aligned}
\|u\|_{X^S_{\delta,b,p}} &= \Big(\sum_{N\ge 1} N^{2\delta}\Big(\sum_{L\ge 1} L^{pb}\|S_L P_N u\|_{L_2}^p\Big)^{\frac{2}{p}}\Big)^{\frac12},\\
\|u\|_{X^W_{\delta,b,p}} &= \Big(\sum_{N\ge 1} N^{2\delta}\Big(\sum_{L\ge 1} L^{pb}\|W_L P_N u\|_{L_2}^p\Big)^{\frac{2}{p}}\Big)^{\frac12},\\
\|u\|_{Y_1} &= \|P_{\lesssim 1}\nabla u\|_{X^S_{\frac12,\frac12,1}} \le \sum_{L\ge 1} L^{1/2}\|S_L P_{\lesssim 1}\nabla u\|_{L_2}\\
\|u\|_{Y_2} &= \|P_{\lesssim 1}\overline{\nabla} u\|_{X^S_{\frac12,\frac12,1}} \le \sum_{L\ge 1} L^{1/2}\|S_L P_{\lesssim 1}\overline{\nabla} u\|_{L_2}\\
\|u\|_{X^S} &:= \|P_{\ge 2}u\|_{X^S_{\frac12,\frac12,1}} + \|u\|_{Y_1} + \|u\|_{Y_2},\\
\|u\|_{X^{\delta,S}} &:= \|P_{\ge 2}u\|_{X^S_{\frac12-\delta,\frac12,1}} + \|u\|_{Y_1} + \|u\|_{Y_2}.
\end{aligned}$$

For $T > 0$, we define the time-localized spaces $X^S_{\delta,b,p}(T)$ and $X^W_{\delta,b,p}(T)$ as

$$\begin{aligned}
\|u\|_{X^S_{\delta,b,p}(T)} &= \inf_{w\in X^S_{\delta,b,p}} \Big\{\|w\|_{X^S_{\delta,b,p}}, \quad w(t) = u(t) \text{ on } [0,T]\Big\},\\
\|u\|_{X^W_{\delta,b,p}(T)} &= \inf_{w\in X^W_{\delta,b,p}} \Big\{\|w\|_{X^W_{\delta,b,p}}, \quad w(t) = u(t) \text{ on } [0,T]\Big\}.
\end{aligned} \tag{2.6}$$

Remark 2.1. The class X_T in the statement of Theorem 1.1 can be chosen as all (φ, n) such that $\varphi \in X^S(T)$, $n \in X^W_{0,1/2,1}(T)$ and $\partial_t n \in X^W_{-1,1/2,1}(T)$.

From the definitions of v in Eq.(1.7)-(1.8), we can see if $v \in X^W_{0,1/2,1}(T)$ is a solution of Eq.(1.7)-(1.8), then $n = \mathrm{Re}v \in X^W_{0,1/2,1}(T)$ and $\partial_t n = \langle\nabla\rangle \mathrm{Im}v \in X^W_{-1,1/2,1}(T)$. Conversely, if $n \in X^W_{0,1/2,1}(T)$ and $\partial_t n \in X^W_{-1,1/2,1}(T)$, then from the definition of v, it is easy to see that $v \in X^W_{0,1/2,1}(T)$.

The class X_T^δ in the statement of Corollary 1.2 can be chosen as all (φ, n) such that $\varphi \in X^{\delta,S}(T)$, $n \in X^W_{\delta,1/2,1}(T)$ and $\partial_t n \in X^W_{-1+\delta,1/2,1}(T)$.

Since Schwartz functions $\mathscr{S}(\mathbb{R}^2 \times \mathbb{R})$ is dense in $X^S_{\delta,b,1}$ and $X^W_{\delta,b,1}$, respectively. It is enough to prove most of our estimates for smooth functions.

3. Linear and multilinear estimates

For $f\in\mathscr{S}(\mathbb{R}_+\times\mathbb{R}^2)$ and $t\in\mathbb{R}_+$, let

$$I^{S_\beta}(f)(t):=\int_0^t S_\beta(t-\tau)f(\tau)d\tau,$$
$$I^{W}(f)(t):=\int_0^t W(t-\tau)f(\tau)d\tau.$$

where $S_\beta(t)=e^{\mathrm{i}t\triangle}e^{\beta t\triangle^{-1}},\beta<0$, $W(t)=e^{-\mathrm{i}\langle\nabla\rangle}$.

We will write Eq. (1.7)–(1.8) into an integral equation

$$\varphi(t)=S_\beta(t)\varphi_0-\mathrm{i}I^{S_\beta}\Big(\alpha\varphi+\mathrm{i}\triangle^{-1}[\nabla\varphi\cdot\overline{\nabla}(\mathrm{Re}v)]\Big), \tag{3.1}$$
$$v(t)=W(t)v_0-\mathrm{i}I^{W}\Big(-\mathrm{i}\omega\langle\nabla\rangle^{-1}[\nabla\varphi^*\cdot\overline{\nabla}\varphi]+\langle\nabla\rangle^{-1}\mathrm{Re}v+\mathrm{i}\gamma(v-\mathrm{Re}v)\Big). \tag{3.2}$$

Set ψ is a smooth time cutoff function satisfying

$$\psi\in C_0^\infty(\mathbb{R}),\quad \mathrm{supp}\ \psi\subset[-2,2],\quad \psi\equiv1 \text{ on } [-1,1]. \tag{3.3}$$

Now we will show linear estimates for equation (3.1), the method is essential due to Molinet and Riboud [13], see also [10] and [8].

Proposition 3.1. *Let $s\in\mathbb{R}$, $\beta<0$, then there exists constant $C>0$ such that*

$$\|\psi(t)S_\beta(t)u_0\|_{X^S_{s,\frac12,1}}\le C\|u_0\|_{H^s}, \tag{3.4}$$

where $\psi(t)$ is defined in (3.3).

Proof. From Plancherel's identity, variable changing, and Young's inequality, we have

$$\begin{aligned}
&\|S_LP_N\psi(t)(e^{\mathrm{i}t\triangle}e^{\beta|t|\triangle^{-1}}u_0)\|_{L^2_{x,t}}\\
=&\|\eta_L(\tau-|\xi|^2)\eta_N(|\xi|)\mathscr{F}_t(e^{\mathrm{i}t|\xi|^2}\psi(t)e^{-|t|\frac{|\beta|}{|\xi|^2}})\widehat{u_0}\|_{L^2_{\xi,\tau}}\\
=&\|\eta_L(\tau)\eta_N(|\xi|)\mathscr{F}_t(\psi(t)e^{-|t|\frac{|\beta|}{|\xi|^2}})\widehat{u_0}\|_{L^2_{\xi,\tau}}\\
\le&\sup_{\xi\sim N}\|\eta_L(\tau)\mathscr{F}_t(\psi(t)e^{-|t|\frac{|\beta|}{|\xi|^2}})\|_{L^2_\tau}\|\eta_N(|\xi|)\widehat{u_0}\|_{L^2_\xi}.
\end{aligned} \tag{3.5}$$

