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ASYMPTOTIC BEHAVIOR OF THE CAHN-HILLIARD-OONO EQUATION

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Abstract Our aim in this article is to study the asymptotic behavior, in terms of finite-dimensional attractors, of the Cahn-Hilliard-Oono equation. This equation differs from the usual Cahn-Hilliard equation by the presence of a term of the form ϵu, $\epsilon > 0$, which takes into account long-ranged interactions. In particular, we prove the existence of a robust family of exponential attractors as ϵ goes to 0.

Keywords Cahn-Hilliard-Oono equation, well-posedness, asymptotic behavior, global attractor, exponential attractor.

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1. Introduction

We consider the following initial and boundary value problem:

$$\frac{\partial u}{\partial t} + \epsilon u + \Delta^2 u - \Delta f(u) = 0, \ \epsilon \geq 0, \tag{1.1}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \text{ on } \Gamma, \tag{1.2}$$

$$u|_{t=0} = u_0, \tag{1.3}$$

in a bounded and regular domain Ω of $\mathbb{R}^3$ with boundary Γ; ν denotes the unit outer normal to Γ.

Here, $u = u^\epsilon$ is the order parameter (it corresponds to a rescaled density of atoms or concentration) and f is the derivative of a double-well potential F whose wells characterize the phases. A thermodynamically relevant potential F is the following logarithmic function which follows from a mean-field model:

$$F(s) = \frac{\theta_c}{2}(1-s^2) + \frac{\theta}{2}[(1-s)\ln(\frac{1-s}{2}) + (1+s)\ln(\frac{1+s}{2})], \ s \in (-1,1), \ 0 < \theta < \theta_c, \tag{1.4}$$

hence

$$f(s) = -\theta_c s + \frac{\theta}{2} \ln \frac{1+s}{1-s}, \tag{1.5}$$

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although, as this will be the case here, such a function is very often approximated by regular ones, typically,

$$F(s)=\frac{1}{4}(s^2-1)^2, \tag{1.6}$$

hence

$$f(s)=s^3-s \tag{1.7}$$

(see [2], [3], [4], [9] and [19]). Such an approximation is reasonable when the quench is shallow, i.e., when the absolute temperature θ is close to the critical one θ_c.

In particular, when $\epsilon=0$, (1.1) reduces to the well-known Cahn-Hilliard equation. When $\epsilon>0$, (1.1) is known as the Oono equation (see [22]) and was introduced to model long-ranged (nonlocal) interactions; actually, this equation was also introduced in order to simplify numerical simulations (see [20]). Short-ranged interactions tend to homogenize the system, whereas long-ranged ones forbid the formation of too large structures; the competition between these two effects translates into the formation of a micro-separated state (also called super-crystal) with a spatially modulated order parameter, defining structures with a uniform size (see [22] for more details and references). Here, the term ϵu, $\epsilon>0$, models the long-ranged interactions.

We can note that (1.1) is a particular (and actually simplified) case of nonlocal Cahn-Hilliard models (see [16], [17] and [18]), obtained by considering the free energy

$$\psi=\frac{1}{2}|\nabla u|^2+F(u)+\int_\Omega u(y)g(y,x)u(x)dy, \tag{1.8}$$

where $|\cdot|$ denotes the usual Euclidean norm and the function g describes the long-ranged interactions. In particular, in Oono's model, one takes

$$g(y,x)=\frac{4\pi\epsilon}{|y-x|},\ \epsilon>0. \tag{1.9}$$

Note that the long-ranged interactions are repulsive when $u(y)$ and $u(x)$ have opposite signs and thus favor the formation of interfaces (see [22] and the references therein). Writing finally, as in the derivation of the classical Cahn-Hilliard equation,

$$\frac{\partial u}{\partial t}=\Delta\frac{\delta\psi}{\delta u}, \tag{1.10}$$

where $\frac{\delta}{\delta u}$ denotes a variational derivative, we find (1.1), noting that $-\frac{1}{|y-x|}$ is the Green function associated with the Laplace operator (see [22] and the references therein for more details).

