THE $C^1$ SOLUTION OF THE HIGH DIMENSIONAL FEIGENBAUM-LIKE FUNCTIONAL EQUATION

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Abstract By constructing a structure operator quite different from before, and using the Schauder’s fixed point theory, the existence and uniqueness of the $C^1$ solutions of the high dimensional Feigenbaum-like functional equations are discussed.

Keywords Feigenbaum-like functional equations; Schauder’s fixed point; existence; uniqueness.

MSC(2000) 35D, 35C.

1. Introduction

The second kind of Feigenbaum functional equation (1.1) with initial value condition in interval $[0, 1]$ is an important form of iterative equations. 

\[
\begin{cases}
    f(x) = \frac{1}{\lambda}f(f(\lambda x)), 0 < \lambda < 1; \\
    f(0) = 1; \\
    0 \leq f(x) \leq 1, x \in [0, 1]
\end{cases}
\] (1.1)

The case of the continuous single-peak solutions was considered in [1-3] and [5-7], also the precise solution was considered in some functional space. Certainly, these discussions are very important to the dynamical systems, but being iterative equations, we also focus on the no-monotone continuous case. In this paper, we consider the $C^1$ solution of the Feigenbaum-like functional equation

\[
f(x) = \frac{1}{\lambda + 1}f^2(\lambda x) + \frac{\lambda}{\lambda + 1}g(x)
\] (1.2)

where $g(x)$ is a given function, $f(x)$ is an unknown function, and $f(x)$, $g(x)$ are the differentiable function of the high dimensional space $\mathbb{R}^n$, $0 < \lambda < 1$. We use the different methods from [1-3] and [5-7].

2. Preliminaries

Suppose $C^0(B, \mathbb{R}^n) = \{f : B \to \mathbb{R}^n, f \text{ is continuous}\}$, where $B$ is a compact convex subset of $\mathbb{R}^n$. let $| \cdot |_n$ be a norm of $\mathbb{R}^n$, $\cdot |_{n \times n}$ be a norm of $\mathbb{R}^{n \times n}$. Clearly

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$C^0(B, \mathbb{R}^n)$ is a Banach space with the norm $\| \cdot \|_{c,0}$, where $\|f\|_{c,0} = \sup_{x \in B} |f(x)|_n$, for $f \in C^0(B, \mathbb{R}^n)$. Let $C^1(B, \mathbb{R}^n) = \{ f : B \to \mathbb{R}^n, \text{f is continuous and continuously differentiable}\}$, then $C^1(B, \mathbb{R}^n)$ is a Banach space with the norm $\| \cdot \|_{c,1}$, where $\|f\|_{c,1} = \sup_{x \in B} |f(x)|_n + \sup_{x \in B} |f'(x)|_{n \times n}$, for $f \in C^1(B, \mathbb{R}^n)$.

Being a closed subset of $C^1(B, \mathbb{R}^n)$, $C^1(B, B)$ defined by $C^1(B, B) = \{ f \in C^1(B, \mathbb{R}^n), f(B) \subseteq B, \forall x \in B \}$ is a complete space.

**Lemma 2.1.** Suppose that $\varphi \in C^1(B, B)$ and

\[ |\varphi'(x)|_{n \times n} \leq M, \forall x \in B, \quad (2.1) \]

\[ |\varphi'(x_1) - \varphi'(x_2)|_{n \times n} \leq M'|x_1 - x_2|_n, \forall x_1, x_2 \in B, \quad (2.2) \]

where $M$ and $M'$ are positive constants. Then

\[ |(\varphi^n(x_1))' - (\varphi^n(x_2))'|_{n \times n} \leq M' \sum_{i=n-1}^{2n-2} M^i |x_1 - x_2|_n, \quad (2.3) \]

where any $x_1, x_2$ in $B$, $(\varphi^n)'$ is the Jacobian matrix denoting $d\varphi^n/dx$.

**Proof.** The methods of proof in (2.3) are similar to paper [8], but the conditions are weaker than that of [8]. Now we prove (2.4).

For any $x_1, x_2$ in $B$,

\[
(\varphi^n(x_1))' - (\varphi^n(x_2))' = |\varphi'((\varphi^n(x_1))' - (\varphi^n(x_2))'|_{n \times n}
\leq |\varphi'((\varphi^n(x_1))' - (\varphi^n(x_2))'|_{n \times n}
+ |\varphi'((\varphi^n(x_2))' - (\varphi^n(x_2))'|_{n \times n}
\leq \lambda M |((\varphi^n(x_1))' - (\varphi^n(x_2))'|_{n \times n} + M |((\varphi^n(x_2))' - (\varphi^n(x_2))'|_{n \times n}
\leq \lambda M M' |\varphi^n(x_1) - \varphi^n(x_2)|_n + \lambda^2 M M' |x_1 - x_2|_n
\leq \lambda^2 M^2 M' |x_1 - x_2|_n + \lambda^2 M M' |x_1 - x_2|_n
\leq \lambda^2 M' (M + M') |x_1 - x_2|_n.