From the definition of $X^S_{s,\frac12,1}$,

$$\begin{aligned}
&\|\psi(t)S_\beta(t)u_0\|_{X^S_{s,\frac12,1}}\\
\le&\Big(\sum_{N\ge1}N^{2s}\Big(\sum_{L\ge1}L^{\frac12}\|\eta_N(|\xi|)\widehat{u_0}\|_{L^2_\xi}\sup_{\xi\sim N}\|\eta_L\mathscr{F}_t(\psi(t)e^{-|t|\frac{|\beta|}{|\xi|^2}})\|_{L^2_\tau}\Big)^2\Big)^{\frac12}\\
\le&\Big(\sum_{N\ge1}N^{2s}\|\eta_N(|\xi|)\widehat{u_0}\|^2_{L^2_\xi}\Big)^{\frac12}\sum_{L\ge1}L^{\frac12}\sup_{\xi\sim N}\|\eta_L\mathscr{F}_t(\psi(t)e^{-|t|\frac{|\beta|}{|\xi|^2}})\|_{L^2_\tau}\\
\le&\|u_0\|_{H^s}\sum_{L\ge1}L^{\frac12}\sup_{\xi\sim N}\|\eta_L\mathscr{F}_t(\psi(t)e^{-|t|\frac{|\beta|}{|\xi|^2}})\|_{L^2_\tau}.
\end{aligned}$$

Since $B^{1/2}_{2,1}$ is a multiplication algebra, $\psi \in \dot{B}^{1/2}_{2,1}$ as well as $e^{-|t|} \in \dot{B}^{1/2}_{2,1}$, and $\dot{B}^{1/2}_{2,1}$ has scaling invariance, we have

$$\begin{aligned}
&\sum_{L\geq 1} L^{\frac{1}{2}} \sup_{\xi\sim N} \|\eta_L \mathscr{F}_t(\psi(t)e^{-|t|\frac{|\beta|}{|\xi|^2}})\|_{L^2_\tau} \\
\lesssim & \|e^{-|t|\frac{|\beta|}{N^2}}\|_{\dot{B}^{1/2}_{2,1}} \|\psi\|_{L^\infty} + \|\psi\|_{\dot{B}^{1/2}_{2,1}} \|e^{-|t|\frac{|\beta|}{|\xi|^2}}\|_{L^\infty_{\xi,t}} \\
\leq & C. \qquad (3.6)
\end{aligned}$$

Then we obtain

$$\|\psi(t)S_\beta(t)u_0\|_{X^S_{s,\frac{1}{2},1}} \leq C\|u_0\|_{H^s}. \qquad (3.7)$$

□

Proposition 3.2. *Let $s \in \mathbb{R}$, $\beta < 0$, then there exists constant $C > 0$ such that*

$$\|\psi(t)I^{S_\beta}f\|_{X^S_{s,\frac{1}{2},1}} \leq C\|f\|_{X^S_{s,-\frac{1}{2},1}} \qquad (3.8)$$

Proof. Assume that $f \in \mathscr{S}(\mathbb{R}^2)$. Taking the x-Fourier transform we get

$$\begin{aligned}
&\chi_{\mathbb{R}_+}(t)\psi(t)\int_0^t S_\beta(t-\tau)f(\tau,x)d\tau \\
=&S_0(t)\Big[\chi_{\mathbb{R}_+}(t)\psi(t)\int_{\mathbb{R}^2} e^{\mathrm{i}x\xi}\int_0^t e^{(-t+\tau)\frac{|\beta|}{|\xi|^2}}\mathscr{F}_x(S_\beta(-\tau)f(\tau,x))(\xi)d\tau d\xi\Big]. \qquad (3.9)
\end{aligned}$$

Setting $w(\tau) = S_\beta(-\tau)f(\tau,x)$, we infer that

$$\begin{aligned}
&\chi_{\mathbb{R}_+}(t)\psi(t)\int_0^t S_\beta(t-\tau)f(\tau,x)d\tau \\
=&S_0(t)\Big[\chi_{\mathbb{R}_+}(t)\psi(t)\int_{\mathbb{R}^3} e^{\mathrm{i}x\xi}\hat{w}(\tau,\xi)\frac{e^{\mathrm{i}t\tau}-e^{\frac{-t|\beta|}{|\xi|^2}}}{\mathrm{i}\tau+\frac{|\beta|}{|\xi|^2}}d\tau d\xi\Big]. \qquad (3.10)
\end{aligned}$$

We have

$$\begin{aligned}
&\|\psi(t)I^{S_\beta}f\|_{X^S_{s,\frac{1}{2},1}} \\
=&\Big(\sum_{N\geq 1} N^{2s}\Big(\sum_{L\geq 1} L^{\frac{1}{2}}\|\eta_L(\tau)\eta_N(|\xi|)\mathscr{F}_t(k_\xi(t))\|_{L_2}\Big)^2\Big)^{\frac{1}{2}}, \qquad (3.11)
\end{aligned}$$

where

$$k_\xi(t) = \psi(t)\int_{\mathbb{R}}\frac{e^{\mathrm{i}t\tau}-e^{\frac{-|t||\beta|}{|\xi|^2}}}{\mathrm{i}\tau+\frac{|\beta|}{|\xi|^2}}\hat{w}(\tau)d\tau. \qquad (3.12)$$

From the definition of $X^S_{s,\frac{1}{2},1}$ we only need to estimate

$$\sum_{L\geq 1} L^{\frac{1}{2}}\|\eta_L(\tau)\eta_N(|\xi|)\mathscr{F}_t(k_\xi(t))\|_{L_2} \lesssim \sum_{L\geq 1} L^{-\frac{1}{2}}\|\eta_L(\tau)\eta_N(|\xi|)f(\xi,\tau)\|_{L^2_{\xi,\tau}}.$$

We will consider it by four different cases.

$$
\begin{aligned}
k_\xi(t) = \psi(t)\int_{|t'|\le 1}\frac{e^{\mathrm{i}tt'}-1}{\mathrm{i}t'+\frac{|\beta|}{|\xi|^2}}\hat{w}(t')dt' + \psi(t)\int_{|t'|\le 1}\frac{1-e^{\frac{-|t||\beta|}{|\xi|^2}}}{\mathrm{i}t'+\frac{|\beta|}{|\xi|^2}}\hat{w}(t')dt' \\
+\psi(t)\int_{|t'|\ge 1}\frac{e^{\mathrm{i}tt'}}{\mathrm{i}t'+\frac{|\beta|}{|\xi|^2}}\hat{w}(t')dt' - \psi(t)\int_{|t'|\ge 1}\frac{e^{\frac{-|t||\beta|}{|\xi|^2}}}{\mathrm{i}t'+\frac{|\beta|}{|\xi|^2}}\hat{w}(t')dt' \\
:= I + II + III - IV.
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{L\ge 1} L^{\frac12}\|\eta_L(\tau)\eta_N(|\xi|)\mathscr{F}_t(IV)\|_{L^2_{\xi,\tau}} \\
\le &\sum_{L\ge 1} L^{\frac12}\sup_{\xi\sim N}\int_{\mathbb{R}}|\eta_N(\xi)\mathscr{F}_t(\psi(t)e^{\frac{-|t||\beta|}{|\xi|^2}})|^2 d\tau\cdot\Big(\int_{|t'|\ge 1}\frac{\|\eta_N(\xi)\hat{w}(\xi,t')\|_{L^2_\xi}}{|t'|}dt'\Big) \\
\le &C\sum_{L\ge 1} L^{-\frac12}\|\eta_L(\tau)\eta_N(|\xi|)\hat{w}(\xi,\tau)\|_{L^2_{\xi,\tau}}, \qquad (3.13)
\end{aligned}
$$

Where in the second inequality, we use equation (3.6).