Integrating now (1.1) over Ω, we have

$$\frac{d<u>}{dt}+\epsilon<u>=0, \tag{1.11}$$

where $<\cdot>=\frac{1}{\mathrm{Vol}(\Omega)}\int_\Omega\cdot dx$ (or $\frac{1}{\mathrm{Vol}(\Omega)}<\cdot,1>_{H^{-1}(\Omega),H^1(\Omega)}$ if one considers functions in $H^{-1}(\Omega)$, $H^{-1}(\Omega)$ denoting the dual space of $H^1(\Omega)$ and $<\cdot,\cdot>_{H^{-1}(\Omega),H^1(\Omega)}$ denoting the duality product) denotes the spatial average. Therefore, when $\epsilon>0$, we no longer have the conservation of mass, i.e., of $<u>$, as in the Cahn-Hilliard equation ($\epsilon = 0$); however, if $< u_0 >= 0$, then $< u(t) >= 0$, $\forall t \geq 0$. Indeed, it follows from (1.11) that

$$< u(t) >= e^{-\epsilon t} < u_0 >, \ t \geq 0. \tag{1.12}$$

We also deduce from (1.12) that

$$| < u(t) > | \leq | < u_0 > |, \ t \geq 0, \tag{1.13}$$

$$\lim_{t \to +\infty} < u(t) >= 0 \ (\epsilon > 0 \text{ fixed}), \tag{1.14}$$

$$\lim_{\epsilon \to 0^+} < u(t) >=< u_0 > \ \ (t \geq 0 \text{ fixed}). \tag{1.15}$$

Our aim in this article is to study the asymptotic behavior of the dynamical system associated with (1.1)-(1.3). In particular, we prove the existence of a robust family of exponential attractors as ϵ goes to 0, i.e., of a family of compact and finite-dimensional sets which attract all bounded sets of initial data exponentially fast and is Hölder continuous at $\epsilon = 0$. This shows that, in some proper sense, the dynamics of the Cahn-Hilliard equation is "close" to that of the Cahn-Hilliard-Oono equation, for $\epsilon > 0$ small. Recall that the Cahn-Hilliard-Oono equation was also introduced for computational purposes.

Throughout this article, the same constant c (and, sometimes, c' or c'') denotes constants which may vary from line to line (and even in a same line).

2. A priori estimates

We assume in what follows that f is the usual cubic nonlinearity (1.7) and we further assume that

$$| < u_0 > | \leq M, \ M \geq 0. \tag{2.1}$$

In particular, we have, a priori,

$$| < u(t) > | \leq M, \ \forall t \geq 0. \tag{2.2}$$

We first note that it follows from (1.1) and (1.11) that

$$\frac{\partial \overline{u}}{\partial t} + \epsilon \overline{u} + \Delta^2 u - \Delta f(u) = 0, \tag{2.3}$$

where $\overline{u} = u- < u >$, which we can rewrite (formally) in the equivalent form

$$\frac{\partial}{\partial t}(-\Delta)^{-1}\overline{u} + \epsilon(-\Delta)^{-1}\overline{u} - \Delta u + f(u)- < f(u) >= 0, \tag{2.4}$$

where $(-\Delta)^{-1}$ denotes the inverse Laplace operator with Neumann boundary conditions and acting on functions with null average.

Multiplying (2.4) by $\overline{u}$, integrating over Ω and integrating by parts, we have

$$\frac{1}{2}\frac{d}{dt}\|\overline{u}\|_{-1}^2 + \epsilon\|\overline{u}\|_{-1}^2 + \|\nabla u\|^2 + ((f(u), \overline{u})) = 0, \tag{2.5}$$

where $((\cdot,\cdot))$ and $\|\cdot\|$ denote the usual L^2-scalar products and norms, respectively, and $\|\cdot\|_{-1} = \|(-\Delta)^{-\frac{1}{2}}\cdot\|$. We then note that

$$((f(u),u)) \geq \frac{3}{4}\|u\|^4_{L^4(\Omega)} - c,$$

$$|<u>\int_\Omega f(u)dx| \leq \frac{1}{4}\|u\|^4_{L^4(\Omega)} + c_M,$$

where, here and below, $\|\cdot\|_X$ denotes the norm on the Banach space X, which yields

$$\frac{d}{dt}\|\overline{u}\|^2_{-1} + \|\nabla u\|^2 + \|u\|^4_{L^4(\Omega)} \leq c_M. \tag{2.6}$$

Noting finally that

$$\|\overline{u}\|_{-1} \leq c\|\nabla u\|, \tag{2.7}$$

we find

$$\frac{d}{dt}\|\overline{u}\|^2_{-1} + c\|\overline{u}\|^2_{-1} + \frac{1}{2}\|\nabla u\|^2 + \|u\|^4_{L^4(\Omega)} \leq c'_M. \tag{2.8}$$