Thus (2.4) holds.

**Lemma 2.2.** Suppose that $\varphi_1, \varphi_2 \in C^0(B, \mathbb{R}^n)$, $\varphi_1(x), \varphi_2(x)$ satisfy (2.1) and $\varphi(\lambda x) := \varphi(\lambda)(x)$. Then

\[ |\varphi_1^{(\lambda)}(x) - \varphi_2^{(\lambda)}(x)|_n \leq (M + 1)|\varphi_1^{(\lambda)} - \varphi_2^{(\lambda)}|_n, \quad (2.5) \]

**Proof.** Since

\[ |\varphi_1^{(\lambda)}(x) - \varphi_2^{(\lambda)}(x)|_n = \sup_{x \in B} |\varphi_1^{(\lambda)}(x) - \varphi_2^{(\lambda)}(x)|_n
\leq \sup_{x \in B} |(\varphi_1^{(\lambda)}(x) - \varphi_1^{(\lambda)}(\lambda x)) - (\varphi_2^{(\lambda)}(x) - \varphi_2^{(\lambda)}(\lambda x))|_n
+ \sup_{x \in B} |(\varphi_2^{(\lambda)}(\lambda x) - \varphi_2^{(\lambda)}(\lambda x)) - (\varphi_1^{(\lambda)}(\lambda x) - \varphi_1^{(\lambda)}(\lambda x))|_n
\leq M |\varphi_1^{(\lambda)} - \varphi_2^{(\lambda)}|_n + |\varphi_1^{(\lambda)} - \varphi_2^{(\lambda)}|_n
= (M + 1)|\varphi_1^{(\lambda)} - \varphi_2^{(\lambda)}|_n.

Thus (2.5) holds.
Lemma 2.3. Suppose that \( \varphi_1, \varphi_2 \in C^1(B, B) \). Then

\[
|\varphi_1^{(2)}(\lambda x)' - \varphi_2^{(2)}(\lambda x)'|_{n \times n} \leq 2\lambda M |\varphi_1^{(\lambda)} - \varphi_2^{(\lambda)}|_n.
\]

Proof. Because

\[
|\varphi_1^{(2)}(\lambda x)' - \varphi_2^{(2)}(\lambda x)'|_{n \times n} \leq \sup_{x \in B} \{ |\varphi_1'((\varphi_1(\lambda x))' - \varphi_2'((\varphi_2(\lambda x))'\varphi_1(\lambda x))'\varphi_2(\lambda x))'|_{n \times n}\}
\]

\[
+ |\varphi_2'((\varphi_2(\lambda x))' - \varphi_2'((\varphi_2(\lambda x))'\varphi_2(\lambda x))'\varphi_2(\lambda x))'|_{n \times n}\}
\]

\[
\leq \lambda M \sup_{x \in B} |\varphi_1'(\varphi_1(\lambda x))' - \varphi_1'(\varphi_2(\lambda x))'|_{n \times n}
\]

\[
+ \lambda M \sup_{x \in B} |\varphi_2'(\varphi_2(\lambda x))' - \varphi_2'(\varphi_2(\lambda x))'|_{n \times n} + \lambda M |\varphi_1'(\lambda x) - \varphi_2'(\lambda x)|_{n \times n}
\]

\[
\leq 2\lambda M |\varphi_1^{(\lambda)} - \varphi_2^{(\lambda)}|_n + \lambda M M' |\varphi_1^{(\lambda)} - \varphi_2^{(\lambda)}|_n.
\]

Thus (2.6) holds.

\[\square\]

3. Main results

For given constants \( M_1 > 0 \) and \( M_2 > 0 \), let

\[
\mathcal{A}(M_1, M_2) = \{ \varphi \in C^1(B, B) : |\varphi'(x)|_{n \times n} \leq M_1, \forall x \in B,
\]

\[
|\varphi'(x) - \varphi'(x)|_{n \times n} \leq M_2 |x_1 - x_2|, \forall x_1, x_2 \in B\}.
\]

Theorem 3.1. (Existence) Given positive constants \( M_1, M_2 \) and \( g \in \mathcal{A}(M_1, M_2) \), if there exists constants \( K_1, K_2, 0 \leq K_1 \leq 1 \) and \( K_2 \geq \frac{1}{g} \), such that

\( (P_1) \quad K_2^2 + M_1 \leq K_1, \)

\( (P_2) \quad K_2(K_1 + K_2^2) + M_2 \leq K_2, \)

then equation (1.2) has a solution \( f \) in \( \mathcal{A}(K_1, K_2) \).