For term III, using the technique in (3.6), we have

$$
\begin{aligned}
&\sum_{L\ge 1} L^{\frac12}\|\eta_L(\tau)\eta_N(|\xi|)\mathscr{F}_t(III)\|_{L^2_{\xi,\tau}} \\
\le &\sum_{L\ge 1} L^{\frac12}\|\eta_L(\tau)\eta_N(|\xi|)\Big[\hat{\psi}*\Big(\frac{\hat{w}(\tau)}{i\tau+\frac{|\beta|}{|\xi|^2}|}\chi_{|\tau|\ge 1}\Big)\Big]\|_{L^2_{\xi,\tau}} \\
\le &\sum_{L\ge 1} L^{\frac12}\Big\|\eta_L(t')\eta_N(|\xi|)\frac{\hat{w}(t')}{it'+\frac{|\beta|}{|\xi|^2}|}\Big\|_{L^2_{\xi,t'}}\cdot\|\hat{\psi}\|_{L_1} \\
\le &\sum_{L\ge 1} L^{\frac12}\Big\|\frac{\eta_L(t')\|\eta_N(|\xi|)\hat{w}(t')\|_{L^2_\xi}}{|t'|}\Big\|_{L^2_{t'}} \\
\le &\sum_{L\ge 1} L^{-\frac12}\|\eta_L(\tau)\eta_N(|\xi|)\hat{w}(\xi,\tau)\|_{L^2_{\xi,\tau}}. \qquad (3.14)
\end{aligned}
$$

For term II, we divide into two cases. When $\frac{|\beta|}{|\xi|^2}\ge 1$, notice $\psi\in C_0^\infty$ we have

$$
\begin{aligned}
&\sum_{L\ge 1} L^{\frac12}\|\eta_L(\tau)\eta_N(|\xi|)\mathscr{F}_t(II)\|_{L^2_{\xi,\tau}} \\
\le &\sum_{L\ge 1} L^{\frac12}\sup_{\xi}\Big\|\eta_N(|\xi|)\eta_L(\tau)\mathscr{F}_t(\psi(t)(1-e^{\frac{-|t||\beta|}{|\xi|^2}}))\Big\|_{L^2_\tau} \\
&\quad\cdot\Big(\int_{|t'|\le 1}\frac{\|\eta_N(\xi)\hat{w}(\xi,t')\|_{L^2_\xi}}{|t'|+1}dt'\Big) \\
\le &C\sum_{L\ge 1} L^{-\frac12}\|\eta_L(\tau)\eta_N(|\xi|)\hat{w}(\xi,\tau)\|_{L^2_{\xi,\tau}}. \qquad (3.15)
\end{aligned}
$$

When $\frac{|\beta|}{|\xi|^2} \le 1$, using Taylor's expansion and notice that $\tau \le 1$, we have

$$\begin{aligned}
&\sum_{L\ge1} L^{\frac12}\|\eta_L(\tau)\eta_N(|\xi|)\mathscr{F}_t(II)\|_{L^2_{\xi,\tau}}\\
\le&\sum_{n\ge1}\frac{1}{n!}\sum_{L\ge1}L^{\frac12}\Big\|\eta_L(\tau)\mathscr{F}_t(|t|^n\psi(t))\int_{|t'|\le1}\Big(\frac{-|\beta|}{|\xi|^2}\Big)^n\frac{\eta_N(|\xi|)\hat{w}(t')}{\mathrm{i}t'+\frac{|\beta|}{|\xi|^2}}dt'\Big\|_{L^2_{\xi,\tau}}\\
\le&\sum_{n\ge1}\frac{1}{n!}\sum_{L\ge1}L^{\frac12}\Big\|\eta_L(\tau)\mathscr{F}_t(|t|^n\psi(t))\Big\|_{L^2_\tau}\Big\|\int_{|t'|\le1}\Big(\frac{-|\beta|}{|\xi|^2}\Big)\frac{\eta_N(|\xi|)\hat{w}(t')}{\mathrm{i}t'+\frac{|\beta|}{|\xi|^2}}dt'\Big\|_{L^2_\xi}\\
\le&\sum_{n\ge1}\frac{1}{n!}\|t^n\psi(t)\|_{B^{1/2}_{2,1}}\sum_{L\ge1}L^{-\frac12}\|\eta_L(\tau)\eta_N(|\xi|)\hat{w}(\xi,\tau)\|_{L^2_{\xi,\tau}}\\
\le&C\sum_{L\ge1}L^{-\frac12}\|\eta_L(\tau)\eta_N(|\xi|)\hat{w}(\xi,\tau)\|_{L^2_{\xi,\tau}},
\end{aligned}\tag{3.16}$$

where in the last inequality, we use that $\||t|^n\psi(t)\|_{B^{1/2}_{2,1}} \le \||t|^n\psi(t)\|_{H^1} \le C2^n$.

Using Taylor's expansion, $I = \psi(t)\int_{|t'|\le1}\sum_{n\ge1}\frac{(\mathrm{i}tt')^n}{n!(\mathrm{i}t'+\frac{|\beta|}{|\xi|^2})}\hat{w}(t')dt'$, so

$$\begin{aligned}
&\sum_{L\ge1} L^{\frac12}\|\eta_L(\tau)\eta_N(|\xi|)\mathscr{F}_t(I)\|_{L^2_{\xi,\tau}}\\
\le&\sum_{n\ge1}\sum_{L\ge1}L^{\frac12}\Big\|\eta_L(\tau)\mathscr{F}_t\Big(\frac{t^n\psi(t)}{n!}\Big)\Big\|_{L^2_\tau}\Big\|\int_{|t'|\le1}\frac{|t'|}{|\mathrm{i}t'+\frac{|\beta|}{|\xi|^2}|}|\eta_N(|\xi|)\hat{w}(\xi,t')|dt'\Big\|_{L^2_\xi}\\
\le&\sum_{n\ge1}\frac{1}{n!}\|t^n\psi(t)\|_{B^{1/2}_{2,1}}\sum_{L\ge1}L^{-\frac12}\|\eta_L(\tau)\eta_N(|\xi|)\hat{w}(\xi,\tau)\|_{L^2_{\xi,\tau}}\\
\le&C\sum_{L\ge1}L^{-\frac12}\|\eta_L(\tau)\eta_N(|\xi|)\hat{w}(\xi,\tau)\|_{L^2_{\xi,\tau}}.
\end{aligned}\tag{3.17}$$

□

For equation (3.2), From [2], we directly have

Lemma 3.1 ([2])**.** *Let $s \in \mathbb{R}$, for all $0 < T \le 1$ there exists constant $C > 0$ such that*

$$\|W(t)u_0\|_{X^W_{s,\frac12,1}(T)} \le C\|u_0\|_{H^s}, \tag{3.18}$$

and such that for $f \in \mathscr{S}(\mathbb{R}^2\times\mathbb{R})$, we have

$$\|I^W f\|_{X^W_{s,\frac12,1}(T)} \le C\|f\|_{X^W_{s,-\frac12,1}(T)}. \tag{3.19}$$

Next we introduce some trilinear estimates. Define

$$I(f,g_1,g_2) = \int f(\zeta_1-\zeta_2)g_1(\zeta_1)g_2(\zeta_2)d\zeta_1d\zeta_2, \tag{3.20}$$

where $\zeta_i = (\xi_i,\tau_i), i = 1,2$, we have:

Lemma 3.2. *Let $f, g_1, g_2 \in L^2$ with $\|f\|_{L^2} = \|g_1\|_{L^2} = \|g_2\|_{L^2} = 1$ and*

$$\mathrm{supp}(f) \subset \Upsilon_L\cap\Gamma_N, \quad \mathrm{supp}(g_k) \subset \Lambda_L\cap\Gamma_N\cap\Omega_j^A \quad (j=1,2). \tag{3.21}$$

The frequencies N, N_1, N_2 satisfy $64 \le N \lesssim N_1 \sim N_2$, then the following estimate

$$I(v^{L,N}, g_1^{L_1,N_1}, g_2^{L_2,N_2}) \le L_1^{\frac{5}{12}} L_2^{\frac{5}{12}} L^{\frac{5}{12}} N^{-\frac{1}{2}} \Big(\frac{N}{N_1}\Big)^{\frac{1}{4}} \tag{3.22}$$

holds.