In particular, in (2.8), all constants are independent of ϵ. We now note that

$$\frac{d}{dt} <u>^2 = 2<u>\frac{d<u>}{dt} = -2\epsilon <u>^2 \leq 0, \tag{2.9}$$

which yields, recalling (2.2),

$$\frac{d}{dt}(\|\overline{u}\|^2_{-1} + <u>^2) + c(\|\overline{u}\|^2_{-1} + <u>^2) + \frac{1}{2}\|\nabla u\|^2 + \|u\|^4_{L^4(\Omega)} \leq c'_M. \tag{2.10}$$

We deduce from (2.10) and Gronwall's lemma a dissipative estimate on $\|\overline{u}\|^2_{-1} + <u>^2$, namely,

$$\|\overline{u}(t)\|^2_{-1} + <u(t)>^2 \leq e^{-ct}(\|\overline{u}_0\|^2_{-1} + <u_0>^2) + c',\ c>0, \tag{2.11}$$

where, here and below, we omit the dependence on M. Noting finally that $v \mapsto (\|\overline{v}\|^2_{-1} + <v>^2)^{\frac{1}{2}}$ is a norm on $H^{-1}(\Omega)$ which is equivalent to the usual H^1-norm, we deduce from (2.11) the existence of a bounded absorbing set for the associated dynamical system on $H^{-1}(\Omega)$, i.e., $\forall R>0$, $\|u_0\|_{H^{-1}(\Omega)} \leq R$, $\exists t_0 = t_0(R) \geq 0$ such that $t \geq t_0$ implies

$$\|u(t)\|_{H^{-1}(\Omega)} \leq c, \tag{2.12}$$

where the constant c is independent of R. We also have

$$\int_t^{t+r} \|\nabla u\|^2 ds \leq c_r,\ t \geq t_0,\ r>0, \tag{2.13}$$

$$\int_t^{t+r} \|u\|^4_{L^4(\Omega)} ds \leq c_r,\ t \geq t_0,\ r>0. \tag{2.14}$$

Again, all constants are independent of ϵ, but may depend on M.

We now multiply (1.1) by u and find, noting that

$$f' \geq -c_0,\ c_0 > 0, \tag{2.15}$$

the following inequation:

$$\frac{d}{dt}\|u\|^2 + \|\Delta u\|^2 \leq c\|\nabla u\|^2. \tag{2.16}$$

Noting that

$$\|u\|^2 \leq 2(\|\overline{u}\|^2 + < u >^2) \leq c(\|\nabla u\|^2 + M^2),$$

we deduce from (2.12), (2.13), (2.16) and the uniform Gronwall's lemma (see, e.g., [21]) that (assuming, as above, that $\|u_0\|_{H^{-1}(\Omega)} \leq R$)

$$\|u(t)\| \leq c,\ t \geq t_1(\geq t_0), \tag{2.17}$$

$$\int_t^{t+r} \|\Delta u\|^2 ds \leq c_r,\ t \geq t_1. \tag{2.18}$$

Note that this also yields the existence of a bounded absorbing set for the associated dynamical system on $L^2(\Omega)$.

We finally multiply (1.1) by $-\Delta u$ to obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + \epsilon\|\nabla u\|^2 + \|\nabla\Delta u\|^2 + ((\Delta f(u), \Delta u)) = 0. \tag{2.19}$$

Noting that

$$\Delta f(u) = f'(u)\Delta u + f''(u)|\nabla u|^2,$$

we have, owing to (2.15),

$$((\Delta f(u), \Delta u)) \geq 6\int_\Omega u|\nabla u|^2 dx - c_0\|\Delta u\|^2. \tag{2.20}$$

Furthermore, owing to Ladyzhenskaya's inequality,

$$\begin{aligned} \textstyle\int_\Omega |u||\nabla u|^2 dx &\leq \|u\|\|\nabla u\|^2_{L^4(\Omega)^3} \leq c\|u\|\|\nabla u\|^{\frac{1}{2}}\|u\|^{\frac{3}{2}}_{H^2(\Omega)} \\ &\leq c\|u\|^4(\|\nabla u\|^2 + 1) + c'\|\Delta u\|^2 + c''. \end{aligned} \tag{2.21}$$

We finally deduce from (2.19), (2.20) and (2.21) that

$$\frac{d}{dt}\|\nabla u\|^2 \leq c\|u\|^4(\|\nabla u\|^2 + 1) + c'\|\Delta u\|^2 + c''. \tag{2.22}$$

We again deduce from the above estimates and the uniform Gronwall's lemma that

$$\|u(t)\|_{H^1(\Omega)} \leq c,\ t \geq t_2(\geq t_1). \tag{2.23}$$

In particular, this yields the existence of a bounded absorbing set for the associated dynamical system on $H^1(\Omega)$.

