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THE C^1 SOLUTION OF THE HIGH DIMENSIONAL FEIGENBAUM-LIKE FUNCTIONAL EQUATION

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Abstract By constructing a structure operator quite different from before, and using the Schauder's fixed point theory, the existence and uniqueness of the C^1 solutions of the high dimensional Feigenbaum-like functional equations are discussed.

Keywords Feigenbaum-like functional equations; Schauder's fixed point; existence; uniqueness.

MSC(2000) 35D, 35C.

1. Introduction

The second kind of Feigenbaum functional equation (1.1) with initial value condition in interval $[0,1]$ is an important form of iterative equations .

$$\begin{cases} f(x) = \frac{1}{\lambda} f(f(\lambda x)), 0 < \lambda < 1; \\ f(0) = 1; \\ 0 \le f(x) \le 1, x \in [0,1] \end{cases} \tag{1.1}$$

The case of the continuous single-peak solutions was considered in [1-3]and[5-7], also the precise solution was considered in some functional space. Certainly, these discussions are very important to the dynamical systems, but being iterative equations, we also focus on the no-monotone continuous case. In this paper, we consider the C^1 solution of the Feigenbaum-like functional equation

$$f(x) = \frac{1}{\lambda+1} f^2(\lambda x) + \frac{\lambda}{\lambda+1} g(x) \tag{1.2}$$

where $g(x)$ is a given function, $f(x)$ is an unknown function, and $f(x)$, $g(x)$ are the differentiable function of the high dimensional space $\mathbf{R}^n$, $0 < \lambda < 1$. We use the different methods from [1-3]and[5-7].

2. Preliminaries

Suppose $C^0(B, \mathbf{R}^n) = \{f : B \to \mathbf{R}^n, f \text{ is continuous}\}$, where B is a compact convex subset of $\mathbf{R}^n$. let $|\cdot|_n$ be a norm of $\mathbf{R}^n$, $|\cdot|_{n\times n}$ be a norm of $\mathbf{R}^{n\times n}$. Clearly

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$C^0(B,\mathbf{R}^n)$ is a Banach space with the norm$\|\cdot\|_{c^0}$, where $\|f\|_{c^0}=\sup_{x\in B}|f(x)|_n$, for $f\in C^0(B,\mathbf{R}^n)$. Let $C^1(B,\mathbf{R}^n)$=$\{f: B\to \mathbf{R}^n,\ f$ is continuous and continuously differentiable$\}$, then $C^1(B,\mathbf{R}^n)$ is a Banach space with the norm $\|.\|_{c^1}$, where$\|f\|_{c^1}=\sup_{x\in B}|f(x)|_n+\sup_{x\in B}|f'(x)|_{n\times n}$, for $f\in C^1(B,\mathbf{R}^n)$.

Being a closed subset of $C^1(B,\mathbf{R}^n)$, $C^1(B,B)$ defined by $C^1(B,B)=\{f\in C^1(B,\mathbf{R}^n), f(B)\subseteq B,\forall x\in B\}$ is a complete space.

Lemma 2.1. *Suppose that* $\varphi\in C^1(B,B)$ *and*

$$|\varphi'(x)|_{n\times n}\le M,\forall x\in B, \tag{2.1}$$

$$|\varphi'(x_1)-\varphi'(x_2)|_{n\times n}\le M'|x_1-x_2|_n,\forall x_1,x_2\in B, \tag{2.2}$$

where M *and* M' *are positive constants. Then*

$$|(\varphi^n(x_1))'-(\varphi^n(x_2))'|_{n\times n}\le M'(\sum_{i=n-1}^{2n-2}M^i)|x_1-x_2|_n, \tag{2.3}$$

$$|(\varphi^2(\lambda x_1))'-(\varphi^2(\lambda x_2))'|_{n\times n}\le \lambda^2M'(M+M^2)|x_1-x_2|_n\ , \tag{2.4}$$

where any x_1,x_2 *in* B, $(\varphi^n)'$ *is the Jacobian matrix denoting* $d\varphi^n/dx$.

Proof. The methods of proof in (2.3) are similar to paper [8],but the conditions are weaker than that of [8]. Now we proof (2.4).