Remark. The detail proof can be found in Proposition 4.2 in [2].

Lemma 3.3 (**[2]**). *Let $f, g_1, g_2 \in L^2$ with $\|f\|_{L^2} = \|g_1\|_{L^2} = \|g_2\|_{L^2} = 1$ and*

$$\operatorname{supp}(f) \subset \Upsilon_L \cap \Gamma_N, \quad \operatorname{supp}(g_k) \subset \Lambda_L \cap \Gamma_N \quad (j = 1, 2). \tag{3.23}$$

The frequencies N, N_1, N_2 and modulations L, L_1, L_2 satisfy $1 \le N_1 \ll N_2$. Then for all $L, L_1, L_2 \ge 1$ we have

$$|I(f, g_1, g_2)| \lesssim L_1^{\frac{5}{12}} L_2^{\frac{5}{12}} L^{\frac{5}{12}} N^{-\frac{1}{2}} \Big(\frac{N_1}{N_2}\Big)^{\frac{1}{6}}. \tag{3.24}$$

Lemma 3.4 (**[2]**). (*Bilinear Strichartz estimates*)
(*a*)*Let $v_1, v_2 \in L^2(\mathbb{R}^3)$ be dyadically Fourier-localized such that*

$$\operatorname{supp}(\mathscr{F} v_i) \subset \Lambda_L \cap \Gamma_N.$$

For $L_1, L_2 \ge 1$, $N_1, N_2 \ge 1$. Then the following estimate holds:

$$\|v_1 v_2\|_{L^2(\mathbb{R}^3)} \lesssim \Big(\frac{N_1}{N_2}\Big)^{\frac{1}{2}} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} \|v_1\|_{L^2} \|v_2\|_{L^2}. \tag{3.25}$$

(b)Let $u, v \in L^2(\mathbb{R}^3)$ be dyadically Fourier-localized such that

$$\operatorname{supp}(\mathscr{F} u) \subset \Upsilon_L \cap \Gamma_N, \quad \operatorname{supp}(\mathscr{F} v) \subset \Lambda_{L_1} \cap \Gamma_{N_1}.$$

For $L, L_1 \ge 1$, $N, N_1 \ge 1$. Then the following estimate holds:

$$\|uv\|_{L^2(\mathbb{R}^3)} \lesssim \Big(\frac{\min\{N, N_1\}}{N_1}\Big)^{\frac{1}{2}} L^{\frac{1}{2}} L_1^{\frac{1}{2}} \|u\|_{L^2} \|v\|_{L^2}. \tag{3.26}$$

Proposition 3.3.

$$\begin{aligned} &\|\langle\nabla\rangle^{-1}(v \cdot \nabla\varphi)\|_{X^S_{\frac{1}{2},-\frac{5}{12},\infty}(T)} \\ \le & C(\|P_{\ge 2}\varphi\|_{X^S_{\frac{1}{2},\frac{5}{12},1}} + \|P_{\lesssim 1}\nabla\varphi\|_{X^S_{\frac{1}{2},\frac{5}{12},1}})\|v\|_{X^W_{0,\frac{5}{12},1}}. \end{aligned} \tag{3.27}$$

Proof. By dulity and $\Big(\overline{X^S_{s,b,p}}\Big)^* = X^S_{-s,-b,p'}$, $\Big(\overline{X^W_{s,b,p}}\Big)^* = X^W_{-s,-b,p'}$, $1 \le p < \infty, s, b \in \infty$, We can deduce Proposition 3.3 to the following trilinear estimates hold for all $v, \varphi, g_2 \in \mathscr{S}(\mathbb{R}^2 \times \mathbb{R})$:

$$I(\mathscr{F} v, \mathscr{F} \nabla P_{\ge 2}\varphi, \mathscr{F} g_2) \le \|P_{\ge 2}\varphi\|_{X^S_{\frac{1}{2},\frac{5}{12},1}} \|v\|_{X^W_{0,\frac{5}{12},1}} \|g_2\|_{X^S_{\frac{1}{2},\frac{5}{12},1}}, \tag{3.28}$$

$$I(\mathscr{F} v, \mathscr{F} P_{\lesssim 1}\nabla\varphi, \mathscr{F} g_2) \le \|P_{\lesssim 1}\nabla\varphi\|_{X^S_{\frac{1}{2},\frac{5}{12},1}} \|v\|_{X^W_{0,\frac{5}{12},1}} \|g_2\|_{X^S_{\frac{1}{2},\frac{5}{12},1}}. \tag{3.29}$$

Let $g_1 = \nabla P_{\ge 2}\varphi$, by definition of the norms, we dyadically decompose

$$v = \sum_{N,L} S_L P_N v, \quad g_i = \sum_{N_i, L_i} S_{L_i} P_{N_i} g_i, \quad i = 1, 2. \tag{3.30}$$

Set $g_i^{L_i,N_i} = \mathscr{F} S_{L_i} P_{N_i} g_i$ ($i = 1, 2$), $v^{L,N} = \mathscr{F} S_L P_N v$ and $\varphi^{L_1,N_1} = \mathscr{F} S_{L_1} P_{N_1}\varphi$, then $g_1^{L_1,N_1} = N_1\varphi^{L_1,N_1}(N_1 \geq 4)$. We have the identity

$$I(\mathscr{F}v, \mathscr{F}\nabla P_{\geq 2}\varphi, \mathscr{F}g_2) = \sum_{\substack{N_1\geq 4,\\ N,N_2\geq 1}} \sum_{L,L_1,L_2\geq 1} I(v^{L,N}, g_1^{L_1,N_1}, g_2^{L_2,N_2}). \tag{3.31}$$

First, we prove (3.28), we divide the summation into three cases:

Case1:(high× high → low interactions) Assume $N \sim N_1 \gtrsim N_2 \geq 1$.

This case is the worst, since we have no method to absorb the derivative in φ. The technique we use here can be called "derivative sharing", where we share the one order derivative in φ to v. From Lemma 3.3 we have

$$\begin{aligned}
&I(v^{L,N}, g_1^{L_1,N_1}, g_2^{L_2,N_2})\\
\lesssim&L_1^{\frac{5}{12}} L_2^{\frac{5}{12}} L^{\frac{5}{12}} N^{-\frac{1}{2}}\Big(\frac{N_2}{N_1}\Big)^{\frac{1}{6}} \|v^{L,N}\|_{L^2}\|g_1^{L_1,N_1}\|_{L^2}\|g_2^{L_2,N_2}\|_{L^2}\\
\lesssim&L_1^{\frac{5}{12}} L_2^{\frac{5}{12}} L^{\frac{5}{12}} N^{-\frac{1}{2}} N_1\Big(\frac{N_2}{N_1}\Big)^{\frac{1}{6}} \|v^{L,N}\|_{L^2}\|\varphi^{L_1,N_1}\|_{L^2}\|g_2^{L_2,N_2}\|_{L^2}\\
\lesssim&L_1^{\frac{5}{12}} L_2^{\frac{5}{12}} L^{\frac{5}{12}} N_1^{\frac{1}{2}-\delta} N^{\delta} N_2^{\frac{1}{2}-\delta}\Big(\frac{N_2}{N_1}\Big)^{\frac{1}{6}} \|v^{L,N}\|_{L^2}\|\varphi^{L_1,N_1}\|_{L^2}\|g_2^{L_2,N_2}\|_{L^2},
\end{aligned} \tag{3.32}$$

where $0 \leq \delta \leq \frac{1}{2}$. When $\delta = 0$, we can prove our Theorem 1.1. Otherwise, we can prove Corollary 1.2. In the following we only consider $\delta = 0$, other case is similar.