Proof. Define \( T : \mathcal{A}(K_1, K_2) \to C^1(B, B) \) as follow:

\[
Tf(x) = \frac{1}{\lambda + 1} f^2(\lambda x) + \frac{\lambda}{\lambda + 1} g(x), \quad \forall x \in B
\]

where \( f \in \mathcal{A}(K_1, K_2) \), then \( f(\lambda x) \in \mathcal{A}(\lambda K_1, \lambda K_2) \). Because \( f \), \( f(\lambda x) \) and \( g \) are continuously differentiable for all \( x \in B \), then \( Tf \) is also continuously differentiable for all \( x \in B \). By condition \( (P_1) \) and \( (P_2) \), for any \( x_1, x_2 \in B \), thus

\[
|(Tf(x))'|_{n \times n} \leq \frac{1}{\lambda + 1} |(f^2(\lambda x))'|_{n \times n} + \frac{\lambda}{\lambda + 1} |g'(x)|_{n \times n}
\]

\[
\leq \frac{1}{\lambda + 1} |(f^2(f(\lambda x))(f(\lambda x))'|_{n \times n} + \frac{\lambda}{\lambda + 1} |g'(x)|_{n \times n}
\]

\[
\leq \frac{\lambda}{\lambda + 1} (K_2^2 + M_1) \leq \lambda K_1 \leq K_1.
\]
Furthermore
\[
|\langle T f(x_1) \rangle' - \langle T f(x_2) \rangle'|_{n \times n}
\]
\[
= |\frac{1}{\lambda + 1} (f^2(\lambda x_1)') + \frac{\lambda}{\lambda + 1} g'(x_1) - \frac{1}{\lambda + 1} (f^2(\lambda x_2)') - \frac{\lambda}{\lambda + 1} g'(x_2)|_{n \times n}
\]
\[
\leq \frac{\lambda}{\lambda + 1} |(f^2(\lambda x_1)') - (f^2(\lambda x_2)')|_{n \times n} + \frac{\lambda}{\lambda + 1} |g'(x_1) - g'(\lambda x_2)|
\]
\[
\leq \frac{\lambda^2}{\lambda + 1} K_2 (K_1 + K_2^2) |x_1 - x_2| + \frac{\lambda}{\lambda + 1} M_2 |x_1 - x_2|
\]
\[
\leq \frac{\lambda}{\lambda + 1} (K_2 (K_1 + K_2^2) + K_2) |x_1 - x_2|
\]
\[
\leq \lambda K_2 |x_1 - x_2|
\]
\[
\leq K_2 |x_1 - x_2|
\]

Thus \( T : \mathcal{A}(K_1, K_2) \rightarrow \mathcal{A}(K_1, K_2) \) is a self-diffeomorphism.

Now we prove the continuity of \( T \) under the norm \( \| \cdot \|_{\text{c}^1} \). For arbitrary \( f_1, f_2 \in \mathcal{A}(K_1, K_2), f_i(\lambda x) \in \mathcal{A}(\lambda K_1, \lambda K_2) \) \( i = 1, 2 \) and \( f(\lambda x) := f^{(\lambda)}(x) \),

\[
\|T f_1 - T f_2\|_{\text{c}^1} = \sup_{x \in B} \|T f_1(x) - T f_2(x)\|_n = \sup_{x \in B} \|T f_1(x)\|_n - \|T f_2(x)\|_n
\]

Let
\[
E_1 = \frac{1}{\lambda + 1} (K_1 + 1) + \frac{\lambda}{\lambda + 1} K_1 K_2, \quad E_2 = \frac{2\lambda}{\lambda + 1} K_1, \quad E = \max\{E_1, E_2\}
\]

Then we have
\[
\|T f_1 - T f_2\|_{\text{c}^1} = \sup_{x \in B} \|T f_1(x) - T f_2(x)\|_n + \sup_{x \in B} \|T f_1(x)\|_n - \|T f_2(x)\|_n
\]
\[
\leq E_1 |f_1^{(\lambda)}| - |f_2^{(\lambda)}| + E_2 |(f_1^{(\lambda)})' - (f_2^{(\lambda)})'|_{n \times n}
\]
\[
\leq E |f_1 - f_2| + E |(f_1)' - (f_2)'|_{n \times n}
\]

(3.1)

which gives continuity of \( T \).