3. Estimates on the difference of two solutions

We first derive an estimate which will yield the uniqueness of solutions (as well as the continuous dependence with respect to the initial data).

Let u_1 and u_2 be two solutions with initial data $u_{0,1}$ and $u_{0,2}$, respectively. We set $u = u_1 - u_2$ and $u_0 = u_{0,1} - u_{0,2}$. We then have

$$\frac{\partial u}{\partial t} + \epsilon u + \Delta^2 u - \Delta(f(u_1) - f(u_2)) = 0, \tag{3.1}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \text{ on } \Gamma, \tag{3.2}$$

$$u|_{t=0} = u_0. \tag{3.3}$$

As in the previous section, we can rewrite (3.1) in the form

$$\frac{\partial \overline{u}}{\partial t} + \epsilon \overline{u} + \Delta^2 u - \Delta(f(u_1) - f(u_2)) = 0. \tag{3.4}$$

We multiply (3.4) by $(-\Delta)^{-1}\overline{u}$ and have

$$\frac{1}{2}\frac{d}{dt}\|\overline{u}\|_{-1}^2 + \epsilon\|\overline{u}\|_{-1}^2 + \|\nabla u\|^2 + ((f(u_1) - f(u_2), \overline{u})) = 0. \tag{3.5}$$

We note that, owing to (2.15),

$$((f(u_1) - f(u_2), \overline{u})) \geq -c_0\|u\|^2 - < u > \int_\Omega (f(u_1) - f(u_2))dx. \tag{3.6}$$

Furthermore,

$$\begin{aligned} \|u\|^2 &\leq 2(\|\overline{u}\|^2 + < u >^2) \leq c(\|\overline{u}\|_{-1}\|\nabla u\| + < u >^2) \\ &\leq \gamma\|\nabla u\|^2 + c(\|\overline{u}\|_{-1}^2 + < u >^2), \ \forall \gamma > 0, \end{aligned} \tag{3.7}$$

hence

$$\begin{aligned} & | < u > \textstyle\int_\Omega (f(u_1) - f(u_2))dx| \\ \leq\ & c| < u > | \textstyle\int_\Omega (|u_1|^2 + |u_2|^2 + 1)|u|dx \\ \leq\ & c(\|u\|^2 + (\|u_1\|_{L^4(\Omega)}^4 + \|u_2\|_{L^4(\Omega)}^4 + 1) < u >^2) \\ \leq\ & \tfrac{1}{4}\|\nabla u\|^2 + c(\|u_1\|_{L^4(\Omega)}^4 + \|u_2\|_{L^4(\Omega)}^4 + 1)(\|\overline{u}\|_{-1}^2 + < u >^2). \end{aligned} \tag{3.8}$$

We thus deduce from the above that

$$\frac{d}{dt}\|\overline{u}\|_{-1}^2 + \|\nabla u\|^2 \leq c(\|u_1\|_{L^4(\Omega)}^4 + \|u_2\|_{L^4(\Omega)}^4 + 1)(\|\overline{u}\|_{-1}^2 + < u >^2),$$

and, noting again that

$$\frac{d}{dt} < u >^2 \leq 0,$$

we finally obtain

$$\frac{d}{dt}(\|\overline{u}\|_{-1}^2 + < u >^2) + \|\nabla u\|^2 \leq c(\|u_1\|_{L^4(\Omega)}^4 + \|u_2\|_{L^4(\Omega)}^4 + 1)(\|\overline{u}\|_{-1}^2 + < u >^2). \tag{3.9}$$

It follows from (2.10), (3.9) and Gronwall's lemma that

$$\|u(t)\|_{H^{-1}(\Omega)} \leq ce^{c't}\|u_0\|_{H^{-1}(\Omega)}, \tag{3.10}$$

where c and c' only depend on $\|u_{0,i}\|_{H^{-1}(\Omega)}$, $i = 1,\ 2$, and M (and are, in particular, independent of ϵ).

Next, we derive a smoothing property on the difference of two solutions which is the key estimate to prove the existence of exponential attractors (see [6]).