noticing that $N \sim N_1$, N_1, N_2 are dyadic number. Let ϵ small enough, from (3.31) and Schwartz's inequality, we have

$$\begin{aligned}
&\sum_{\substack{N_1\geq 4\\ N,N_2\geq 1,}} \sum_{L,L_1,L_2\geq 1} I(v^{L,N}, g_1^{L_1,N_1}, g_2^{L_2,N_2})\\
\leq&C \sum_{N_1\geq 4, N_2\geq 1} \Big(\frac{N_2}{N_1}\Big)^{\frac{1}{6}} N_1^{\epsilon}\Big(N_1^{\frac{1}{2}} \sum_{L_1\geq 1} L_1^{\frac{5}{12}}\|\varphi^{L_1,N_1}\|_{L^2}\Big)\\
&\qquad\Big(N_2^{\frac{1}{2}} \sum_{L_2\geq 1} L_2^{\frac{5}{12}}\|g_2^{L_2,N_2}\|_{L^2}\Big)\|v\|_{X^S_{0,\frac{5}{12},1}}\\
\leq& \sum_{N_1\geq 4} N_1^{-\epsilon}\Big(N_1^{\frac{1}{2}} \sum_{L_1\geq 1} L_1^{\frac{5}{12}}\|\varphi^{L_1,N_1}\|_{L^2}\Big)\\
&\qquad\sum_{1\leq N_2\lesssim N_1} \Big(\frac{N_2}{N_1}\Big)^{\frac{1}{6}} N_1^{2\epsilon}\Big(N_2^{\frac{1}{2}} \sum_{L_2\geq 1} L_2^{\frac{5}{12}}\|g_2^{L_2,N_2}\|_{L^2}\Big)\|v\|_{X^W_{0,\frac{5}{12},1}}\\
\leq&C\|P_{\geq 2}\varphi\|_{X^S_{\frac{1}{2},\frac{5}{12},1}}\|g_2\|_{X^S_{\frac{1}{2},\frac{5}{12},1}}\|v\|_{X^W_{0,\frac{5}{12},1}}.
\end{aligned} \tag{3.33}$$

Similar proof will be used in the following many times.

Case2:(low × high → high interactions) Assume $2^{10} \leq N \lesssim N_1 \sim N_2$. From Lemma 3.2, we obtain

$$\begin{aligned}
&I(v^{L,N}, g_1^{L_1,N_1}, g_2^{L_2,N_2})\\
\leq&L_1^{\frac{5}{12}} L_2^{\frac{5}{12}} L^{\frac{5}{12}} N^{-\frac{1}{2}}\Big(\frac{N}{N_1}\Big)^{\frac{1}{4}} \|v^{L,N}\|_{L^2}\|g_1^{L_1,N_1}\|_{L^2}\|g_2^{L_2,N_2}\|_{L^2}.
\end{aligned}$$

Notice $N_1 \sim N_2$, we have

$$\begin{aligned}
&I(v^{L,N}, g_1^{L_1,N_1}, g_2^{L_2,N_2})\\
\leq &L_1^{\frac{5}{12}} L_2^{\frac{5}{12}} L^{\frac{5}{12}} N^{-\frac{1}{2}} \left(\frac{N}{N_1}\right)^{\frac{1}{4}} N_1 \|v^{L,N}\|_{L^2} \|\varphi^{L_1,N_1}\|_{L^2} \|g_2^{L_2,N_2}\|_{L^2}\\
\leq &L_1^{\frac{5}{12}} L_2^{\frac{5}{12}} L^{\frac{5}{12}} N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} N^{-\frac{1}{2}} \left(\frac{N}{N_1}\right)^{\frac{1}{4}} \|v^{L,N}\|_{L^2} \|\varphi^{L_1,N_1}\|_{L^2} \|g_2^{L_2,N_2}\|_{L^2} \qquad (3.34)
\end{aligned}$$

The case low $\times$ high $\to$ high interactions is similar to case 2, we omit the detail here.

Case3:(low frequency) Assume $N \lesssim 1$.

In this case, we must have $N_1 \sim N_2$. First, assume $L = \max\{L, L_1, L_2\}$, from (3.25), we have

$$\begin{aligned}
&|I(v^{L,N}, g_1^{L_1,N_1}, g_2^{L_2,N_2})|\\
\leq &\|v^{L,N}\|_{L^2} \|\mathscr{F}^{-1} g_1^{L_1,N_1} \overline{\mathscr{F}^{-1} g_2^{L_2,N_2}}\|_{L^2}\\
\leq &L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} \left(\frac{N_1}{N_2}\right)^{\frac{1}{2}} N_1 \|v^{L,N}\|_{L^2} \|\varphi^{L_1,N_1}\|_{L^2} \|g_2^{L_2,N_2}\|_{L^2}\\
\leq &L_1^{\frac{5}{12}} L_2^{\frac{5}{12}} L^{\frac{5}{12}} N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} \left(\frac{N_1}{N_2}\right)^{\frac{1}{2}} \|v^{L,N}\|_{L^2} \|\varphi^{L_1,N_1}\|_{L^2} \|g_2^{L_2,N_2}\|_{L^2} \qquad (3.35)
\end{aligned}$$

Second, assume $L_1 = \max\{L, L_1, L_2\}$, from (3.26), we have

$$\begin{aligned}
&|I(v^{L,N}, g_1^{L_1,N_1}, g_2^{L_2,N_2})|\\
\leq &\|g_1^{L_1,N_1}\|_{L^2} \|\mathscr{F}^{-1} v^{L,N} \overline{\mathscr{F}^{-1} g_2^{L_2,N_2}}\|_{L^2}\\
\leq &L^{\frac{1}{2}} L_2^{\frac{1}{2}} \left(\frac{N}{N_2}\right)^{\frac{1}{2}} N_1 \|v^{L,N}\|_{L^2} \|\varphi^{L_1,N_1}\|_{L^2} \|g_2^{L_2,N_2}\|_{L^2}\\
\leq &L_1^{\frac{5}{12}} L_2^{\frac{5}{12}} L^{\frac{5}{12}} N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} \left(\frac{N}{N_2}\right)^{\frac{1}{2}} \|v^{L,N}\|_{L^2} \|\varphi^{L_1,N_1}\|_{L^2} \|g_2^{L_2,N_2}\|_{L^2}. \qquad (3.36)
\end{aligned}$$

From symmetry, when $L_2 = \max\{L, L_1, L_2\}$, we also have

$$\begin{aligned}
&|I(v^{L,N}, g_1^{L_1,N_1}, g_2^{L_2,N_2})|\\
\leq &L_1^{\frac{5}{12}} L_2^{\frac{5}{12}} L^{\frac{5}{12}} N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} \left(\frac{N}{N_1}\right)^{\frac{1}{2}} \|v^{L,N}\|_{L^2} \|\varphi^{L_1,N_1}\|_{L^2} \|g_2^{L_2,N_2}\|_{L^2}. \qquad (3.37)
\end{aligned}$$

To summation, from the proof in (3.33), we obtain (3.28) as desired.