For any x_1,x_2 in B,

$$\begin{aligned}
&|(\varphi^2(\lambda x_1))'-(\varphi^2(\lambda x_2))'|_{n\times n}\\
=\ &|\varphi'(\varphi(\lambda x_1))(\varphi(\lambda x_1))'-\varphi'(\varphi(\lambda x_2))(\varphi(\lambda x_2))'|_{n\times n}\\
\le\ &|\varphi'(\varphi(\lambda x_1))(\varphi(\lambda x_1))'-\varphi'(\varphi(\lambda x_2))(\varphi(\lambda x_1))'|_{n\times n}\\
&+|\varphi'(\varphi(\lambda x_2))(\varphi(\lambda x_1))'-\varphi'(\varphi(\lambda x_2))(\varphi(\lambda x_2))'|_{n\times n}\\
\le\ &\lambda M|(\varphi'(\varphi(\lambda x_1))-(\varphi'(\varphi(\lambda x_2))|_{n\times n}+M|(\varphi(\lambda x_1))'-(\varphi(\lambda x_2))'|_{n\times n}\\
\le\ &\lambda MM'|\varphi(\lambda x_1)-\varphi(\lambda x_2)|_n+\lambda^2MM'|x_1-x_2|_n\\
\le\ &\lambda^2M^2M'|x_1-x_2|_n+\lambda^2MM'|x_1-x_2|_n\\
\le\ &\lambda^2M'(M+M')|x_1-x_2|_n.
\end{aligned}$$

Thus (2.4)holds. □

Lemma 2.2. *Suppose that* φ_1 , $\varphi_2\in C^0(B,\boldsymbol{R}^n)$, $\varphi_1(x)$, $\varphi_2(x)$ *satisfy (2.1) and* $\varphi(\lambda x):=\varphi^{(\lambda)}(x)$. *Then*

$$|\varphi_1^{2(\lambda)}-\varphi_2^{2(\lambda)}|_n\le(M+1)|\varphi_1^{(\lambda)}-\varphi_2^{(\lambda)}|_n. \tag{2.5}$$

Proof. Since

$$\begin{aligned}
|\varphi_1^{2(\lambda)}-\varphi_2^{2(\lambda)}|_n &= \sup_{x\in B}|\varphi_1^2(\lambda x)-\varphi_2^2(\lambda x)|_n\\
&\le \sup_{x\in B}|(\varphi_1(\varphi_1(\lambda x))-\varphi_1(\varphi_2(\lambda x))|_n\\
&\quad+\sup_{x\in B}|\varphi_1(\varphi_2(\lambda x))-\varphi_2(\varphi_2(\lambda x))|_n\\
&\le M|\varphi_1^{(\lambda)}-\varphi_2^{(\lambda)}|_n+|\varphi_1^{(\lambda)}-\varphi_2^{(\lambda)}|_n\\
&= (M+1)|\varphi_1^{(\lambda)}-\varphi_2^{(\lambda)}|_n.
\end{aligned}$$

Thus(2.5)holds. □

Lemma 2.3. *Suppose that* $\varphi_1, \varphi_2 \in C^1(B, B)$. *Then*

$$\begin{aligned} & |(\varphi_1^{2(\lambda)})' - (\varphi_2^{2(\lambda)})'|_{n\times n} \\ \leq \; & 2\lambda M|(\varphi_1^{(\lambda)})' - (\varphi_2^{(\lambda)})'|_{n\times n} + \lambda M M'|\varphi_1^{(\lambda)} - \varphi_2^{(\lambda)}|_n. \end{aligned} \tag{2.6}$$

Proof. Because

$$\begin{aligned} & |(\varphi_1^{2(\lambda)})' - (\varphi_2^{2(\lambda)})'|_{n\times n} \\ \leq \; & \sup_{x\in B}\{|\varphi_1'(\varphi_1(\lambda x))(\varphi_1(\lambda x))' - \varphi_2'(\varphi_2(\lambda x))(\varphi_1(\lambda x))'|_{n\times n}\} \\ & + \sup_{x\in B}\{|\varphi_2'(\varphi_2(\lambda x))(\varphi_1(\lambda x))' - \varphi_2'(\varphi_2(\lambda x))(\varphi_2(\lambda x))'|_{n\times n}\} \\ \leq \; & \lambda M \sup_{x\in B}\{|\varphi_1'(\varphi_1(\lambda x)) - \varphi_2'(\varphi_2(\lambda x))|_{n\times n} + \lambda^n M|\varphi_1'(\lambda x) - \varphi_2'(\lambda x)|_{n\times n}\} \\ \leq \; & \lambda M \sup_{x\in B}|\varphi_1'(\varphi_1(\lambda x)) - \varphi_1'(\varphi_2(\lambda x))|_{n\times n} \\ & + \lambda M \sup_{x\in B}|\varphi_1'(\varphi_2(\lambda x) - \varphi_2'(\varphi_2(\lambda x))|_{n\times n} + \lambda M|\varphi_1'(\lambda x) - \varphi_2'(\lambda x)|_{n\times n} \\ \leq \; & 2\lambda M|(\varphi_1^{(\lambda)})' - (\varphi_2^{(\lambda)})'|_{n\times n} + \lambda M M'|\varphi_1^{(\lambda)} - \varphi_2^{(\lambda)}|_n. \end{aligned}$$