Keeping the above notation, we multiply (3.1) by tu to find, owing to (2.15),

$$\frac{d}{dt}(t\|u\|^2) \leq \|u\|^2 + ct\|\nabla u\|^2. \tag{3.11}$$

Integrating (3.11) between 0 and t, we have

$$\|u(t)\|^2 \leq c\frac{1+t}{t}\int_0^t \|u\|_{H^1(\Omega)}^2 ds,\ t > 0. \tag{3.12}$$

It now follows from (2.10), (3.9) and (3.10) that

$$\int_0^t \|\nabla u\|^2 ds \leq ce^{c't}\|u_0\|_{H^{-1}(\Omega)}^2. \tag{3.13}$$

Furthermore, recalling again that

$$\|u\|^2 \leq c(\|\nabla u\|^2 + < u >^2)$$

and noting that

$$< u >^2 \leq < u_0 >^2, \tag{3.14}$$

we deduce from (3.12) and (3.13) that

$$\|u(t)\|^2 \leq c\frac{1+t}{t}e^{c't}\|u_0\|_{H^{-1}(\Omega)}^2, \tag{3.15}$$

where all constants are independent of ϵ.

Let finally u^ϵ and u^0 be two solutions to (1.1) for $\epsilon > 0$ and $\epsilon = 0$, respectively, with the same initial datum u_0. We set $u = u^\epsilon - u^0$. We then have

$$\frac{\partial u}{\partial t} + \epsilon u + \Delta^2 u - \Delta(f(u^\epsilon) - f(u^0)) = -\epsilon u^0, \tag{3.16}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \text{ on } \Gamma, \tag{3.17}$$

$$u|_{t=0} = u_0. \tag{3.18}$$

We recall that

$$\frac{d < u^\epsilon >}{dt} + \epsilon < u^\epsilon >= 0\ (\epsilon > 0)$$

and we have

$$\frac{d < u^0 >}{dt} = 0,$$

which yields

$$\frac{d < u >}{dt} + \epsilon < u >= -\epsilon < u^0 > . \tag{3.19}$$

We thus deduce from (3.16) and (3.19) that

$$\frac{\partial \overline{u}}{\partial t} + \epsilon \overline{u} + \Delta^2 u - \Delta(f(u^\epsilon) - f(u^0)) = -\epsilon \overline{u^0}. \tag{3.20}$$

We multiply (3.20) by $(-\Delta)^{-1}\overline{u}$ to have

$$\frac{1}{2}\frac{d}{dt}\|\overline{u}\|_{-1}^2 + \epsilon\|\overline{u}\|_{-1}^2 + \|\nabla u\|^2 + ((f(u^\epsilon) - f(u^0), \overline{u})) = -\epsilon(((-\Delta)^{-\frac{1}{2}}\overline{u^0}, (-\Delta)^{-\frac{1}{2}}\overline{u})), \tag{3.21}$$

which yields, owing to (2.15) and recalling that

$$\|(-\Delta)^{-\frac{1}{2}}\overline{u}\| \leq c\|\nabla u\|,$$

the following inequation:

$$\frac{d}{dt}\|\overline{u}\|_{-1}^2 + \|\nabla u\|^2 \leq c\epsilon^2\|\overline{u^0}\|_{-1}^2 + c'\|u\|^2 + 2 < u > \int_\Omega (f(u^\epsilon) - f(u^0))dx. \tag{3.22}$$

We then recall that

$$\|u\|^2 \leq 2(\|\overline{u}\|^2 + < u >^2) \leq c(\|\overline{u}\|_{-1}\|\nabla u\| + < u >^2) \tag{3.23}$$

$$\leq \gamma\|\nabla u\|^2 + c(\|\overline{u}\|_{-1}^2 + < u >^2), \ \forall \gamma > 0,$$

hence

$$\begin{array}{ll} & | < u > \int_\Omega (f(u^\epsilon) - f(u^0))dx| \\ \leq & c| < u > | \int_\Omega (|u^\epsilon|^2 + |u^0|^2 + 1)|u|dx \\ \leq & c(\|u\|^2 + (\|u^\epsilon\|_{L^4(\Omega)}^4 + \|u^0\|_{L^4(\Omega)}^4 + 1) < u >^2) \\ \leq & \frac{1}{2}\|\nabla u\|^2 + c(\|u^\epsilon\|_{L^4(\Omega)}^4 + \|u^0\|_{L^4(\Omega)}^4 + 1)(\|\overline{u}\|_{-1}^2 + < u >^2). \end{array} \tag{3.24}$$