For (3.29), Set $g_3 = \nabla P_{\leq 1}\varphi$. The proof is similar to above. Notice that $N_1 \lesssim 1$, then the integral vanishes unless $N \sim N_2$. We divide the summation into two cases: (a) $N \sim N_2 \lesssim 1$, this case reduces to case 3 above(see (3.35) and (3.36)), we have

$$\begin{aligned}
&|I(v^{L,N}, g_3^{L_1,N_1}, g_2^{L_2,N_2})|\\
\leq &L_1^{\frac{5}{12}} L_2^{\frac{5}{12}} L^{\frac{5}{12}} \|v^{L,N}\|_{L^2} \|P_{\leq 1}\nabla\varphi^{L_1,N_1}\|_{L^2} \|g_2^{L_2,N_2}\|_{L^2}. \qquad (3.38)
\end{aligned}$$

(b) $N \sim N_2 \gg 1$, this case reduces to case 1 above(see (3.32)), we have

$$\begin{aligned}
&|I(v^{L,N}, g_3^{L_1,N_1}, g_2^{L_2,N_2})|\\
\lesssim &L_1^{\frac{5}{12}} L_2^{\frac{5}{12}} L^{\frac{5}{12}} N_1^{\frac{1}{2}} \left(\frac{N_1}{N_2}\right)^{\frac{1}{6}} \|v^{L,N}\|_{L^2} \|P_{\leq 1}\nabla\varphi^{L_1,N_1}\|_{L^2} \|g_2^{L_2,N_2}\|_{L^2}.
\end{aligned}$$

Similarly, from (3.33), we obtain (3.29) as desired.

Proposition 3.4.

$$\begin{aligned}&\Big\|\frac{1}{\langle\nabla\rangle}\nabla\varphi^*\cdot\overline{\nabla}\varphi\Big\|_{X^W_{0,-\frac{5}{12},\infty}}\\ \le&(\|P_{\ge 2}\varphi\|_{X^S_{\frac12,\frac{5}{12},1}}+\|P_{\lesssim 1}\nabla\varphi\|_{X^S_{\frac12,\frac{5}{12},1}}+\|P_{\lesssim 1}\overline{\nabla}\varphi\|_{X^S_{\frac12,\frac{5}{12},1}})^2. \end{aligned}\tag{3.39}$$

Proof. We also apply the dyadic decomposition in Proposition 3.3. Let $g_1 = \varphi, g_2 = \varphi^*$, by definition of the norms, we have

$$v=\sum_{N,L}S_LP_Nv,\quad g_i=\sum_{N_i,L_i}S_{L_i}P_{N_i}g_i,\quad i=1,2. \tag{3.40}$$

then we have

$$\begin{aligned}&\Big\|\frac{1}{\langle\nabla\rangle}\nabla\varphi^*\cdot\overline{\nabla}\varphi\Big\|_{X^W_{0,-\frac{5}{12},\infty}(T)}\\ \le&\sum_{N_1 or N_2\sim 1}\sum_{L_1,L_2\ge 1}\Big\|\frac{1}{\langle\nabla\rangle}(\nabla S_{L_2}P_{N_2}\varphi^*\cdot\overline{\nabla}S_{L_1}P_{N_1}\varphi)\Big\|_{X^W_{0,-\frac{5}{12},\infty}(T)}\\ &+\sum_{N_1,N_2\gg 1}\sum_{L_1,L_2\ge 1}\Big\|\frac{1}{\langle\nabla\rangle}(\nabla S_{L_2}P_{N_2}\varphi^*\cdot\overline{\nabla}S_{L_1}P_{N_1}\varphi)\Big\|_{X^W_{0,-\frac{5}{12},\infty}(T)}\\ :=&\sum_{N_1 or N_2\sim 1}\sum_{L_1,L_2\ge 1}A+\sum_{N_1,N_2\gg 1}\sum_{L_1,L_2\ge 1}B \end{aligned}\tag{3.41}$$

We will estimate term A directly. Without loss of generality, assume $N_1\sim 1$.

$$\begin{aligned}&\Big\|\frac{1}{\langle\nabla\rangle}(\nabla S_{L_2}P_{N_2}\varphi^*\cdot\overline{\nabla}S_{L_1}P_{N_1}\varphi)\Big\|_{X^W_{0,-\frac12,1}(T)}\\ \le&\Big\|\nabla S_{L_2}P_{N_2}\varphi^*\cdot\overline{\nabla}S_{L_1}P_{N_1}\varphi\Big\|_{X^W_{-1,-\frac12,1}(T)}.\end{aligned}$$

Set $g_1^{L_1N_1}=\mathscr{F}S_{L_1}P_{N_1}\overline{\nabla}\varphi$ and $g_2^{L_2,N_2}=\mathscr{F}S_{L_2}P_{N_2}\nabla\varphi^*$, $v^{L,N}=\mathscr{F}S_LP_Nv$, then similar to the proof of (3.29), we divide into two cases:

(a) $N\sim N_2\lesssim 1$, this case reduces to (3.38), we have

$$\begin{aligned}&|I(v^{L,N},g_1^{L_1,N_1},g_2^{L_2,N_2})|\\ \le&L_1^{\frac{5}{12}}L_2^{\frac{5}{12}}L^{\frac{5}{12}}\|v^{L,N}\|_{L^2}\|P_{\lesssim 1}\overline{\nabla}\varphi^{L_1,N_1}\|_{L^2}\|P_{\lesssim 1}\nabla\varphi^{*L_2,N_2}\|_{L^2}.\end{aligned}\tag{3.42}$$

(b) $N\sim N_2\gg 1$, this case reduces to equation (3.32). From Lemma 3.3, we have

$$\begin{aligned}&|I(v^{L,N},g_1^{L_1,N_1},g_2^{L_2,N_2})|\\ \lesssim&L_1^{\frac{5}{12}}L_2^{\frac{5}{12}}L^{\frac{5}{12}}N^{-\frac12}\Big(\frac{N_1}{N_2}\Big)^{\frac16}\|v^{L,N}\|_{L^2}\|g_1^{L_1,N_1}\|_{L^2}\|g_2^{L_2,N_2}\|_{L^2}\\ \le&L_1^{\frac{5}{12}}L_2^{\frac{5}{12}}L^{\frac{5}{12}}N_2^{\frac12}\Big(\frac{N_1}{N_2}\Big)^{\frac16}\|v^{L,N}\|_{L^2}\|P_{\lesssim 1}\overline{\nabla}\varphi^{L_1,N_1}\|_{L^2}\|\varphi^{*L_2,N_2}\|_{L^2}.\end{aligned}\tag{3.43}$$

From (3.42) and (3.43), we obtain

$$\sum_{N_1\sim 1}\sum_{L_1,L_2\ge 1}A\le\|\varphi\|_{X^S_{\frac12,\frac12,1}}\|\varphi\|_{Y_2}. \tag{3.44}$$

If we assume $N_2 \sim 1$, then we have

$$\sum_{N_2\sim 1}\sum_{L_1,L_2\geq 1} A \leq \|\varphi\|_{X^S_{\frac{1}{2},\frac{1}{2},1}}\|\varphi\|_{Y_1}. \tag{3.45}$$

To summation, (3.51), (3.44) and (3.45) means (3.39) as desired.