Thus (2.6)holds. ☐

3. Main results

For given constants $M_1 > 0$ and $M_2 > 0$, let

$$\begin{aligned} \mathcal{A}(M_1, M_2) = \{\varphi \in C^1(B, B) : & |\varphi'(x)|_{n\times n} \leq M_1, \forall x \in B, \\ & |\varphi'(x_1) - \varphi'(x_2)|_{n\times n} \leq M_2|x_1 - x_2|_n \forall x_1, x_2 \in B\}. \end{aligned}$$

Theorem 3.1. *(***Existence)** *Given positive constants* M_1 , M_2 *and* $g \in \mathcal{A}(M_1, M_2)$, *if there exists constants* K_1, K_2, $0 \leq K_1 \leq 1$ *and* $K_2 \geq \frac{1}{2}$, *such that*

(P_1) $\quad K_1^2 + M_1 \leq K_1,$

(P_2) $\quad K_2(K_1 + K_1^2) + M_2 \leq K_2,$

then equation(1.2) has a solution f *in* $\mathcal{A}(K_1, K_2)$.

Proof. Define $T : \mathcal{A}(K_1, K_2) \to C^1(B, B)$ as follow:

$$Tf(x) = \frac{1}{\lambda+1}f^2(\lambda x) + \frac{\lambda}{\lambda+1}g(x), \quad \forall x \in B$$

where $f \in \mathcal{A}(K_1, K_2)$, then $f(\lambda x) \in \mathcal{A}(\lambda K_1, \lambda K_2)$. Because f , $f(\lambda x)$ and g are continuously differentiable for all $x \in B$, then Tf is also continuously differentiable for all $x \in B$. By condition (P_1) and (P_2), for any x_1, x_2 in B, thus

$$\begin{aligned} |(Tf(x))'|_{n\times n} & \leq \frac{1}{\lambda+1}|(f^2(\lambda x))'|_{n\times n} + \frac{\lambda}{\lambda+1}|g'(x)|_{n\times n} \\ & \leq \frac{1}{\lambda+1}|(f'(f(\lambda x))(f(\lambda x))'|_{n\times n} + \frac{\lambda}{\lambda+1}|g'(x)|_{n\times n} \\ & \leq \frac{\lambda}{\lambda+1}(K_1^2 + M_1) \leq \lambda K_1 \leq K_1. \end{aligned}$$

Furthermore

$$\begin{aligned}
& |(Tf(x_1))' - (Tf(x_2))'|_{n\times n} \\
= & \ |\frac{1}{\lambda+1}(f^2(\lambda x_1))' + \frac{\lambda}{\lambda+1}g'(x_1) - \frac{1}{\lambda+1}(f^2(\lambda x_2))' - \frac{\lambda}{\lambda+1}g'(x_2)|_{n\times n} \\
\leq & \ \frac{\lambda}{\lambda+1}|(f^2(\lambda x_1))' - (f^2(\lambda x_2))'|_{n\times n} + \frac{\lambda}{\lambda+1}|g'(x_1)) - g'(\lambda x_2))|_{n\times n} \\
\leq & \ \frac{\lambda^2}{\lambda+1}K_2(K_1+K_1^2)|x_1-x_2|_n + \frac{\lambda}{\lambda+1}M_2|x_1-x_2|_n \\
\leq & \ \frac{\lambda}{\lambda+1}(K_2(K_1+K_1^2)+K_2)|x_1-x_2|_n \\
\leq & \ \lambda K_2|x_1-x_2|_n \\
\leq & \ K_2|x_1-x_2|_n.
\end{aligned}$$