We thus deduce from (3.22), (3.23) and (3.24) that

$$\frac{d}{dt}\|\overline{u}\|_{-1}^2 \leq c\epsilon^2\|\overline{u^0}\|_{-1}^2 + c'(\|u^\epsilon\|_{L^4(\Omega)}^4 + \|u^0\|_{L^4(\Omega)}^4 + 1)(\|\overline{u}\|_{-1}^2 + < u >^2). \tag{3.25}$$

Noting finally that

$$\frac{d}{dt} < u >^2 = 2 < u > (-\epsilon < u > -\epsilon < u^0 >) \leq -2\epsilon < u >< u^0 > \tag{3.26}$$

$$\leq \epsilon^2 < u^0 >^2 + < u >^2,$$

we find

$$\begin{aligned} & \frac{d}{dt}(\|\overline{u}\|_{-1}^2 + < u >^2) \\ \leq \quad & c\epsilon^2(\|\overline{u^0}\|_{-1}^2 + < u^0 >^2) \\ & + c'(\|u^\epsilon\|_{L^4(\Omega)}^4 + \|u^0\|_{L^4(\Omega)}^4 + 1)(\|\overline{u}\|_{-1}^2 + < u >^2). \end{aligned} \tag{3.27}$$

Now, it follows from (2.10) that

$$\int_0^t \|u^\epsilon\|_{L^4(\Omega)}^4 ds \leq ce^{c't}, \tag{3.28}$$

and it is well-known that (see, e.g., [9], [15] and [21]; recall that, for $\epsilon = 0$, (1.1) reduces to the Cahn-Hilliard equation)

$$\int_0^t \|u^0\|_{L^4(\Omega)}^4 ds \leq ce^{c't}. \tag{3.29}$$

We further have (this is again well-known)

$$\|u^0\|_{H^{-1}(\Omega)} \leq c. \tag{3.30}$$

Here, all constants only depend on M and $\|u_0\|_{H^{-1}(\Omega)}$. We finally deduce from (3.27), (3.28), (3.29), (3.30) and Gronwall's lemma that

$$\|u(t)\|_{H^{-1}(\Omega)} \leq c\epsilon e^{c't}, \tag{3.31}$$

where the constants c and c' only depend on M and $\|u_0\|_{H^{-1}(\Omega)}$.

4. The dissipative semigroup

First, it follows from the a priori estimates obtained in Section 2, (3.10) and standard techniques that we have the existence and uniqueness of solutions to (1.1)-(1.3). Here, the proof essentially is the same as that of the classical Cahn-Hilliard equation (see, e.g., [7], [15] and [21]). In particular, this allows to define the family of solution operators

$$S_\epsilon(t) : H^{-1}(\Omega) \to H^{-1}(\Omega),\ u_0 \mapsto u(t),\ t \geq 0,\ \epsilon \geq 0,$$

which maps the initial datum onto the solution at time t. This family of operators forms a continuous semigroup, i.e.,

$$x \mapsto S_\epsilon(t)x \text{ is continuous},\ t \geq 0,$$

$$S_\epsilon(0) = \mathrm{I},\ S_\epsilon(t+s) = S_\epsilon(t) \circ S_\epsilon(s),\ t,\ s \geq 0,$$

where I denotes the identity operator. Finally, we have

$$S(t) : H^{-1}(\Omega) \to H^1(\Omega),\ t > 0.$$

Setting now

$$\Phi_M = \{v \in H^{-1}(\Omega),\ |<v>| \leq M\},\ M \geq 0,$$

it follows from the uniform estimates obtained in Section 2 that we have the dissipative semigroup (still denoted by $S_\epsilon(t)$) acting on the phase space Φ_M,

$$S_\epsilon(t) : \Phi_M \to \Phi_M,\ t \geq 0.$$

Finally, it follows, again from the uniform estimates obtained in Section 2 and from standard results (see, e.g., [1], [14] and [21]), that we have the

Theorem 4.1. *The semigroup $S_\epsilon(t)$ possesses the global attractor $\mathcal{A}_\epsilon^M$ on the phase space Φ_M which is compact in $L^2(\Omega)$ and bounded in $H^1(\Omega)$.*

Remark 4.1. We recall that the global attractor $\mathcal{A}_\epsilon^M$ is the smallest (for the inclusion) compact set of the phase space which is invariant by the flow (i.e., $S_\epsilon(t)\mathcal{A}_\epsilon^M = \mathcal{A}_\epsilon^M,\ \forall t \geq 0$) and attracts all bounded sets of initial data as time goes to infinity; it thus appears as a suitable object in view of the study of the asymptotic behavior of the system.