Noticing that

$$\nabla f_1 \cdot \overline{\nabla} f_2 = \overline{\nabla}\cdot (f_2 \nabla f_1), \tag{3.46}$$

so

$$B = \Big\| \frac{\overline{\nabla}}{\langle\nabla\rangle}\cdot (S_{L_1}P_{N_1}\cdot \varphi \nabla S_{L_2}P_{N_2}\varphi^*)\Big\|_{X^W_{0,-\frac{1}{2},1}(T)} \tag{3.47}$$

$$\leq \Big\| S_{L_1}P_{N_1}\varphi \cdot \nabla S_{L_2}P_{N_2}\varphi^*\Big\|_{X^W_{0,-\frac{1}{2},1}(T)}. \tag{3.48}$$

for term B, we treat it using the same way as in Proposition 3.3.

Set $g_1^{L_1N_1} = \mathscr{F} S_{L_1}P_{N_1}\varphi$ and $g_2^{L_2,N_2} = \mathscr{F} S_{L_2}P_{N_2}\nabla\varphi^*$, $v^{L,N} = \mathscr{F} S_L P_N v$, similar to Proposition 3.3, we divide into two cases:
$(a) N \sim N_2 \gtrsim N_1 \geq 1$, this case reduces to (3.32).

$$\begin{aligned} & I(v, g_1^{L_1,N_1}, g_2^{L_2,N_2}) \\ \leq & \sum_{L\geq 1, N\geq 1} I(v^{L,N}, g_1^{L_1,N_1}, g_2^{L_2,N_2}) \\ \leq & \sum_{L\geq 1, N\geq 1} L_1^{\frac{5}{12}} L_2^{\frac{5}{12}} L^{\frac{5}{12}} N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} \Big(\frac{N_1}{N_2}\Big)^{\frac{1}{2}} \|v^{L,N}\|_{L^2}\|\varphi^{*L_2,N_2}\|_{L^2}\|\varphi^{L_1,N_1}\|_{L^2} \end{aligned} \tag{3.49}$$

$(b) 2^{10} \lesssim N \lesssim N_1 \sim N_2$, this case reduces to (3.34).

$$\begin{aligned} & I(v, g_1^{L_1,N_1}, g_2^{L_2,N_2}) \\ \leq & \sum_{L\geq 1, N\geq 1} I(v^{L,N}, g_1^{L_1,N_1}, g_2^{L_2,N_2}) \\ \leq & \sum_{L\geq 1, N\geq 1} L_1^{\frac{5}{12}} L_2^{\frac{5}{12}} L^{\frac{5}{12}} N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} N^{-\frac{1}{2}} \Big(\frac{N}{N_1}\Big)^{\frac{1}{4}} \|v^{L,N}\|_{L^2}\|\varphi^{*L_2,N_2}\|_{L^2}\|\varphi^{L_1,N_1}\|_{L^2} \end{aligned} \tag{3.50}$$

Similar to (3.33), from duality we have

$$\sum_{N_1,N_2\gg 1}\sum_{L_1,L_2\geq 1} B \leq \|\varphi\|^2_{X^S_{\frac{1}{2},\frac{1}{2},1}}. \tag{3.51}$$

Proposition 3.5. *For $i = 1, 2$, we have*

$$\|\langle\nabla\rangle^{-1}\cdot (v\nabla\varphi)\|_{Y_i} \leq C(\|P_{\geq 2}\varphi\|_{X^S_{\frac{1}{2},\frac{5}{12},1}} + \|P_{\lesssim 1}\nabla\varphi\|_{X_{X^S_{\frac{1}{2},\frac{5}{12},1}}})\|v\|^W_{0,\frac{5}{12},1}. \tag{3.52}$$

Proof. From the definition of Y_i, we have

$$\|\langle\nabla\rangle^{-1}\cdot (v\nabla\varphi)\|_{Y_i} \lesssim \|P_{\lesssim 1}(v\nabla\varphi)\|_{X^S_{0,\frac{5}{12},1}}.$$

So (3.52) deduces to the following trilinear estimates hold for all $v, \varphi, g_2 \in \mathscr{S}(\mathbb{R}^2 \times \mathbb{R})$:

$$I(\mathscr{F}v, \mathscr{F}P_{\geq 2}\nabla\varphi, \mathscr{F}P_{\lesssim 1}g_2) \leq \|P_{\geq 2}\varphi\|_{X^S_{\frac{1}{2},\frac{5}{12},1}} \|v\|_{X^W_{0,\frac{5}{12},1}} \|g_2\|_{X^S_{0,\frac{5}{12},1}},$$
$$I(\mathscr{F}v, \mathscr{F}P_{\lesssim 1}\nabla\varphi, \mathscr{F}P_{\lesssim 1}g_2) \leq \|P_{\lesssim 1}\nabla\varphi\|_{X^S_{\frac{1}{2},\frac{5}{12},1}} \|v\|_{X^W_{0,\frac{5}{12},1}} \|g_2\|_{X^S_{0,\frac{5}{12},1}}.$$

The idea is similar to the estimates for term A in Proposition 3.4, see (3.49), (3.50), we do not show detail proof here.

□

4. Proof of Theorem1.1

Lemma 4.1. *Let $s, b \in \mathbb{R}$, $0 < b < \frac{1}{2}$. There exists a constant $C > 0$ such that for all $T \in (0, 1]$ the estimate*

$$\|f\|_{X^S_{s,b,1}(T)} \leq CT^{\frac{1}{2}-b}\|f\|_{X^S_{s,\frac{1}{2},1}(T)} \tag{4.1}$$

holds for all $u \in X_{s,b,1}(T)$. Specially, we have

$$\|f\|_{X^S_{s,-\frac{1}{2},1}(T)} \leq CT^{\frac{1}{2}}\|f\|_{X^S_{s,\frac{1}{2},1}(T)}. \tag{4.2}$$

Moreover, the embedding $X^S_{s,\frac{1}{2},1}(T) \subset C([0,T]; H^s)$ is continuous, i. e. there exists a constant $C > 0$ such that for all $T \in (0, 1]$ it holds

$$\sup_{0\leq t\leq T} \|f(t)\|_{H^s} \leq C\|f\|_{X^S_{s,\frac{1}{2},1}(T)}. \tag{4.3}$$

for all $f \in X^S_{s,\frac{1}{2},1}(T)$.

The results are completely right, if we substitute $X^S_{s,\frac{1}{2},1}(T)$ with $X^W_{s,\frac{1}{2},1}(T)$.

Proof. (4.1) and (4.3) can be found in [2].