It is easy to show that \( \mathcal{A}(K_1, K_2) \) is a compact convex subset of \( C^1(B, B) \). By the
Schauder’s fixed point theorem, we assert that there is a mapping $f \in \mathcal{A}(K_1, K_2)$, such that

$$f(x) = Tf(x) = \frac{1}{\lambda + 1} f^2(\lambda x) + \frac{\lambda}{\lambda + 1} g(x), \quad \forall x \in B.$$  

This completes the proof.

**Theorem 3.2. (Uniqueness)** Suppose that $(P_1)$ and $(P_2)$ are satisfied, also we suppose that $(P_3) : E < 1$, then for arbitrary function $g$ in $\mathcal{A}(M_1, M_2)$, equation (1.2) has a unique solution in $\mathcal{A}(K_1, K_2)$.

**Proof.** The existence of equation (1.2) in $\mathcal{A}(K_1, K_2)$ is given by theorem 3.1, from the proof of theorem 3.1, we see that $\mathcal{A}(K_1, K_2)$ is a closed subset of $C^1(B, B)$, by (3.1) and (P_3), we see that

$$T : \mathcal{A}(K_1, K_2) \rightarrow \mathcal{A}(K_1, K_2)$$

is a contraction. Therefore $T$ has a unique fixed point $f(x)$ in $\mathcal{A}(K_1, K_2)$, that is, equation (1.2) has a unique solution in $\mathcal{A}(K_1, K_2)$, this proves theorem.

**4. Example**

Suppose $B := \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1, x_i \in \mathbb{R}, i = 1, 2\} \subset \mathbb{R}^2$. Clearly $B$ is a compact convex subset of $\mathbb{R}^2$. We can choose

$$g(x) := g(x_1, x_2) = (g_1(x_1, x_2), g_2(x_1, x_2)) = (\frac{1}{24} x_1^2, \frac{1}{16} x_1 x_2),$$

then $g(x) \in C^1(B, B)$.

Consider the equation

$$f(x) = \frac{2}{3} f^2(\frac{1}{2} x) + \frac{1}{3} g(x), x \in B. \quad (4.1)$$

It is in the form of (1.2) where $\lambda = \frac{1}{2}$.

Since

$$g'(x) = \left( \begin{array}{cc} \frac{1}{12} x_1 & 0 \\ \frac{1}{16} x_2 & \frac{1}{16} x_1 \end{array} \right),$$

then

$$|g'(x)|_{2 \times 2} = \max_{x \in B} \left( \frac{1}{12} |x_1|, \frac{1}{16} (|x_1| + |x_2|) \right) \leq \frac{1}{8}.$$  

For any $x, y \in B$, let $x = (x_1, x_2), y = (y_1, y_2)$, we can obtain

$$g'(x) = \left( \begin{array}{cc} \frac{1}{12} x_1 & 0 \\ \frac{1}{16} x_2 & \frac{1}{16} x_1 \end{array} \right), g'(y) = \left( \begin{array}{cc} \frac{1}{12} y_1 & 0 \\ \frac{1}{16} y_2 & \frac{1}{16} y_1 \end{array} \right),$$

then

$$|g'(x) - g'(y)|_{2 \times 2} = \max_{x, y \in B} \left( \frac{1}{12} |x_1 - y_1|, \frac{1}{16} (|x_1 - y_1| + |x_2 - y_2|) \right) \leq \frac{\sqrt{2}}{12} |x - y|_2,$$

thus $g \in \mathcal{A}(\frac{1}{8}, \frac{\sqrt{2}}{12})$.

By condition $(P_1)$, there is

$$K^2 - K_1 + \frac{1}{8} \leq 0,$$
then $K_1 \in \left( \frac{1}{2} - \frac{\sqrt{2}}{2}, \frac{1}{2} + \frac{\sqrt{2}}{2} \right)$, we can choose $K_1 = 0.5$.

Similarly, by condition $(P_2)$, there is

$$K_2 \left( \frac{1}{2} + \frac{1}{4} \right) + \frac{\sqrt{2}}{12} \leq K_2,$$

then $K_2 \in \left( \frac{\sqrt{2}}{3}, +\infty \right)$, we can choose $K_2 = 0.6$. Then by theorem 3.1, there is a continuously differentiable solution of equation (4.1) in $\mathcal{A}(0.5, 0.6)$.

**References**