Thus $T : \mathcal{A}(K_1, K_2) \to \mathcal{A}(K_1, K_2)$ is a self-diffeomorphism.
Now we prove the continuity of T under the norm $\|.\|_{c^1}$. For arbitrary $f_1, f_2 \in \mathcal{A}(K_1, K_2)$, $f_i(\lambda x) \in \mathcal{A}(\lambda K_1, \lambda K_2)$ $i = 1, 2$ and $f(\lambda x) := f^{(\lambda)}(x)$,

$$\begin{aligned}
& |Tf_1 - Tf_2|_n \\
\leq & \ \sup_{x\in B}\{|\frac{1}{\lambda+1}f_1^2(\lambda x) + \frac{\lambda}{\lambda+1}g(x) - \frac{1}{\lambda+1}f_2^2(\lambda x) - \frac{\lambda}{\lambda+1}g(x)|_n\} \\
= & \ \frac{1}{\lambda+1}\sup_{x\in B}\{|f_1^2(\lambda x) - f_2^2(\lambda x)|_n\} \\
\leq & \ \frac{1}{\lambda+1}(K_1+1)|f_1^{(\lambda)} - f_2^{(\lambda)}|_n, \\
& |(Tf_1)' - (Tf_2)'|_{n\times n} \\
\leq & \ \sup_{x\in B}\{|\frac{1}{\lambda+1}(f_1^2(\lambda x))' + \frac{\lambda}{\lambda+1}g'(x) - \frac{1}{\lambda+1}(f_2^2(\lambda x))' - \frac{\lambda}{\lambda+1}g'(x)|_{n\times n}\} \\
= & \ \sup_{x\in B}\{\frac{1}{\lambda+1}|(f_1^2(\lambda x))' - (f_2^2(\lambda x))'|_{n\times n}\} \\
\leq & \ \frac{1}{\lambda+1}\{\lambda K_1K_2|f_1^{(\lambda)} - f_2^{(\lambda)}|_n + 2\lambda K_1|(f_1^{(\lambda)})' - (f_2^{(\lambda)})'|_{n\times n}\}.
\end{aligned}$$

Let

$$E_1 = \frac{1}{\lambda+1}(K_1+1) + \frac{\lambda}{\lambda+1}K_1K_2,\ E_2 = \frac{2\lambda}{\lambda+1}K_1,\ E = \max\{E_1, E_2\}.$$

Then we have

$$\begin{aligned}
\|Tf_1 - Tf_2\|_{c^1} &= \sup_{x\in B}|Tf_1(x) - Tf_2(x)|_n + \sup_{x\in B}|(Tf_1(x))' - (Tf_2(x))'|_{n\times n} \\
&\leq E_1|f_1^{(\lambda)} - f_2^{(\lambda)}|_n + E_2|(f_1^{(\lambda)})' - (f_2^{(\lambda)})'|_{n\times n} \\
&\leq E|f_1 - f_2|_n + E|(f_1)' - (f_2)'|_{n\times n} \\
&= E\|f_1 - f_2\|_{c^1}, \qquad (3.1)
\end{aligned}$$

which gives continuity of T.
It is easy to show that $\mathcal{A}(K_1, K_2)$ is a compact convex subset of $C^1(B, B)$. By the

Schauder's fixed point theorem, we assert that there is a mapping $f \in \mathcal{A}(K_1, K_2)$, such that

$$f(x) = Tf(x) = \frac{1}{\lambda+1} f^2(\lambda x) + \frac{\lambda}{\lambda+1} g(x), \quad \forall x \in B.$$

This completes the proof . □

Theorem 3.2. (**Uniqueness**) *Suppose that (P_1) and (P_2) are satisfied, also we suppose that (P_3) : $E < 1$, then for arbitrary function g in $\mathcal{A}(M_1, M_2)$, equation (1.2) has a unique solution $f \in \mathcal{A}(K_1, K_2)$.*

Proof. The existence of equation(1.2) in $\mathcal{A}(K_1, K_2)$ is given by theorem 3.1, from the proof of theorem 3.1, we see that $\mathcal{A}(K_1, K_2)$ is a closed subset of $C^1(B, B)$, by (3.1) and (P_3), we see that

$$T : \mathcal{A}(K_1, K_2) \to \mathcal{A}(K_1, K_2)$$

is a contraction . Therefore T has a unique fixed point $f(x)$ in $\mathcal{A}(K_1, K_2)$, that is, equation(1.2) has a unique solution in $\mathcal{A}(K_1, K_2)$, this proves theorem. □