For (4.2), from embedding Theorem , we have

$$\|u\|_{X^S_{s,-\frac{1}{2},1}(T)} \leq \Big(\sum_{N\geq 2} N^{2s}\|P_N u\|^2_{L_2}\Big)^{\frac{1}{2}} \leq C\|u\|_{L^2_T H^s_x}$$
$$\leq CT^{\frac{1}{2}}\|u\|_{L^\infty_T H^s_x} \leq CT^{1/2}\|u\|_{X^S_{s,\frac{1}{2},1}(T)}.$$

□

Proof of Theorem1.1. We consider the following mapping:

$$\mathscr{T}_1 : \varphi(t) \to S_\beta(t)\varphi_0 - \mathrm{i}I^{S_\beta}\Big(\alpha\varphi + \mathrm{i}\triangle^{-1}[\nabla\varphi \cdot \overline{\nabla}(\mathrm{Re}v)]\Big),$$
$$\mathscr{T}_2 : v(t) \to W(t)v_0 - \mathrm{i}I^W\Big(-\mathrm{i}\omega\langle\nabla\rangle^{-1}[\nabla\varphi^* \cdot \overline{\nabla}\varphi] + \langle\nabla\rangle^{-1}\mathrm{Re}v + \mathrm{i}\gamma(v - \mathrm{Re}v)\Big).$$

From Proposition 3.1, we have

$$\|P_{\geq 2}S_\beta(t)\varphi_0\|_{X^S_{\frac{1}{2},\frac{1}{2},1}(T)} \leq \|P_{\geq 2}\varphi_0\|_{H^{\frac{1}{2}}}. \tag{4.4}$$

From Proposition 3.2 and equation (4.2), we have

$$\begin{aligned}\|P_{\geq 2}I^{S_\beta}(\alpha\varphi)\|_{X^S_{\frac12,\frac12,1}(T)} &\leq C\|P_{\geq 2}\varphi\|_{X^S_{\frac12,-\frac12,1}(T)}\\ &\leq CT^{\frac12}\|P_{\geq 2}\varphi\|_{X^S_{\frac12,\frac12,1}(T)}.\end{aligned} \tag{4.5}$$

Notice the definition of $\overline{\nabla}$, we have

$$\nabla\varphi\cdot\overline{\nabla}(\mathrm{Re}v) = \overline{\nabla}\cdot(\mathrm{Re}v\nabla\varphi),$$

then from Proposition 3.2, lemma 4.1 and Proposition (3.4) we have

$$\begin{aligned}&\|P_{\geq 2}I^{S_\beta}\Big(\triangle^{-1}[\nabla\varphi\cdot\overline{\nabla}(\mathrm{Re}v)]\Big)\|_{X^S_{\frac12,\frac12,1}(T)}\\ \leq&\|I^{S_\beta}P_{\geq 2}\Big(\triangle^{-1}\overline{\nabla}(\mathrm{Re}v\nabla\varphi)\Big)\|_{X^S_{\frac12,\frac12,1}(T)}\\ \leq&\|\langle\nabla\rangle^{-1}\cdot(\mathrm{Re}v\nabla\varphi)\|_{X^S_{\frac12,-\frac12,1}(T)}\\ \leq&CT^{\frac{1}{12}}\|\langle\nabla\rangle^{-1}(\mathrm{Re}v\nabla\varphi)\|_{X^S_{\frac12,-\frac{5}{12},\infty}(T)}\\ \leq&CT^{\frac14}\|\varphi\|_{X^S(T)}\|v\|_{X^W_{0,\frac12,1}(T)}.\end{aligned} \tag{4.6}$$

From (4.4)–(4.6), we obtain

$$\|P_{\geq 2}\mathscr{T}_1\varphi\|_{X^S_{\frac12,\frac12,1}(T)} \leq \|P_{\geq 2}\varphi_0\|_{H^{\frac12}} + T^{\frac14}\|\varphi\|_{X^S(T)}(\|v\|_{X^W_{0,\frac12,1}(T)}+1). \tag{4.7}$$

Next,

$$\|S_\beta(t)\varphi_0\|_{Y_1} \leq \|P_{\leq 1}\nabla\varphi_0\|_{L^2} \leq \|P_{\leq 1}\varphi_0\|_{\dot{H}^1}. \tag{4.8}$$

Similar to (4.5), we have

$$\|I^{S_\beta}(\alpha\varphi)\|_{Y_1} \leq CT^{\frac12}\|\varphi\|_{Y_1}. \tag{4.9}$$

Similar to (4.6), from Proposition 3.5, we have

$$\|I^{S_\beta}\Big(\frac{1}{\triangle}[\nabla\varphi\cdot\overline{\nabla}(\mathrm{Re}v)]\Big)\|_{Y_1} \leq T^{\frac14}\|\varphi\|_{X^S(T)}\|v\|_{X^W_{0,\frac12,1}(T)}. \tag{4.10}$$

From (4.8)–(4.10), we obtain

$$\|\mathscr{T}_1\varphi\|_{Y_1} \leq \|P_{\leq 1}\varphi_0\|_{\dot{H}^1} + T^{\frac14}\|\varphi\|_{X^S(T)}(\|v\|_{X^W_{0,\frac12,1}(T)}+1). \tag{4.11}$$

By the same way, we have

$$\|\mathscr{T}_1\varphi\|_{Y_2} \leq \|P_{\leq 1}\varphi_0\|_{\dot{H}^1} + T^{\frac14}\|\varphi\|_{X^S(T)}(\|v\|_{X^W_{0,\frac12,1}(T)}+1). \tag{4.12}$$

Combine (4.7), (4.11) and (4.12), we have

$$\|\mathscr{T}_1\varphi\|_{X^S} \leq \|P_{\leq 1}\varphi_0\|_{\dot{H}^1} + \|P_{\geq 2}\varphi_0\|_{H^{\frac12}} + T^{\frac14}\|\varphi\|_{X^S(T)}(\|v\|_{X^W_{0,\frac12,1}(T)}+1). \tag{4.13}$$

Finally, we consider the second equation (3.2). From Lemma 3.1, we have

$$\|W(t)v_0\|_{X^W_{0,\frac{1}{2},1}(T)} \le \|v_0\|_{L^2}. \tag{4.14}$$

From Lemma 4.1 and Lemma 3.1, we have

$$\begin{aligned}&\Big\|I^W\Big(\frac{1}{\langle\nabla\rangle}\mathrm{Re}v + \mathrm{i}\gamma(v-\mathrm{Re}v)\Big)\Big\|_{X^W_{0,\frac{1}{2},1}(T)}\\ \le&\Big\|\frac{1}{\langle\nabla\rangle}\mathrm{Re}v + \mathrm{i}\gamma(v-\mathrm{Re}v)\Big\|_{X^W_{0,-\frac{1}{2},1}(T)}\\ \le&CT^{\frac{1}{2}}\|v\|_{X^W_{0,\frac{1}{2},1}(T)}.\end{aligned} \tag{4.15}$$

From Lemma 3.1 and Proposition 3.4

$$\begin{aligned}\Big\|I^W\Big(\frac{\omega}{\mathrm{i}\langle\nabla\rangle}\nabla\varphi^*\cdot\overline{\nabla}\varphi\Big)\Big\|_{X^W_{0,\frac{1}{2},1}(T)} &\le C\Big\|\frac{1}{\langle\nabla\rangle}\nabla\varphi^*\cdot\overline{\nabla}\varphi\Big\|_{X^W_{0,-\frac{1}{2},1}(T)}\\ &\le T^{\frac{1}{4}}\|\varphi\|^2_{X^S(T)}.\end{aligned} \tag{4.16}$$

So we have

$$\|\mathscr{T}_2 v\|_{X^W_{0,\frac{1}{2},1}(T)} \le \|v_0\|_{L^2} + T^{\frac{1}{4}}\|\varphi\|_{X^S(T)}(\|\varphi\|_{X^S(T)}+1). \tag{4.17}$$

In conclusion, from (4.13), (4.17) and standard iteration argument, we can construct a unique solution $(\varphi, v) \in B_{X^S(T)(0,\ C\|\varphi_0\|_{H^{\frac{1}{2}}\cup\dot{H}^1})} \times B_{X^W_{0,\frac{1}{2},1}(T)(0,\ C\|v_0\|_{L^2})}$ for Eq. (3.1)-(3.2). In addition, we can also show local Lipschitz continuity for the map $(\varphi_0, v_0) \mapsto (\varphi, v)$.

The proof for Corollary 1.2 is similar, we do not show the detail here. □

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