5. Robust exponential attractors

We first recall the following result concerning the construction of a robust family of exponential attractors for a discrete dynamical system (see [7]; see also [8], [10], [11], [12] and [13] for generalizations):

Proposition 5.1. *Let H and H_1 be two Banach spaces such that the injection $H_1 \subset H$ is compact, B be a bounded subset of H and $L_\epsilon : B \to B$, $\epsilon \in [0, \epsilon_0]$, $\epsilon_0 > 0$, be a family of operators such that*
a) For every x_1, $x_2 \in B$ and every $\epsilon \in [0, \epsilon_0]$,

$$\|L_\epsilon x_1 - L_\epsilon x_2\|_{H_1} \leq c\|x_1 - x_2\|_H,$$

where the constant c is independent of ϵ.
b) For every $\epsilon \in [0, \epsilon_0]$, every $i \in \mathbb{N}$ and every $x \in B$,

$$\|L_\epsilon^i x - L_0^i x\|_H \leq c^i \epsilon,$$

where the constant c is independent of ϵ.
Then, there exists a family $\mathcal{M}_\epsilon \subset B$, $\epsilon \in [0, \epsilon_0]$, such that $\mathcal{M}_\epsilon$ is an exponential attractor for the discrete dynamical system generated by L_ϵ, i.e.,
(i) The set $\mathcal{M}_\epsilon$ is compact in H and has finite fractal dimension in H,

$$\dim_F \mathcal{M}_\epsilon \leq c.$$

(ii) The set $\mathcal{M}_\epsilon$ is positively invariant,

$$L_\epsilon \mathcal{M}_\epsilon \subset \mathcal{M}_\epsilon.$$

(iii) The set $\mathcal{M}_\epsilon$ attracts B exponentially fast,

$$\mathrm{dist}_H(L^i B, \mathcal{M}_\epsilon) \leq c e^{-c' i},\ i \in \mathbb{N},\ c' > 0,$$

where dist_H *denotes the Hausdorff semidistance between sets defined by*

$$\mathrm{dist}_H(A,B) = \sup_{a\in A}\inf_{b\in B}\|a-b\|_H.$$

(iv) Furthermore, the family $\mathcal{M}_\epsilon$ *is Hölder continuous at* $\epsilon = 0$,

$$\mathrm{dist}_{\mathrm{sym}}(\mathcal{M}_\epsilon,\mathcal{M}_0) \le c\epsilon^{c'},\ c' \in (0,1),$$

where $\mathrm{dist}_{\mathrm{sym}}$ *denotes the Hausdorff symmetric distance between sets defined by*

$$\mathrm{dist}_{\mathrm{sym}}(A,B) = \max(\mathrm{dist}_H(A,B),\mathrm{dist}_H(B,A)).$$

Finally, all constants are independent of ϵ *and can be computed explicitly.*

Based on Proposition 5.1, we can prove the

Theorem 5.1. *For every* $\epsilon \in [0,\epsilon_0]$, $\epsilon_0 > 0$, *the semigroup* $S_\epsilon(t)$ *acting on* Φ_M *possesses an exponential attractor* $\mathcal{M}_\epsilon^M$ *on* Φ_M *such that*
1. The set $\mathcal{M}_\epsilon^M$ *has finite fractal dimension in* $H^{-1}(\Omega)$,

$$\mathrm{dim}_{\mathrm{F}}\mathcal{M}_\epsilon^M \le c.$$

2. The set $\mathcal{M}_\epsilon^M$ *is positively invariant by* $S_\epsilon(t)$,

$$S_\epsilon(t)\mathcal{M}_\epsilon^M \subset \mathcal{M}_\epsilon^M,\ t \ge 0.$$

3. The set $\mathcal{M}_\epsilon^M$ *attracts all bounded subsets of* Φ_M *exponentially fast, i.e., for every bounded subset* B *of* Φ_M, *there exists a constant* $c = c(B)$ *such that*

$$\mathrm{dist}_{H^{-1}(\Omega)}(S_\epsilon(t)B,\mathcal{M}_\epsilon^M) \le ce^{-c't},\ t \ge 0,\ c' > 0.$$

4. The family of sets $\mathcal{M}_\epsilon^M$ *is Hölder continuous at* 0,

$$\mathrm{dist}_{\mathrm{sym}}(\mathcal{M}_\epsilon^M,\mathcal{M}_0^M) \le c\epsilon^{c'},\ c' \in (0,1).$$