4. Example

Suppose $B := \{(x_1, x_2) : x_1^2 + x_2^2 \le 1, x_i \in \mathbf{R}, i = 1, 2\} \subset \mathbf{R}^2$. Clearly B is a compact convex subset of $\mathbf{R}^2$.We can choose

$$g(x) := g(x_1, x_2) = (g_1(x_1, x_2), g_2(x_1, x_2)) = (\frac{1}{24} x_1^2, \frac{1}{16} x_1 x_2),$$

then $g(x) \in C^1(B, B)$.
Consider the equation

$$f(x) = \frac{2}{3} f^2(\frac{1}{2} x) + \frac{1}{3} g(x), x \in B. \tag{4.1}$$

It is in the form of (1.2) where $\lambda = \frac{1}{2}$.
Since

$$g'(x) = (\begin{matrix} \frac{1}{12} x_1 & 0 \\ \frac{1}{16} x_2 & \frac{1}{16} x_1 \end{matrix}),$$

then

$$|g'(x)|_{2\times 2} = \max_{x \in B} (\frac{1}{12}|x_1|, \frac{1}{16}(|x_1| + |x_2|)) \le \frac{1}{8}.$$

For any $x, y \in B$, let $x = (x_1, x_2), y = (y_1, y_2)$, we can obtain

$$g'(x) = \begin{pmatrix} \frac{1}{12} x_1 & 0 \\ \frac{1}{16} x_2 & \frac{1}{16} x_1 \end{pmatrix}, g'(y) = \begin{pmatrix} \frac{1}{12} y_1 & 0 \\ \frac{1}{16} y_2 & \frac{1}{16} y_1 \end{pmatrix},$$

then

$$|g'(x) - g'(y)|_{2\times 2} = \max_{x,y \in B} (\frac{1}{12}|x_1 - y_1|, \frac{1}{16}(|x_1 - y_1| + |x_2 - y_2|)) \le \frac{\sqrt{2}}{12}|x - y|_2,$$

thus $g \in \mathcal{A}(\frac{1}{8}, \frac{\sqrt{2}}{12})$.
By condition (P_1) , there is

$$K_1^2 - K_1 + \frac{1}{8} \le 0,$$

then $K_1 \in (\frac{1-\frac{\sqrt{2}}{2}}{2}, \frac{1+\frac{\sqrt{2}}{2}}{2})$, we can choose $K_1 = 0,5$.
Simily,by condition (P_2) , there is

$$K_2(\frac{1}{2}+\frac{1}{4})+\frac{\sqrt{2}}{12} \leq K_2,$$

then $K_2 \in (\frac{\sqrt{2}}{3}, +\infty)$, we can choose $K_2 = 0.6$. Then by theorem 3.1 , there is a continuously differentiable solution of equation(4.1) in $\mathcal{A}(0.5, 0.6)$.

References

[1] H. Epstein, *New proofs of the existence of the Feigenbaum function*, Comm., Math. phys., 106 (1986), 395-426.

[2] J. P. Eokmann and P. Wittwer, *A computer assisted proof of the Feigenbaum conjectures*, J. Stat. phys., 46 (1987), 455-477.

[3] O. E. Lanford III , *A computer assisted proof of the Feigenbaum conjectures*, Bull. Amer. Math Soc., 6 (1987), 427-434.

[4] X. P. Li, *A Class of Iterative Equation on a Banach Space*,Journal of Sichuan University(N. SCI.E), 41 (3)(2004), 505-510.

[5] B. D. Mestel, A. H. Osbaldestin and A. V. Tstgvintsev, *Continued fractions and solutions of the Feigenbaum-Cvitanovie equation*, C. R. Acad. Sci. Paris, 334 (2002), 683-688. Sci., 125B:1 (2005), 130-136.

[6] P. J. Mccarthy, *The general exact bijective continuous solution of Feigenbaum's functional equation*, Comm. Math.Phys., 91 (1983), 431-443.

[7] C. J. Thompson and J. B. Mcguire, *Asymptotic and essentially singular solution of the Feigenbaum equation*, J. Stat. phys., 27 (1982), 183-200.

[8] W. N. Zhang, *Discussion on the differentiable solutions of the iterated equation* , Nonlinear Analysis:Theory, Mathods & Applications, vol.15, 4(1990), 387-398.