Furthermore, all constants are independent of ϵ *and can be computed explicitly.*

Proof. We first note that, owing to the uniform estimates obtained in Section 2, we have the existence of a uniform (with respect to ϵ) absorbing set $B_0 \subset \Phi_M \cap H^1(\Omega)$, i.e., $\forall B \subset \Phi_M$ bounded, $\exists t_0 = t_0(B) > 0$ independent of $\epsilon \in [0,\epsilon_0]$ such that

$$S_\epsilon(t)B \subset B_0,\ t \ge t_0,\ \epsilon \in [0,\epsilon_0].$$

It is thus sufficient to construct the exponential attractor $\mathcal{M}_\epsilon^M$ on B_0.

To do so, as usual (see [5]), we first construct exponential attractors for a proper family of discrete semigroups and then pass to the continuous case.

It is easy to show that there exists $t_1 > 0$ independent of $\epsilon \in [0,\epsilon_0]$ such that

$$S_\epsilon(t)B_0 \subset B_0,\ t \ge t_1,\ \epsilon \in [0,\epsilon_0].$$

We then set

$$L_\epsilon = S_\epsilon(t_1)$$

and consider the spaces $H = H^{-1}(\Omega)$ and $H_1 = L^2(\Omega)$. It follows from (3.15) and (3.31) that the assumptions of Proposition 5.1 are satisfied, hence the existence of a robust family of exponential attractors $\mathcal{M}_\epsilon^{M,d}$ for the discrete dynamical systems generated by the operators L_ϵ.

We finally set

$$\mathcal{M}_\epsilon^M = \cup_{t\in[0,t_1]} S_\epsilon(t)\mathcal{M}_\epsilon^{M,d}.$$

To finish the proof, it suffices to prove that the mapping $(t, x) \mapsto S_\epsilon(t)x$ is Hölder continuous on $[0, t_1] \times B$, uniformly with respect to $\epsilon \in [0, \epsilon_0]$ (see [5] and [7]). The Hölder (and, actually, Lipschitz) continuity with respect to x follows from (3.10). To prove the Hölder continuity with respect to t, we note that

$$\|S_\epsilon(t+s)u_0 - S_\epsilon(t)u_0\|_H \leq |s|^{\frac{1}{2}}\left(\int_t^{t+s} \|\frac{\partial u}{\partial t}\|^2_{H^{-1}(\Omega)} d\tau\right)^{\frac{1}{2}}$$

$$\leq \quad c|s|^{\frac{1}{2}}\left(\int_t^{t+s} (\|u\|^2_{H^1(\Omega)} + \|f(u)\|^2) d\tau\right)^{\frac{1}{2}}$$

and we easily conclude in view of the estimates obtained in Section 2.

□

As a consequence of this result, we have the

Corollary 5.1. *The global attractor $\mathcal{A}_\epsilon^M$ has finite fractal dimension in $H^{-1}(\Omega)$.*

Indeed, an exponential attractor always contains the global attractor.

Remark 5.1. (i) The finite dimensionality means, roughly speaking, that, even though the initial phase space is infinite dimensional, the reduced dynamics is, in some proper sense, finite dimensional and can be described by a finite number of parameters. We refer the reader to [1], [14] and [21] for more details and discussions on this.
(ii) Proceeding as in [7], we can actually prove that all the above results hold for the topology of $L^2(\Omega)$ instead of that of $H^{-1}(\Omega)$.

Remark 5.2. (i) Here, we have the Hölder continuity only at $\epsilon = 0$. However, proceeding as in [8], we can construct a robust family of exponential attractors which is Hölder continuous at every $\epsilon \in [0, \epsilon_0]$.
(ii) We can consider more general nonlinearities f, typically, polynomials of the form

$$f(s) = \sum_{i=0}^{2p+1} a_i s^i, \ a_{2p+1} > 0, \ p \in \mathbb{N}.$$

In that case, it is not difficult to adapt the estimates of Section 2. However, when $p \geq 2$, we are not able to obtain Lipschitz type estimates as in Section 3, but only Hölder type ones (in particular, in the equivalent of (3.10)). Actually, here the situation is similar to that in [13] for the Cahn-Hilliard equation with singular potentials and the construction of a robust family of exponential attractors on Φ_M should be much more involved (see [13], Remark 6.2; one can nevertheless treat the case when the average of the order parameter vanishes (we recall that, in that case, the average of the order parameter is conserved)).

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