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LIE SYMMETRIES AND THE CENTER PROBLEM*

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Abstract In this short survey we discuss the narrow relation between the center problem and the Lie symmetries. It is well known that an analytic vector field $\mathcal{X}$ having a non–degenerate center has a non–trivial analytic Lie symmetry in a neighborhood of it, i.e. there exists an analytic vector field $\mathcal{Y}$ such that $[\mathcal{X}, \mathcal{Y}] = \mu \mathcal{X}$. The same happens for a nilpotent center with an analytic first integral as can be seen from the recent results about nilpotent centers. From the recent results for nilpotent and degenerate centers it also can be proved that any nilpotent or degenerate center has a trivial smooth (of class $\mathcal{C}^{\infty}$) Lie symmetry. It remains as open problem if there always exists also a non–trivial Lie symmetry for any nilpotent and degenerate center.

Keywords Lie symmetries, center problem, vector fields.

MSC(2000) Primary 34C14; Secondary 34A26, 37C27, 34C25.

1. Introduction

One of the most important applications of the Lie's group theory is to the integrability problem of ordinary differential equations in the sense that knowledge of sufficient large group of symmetries of system of ordinary differential equations allows one to integrate the system by quadratures. On the other hand, the narrow relation between the center problem and the integrability problem shown first by Poincaré [20] suggests, as it is, that there is a narrow relation between the center problem and the Lie symmetries. In this paper we study the relation between systems that have a center type singularity at the origin and the existence of a Lie symmetry defined in a neighborhood of it. Actually we try to respond if the Lie symmetries is what characterize the existence of a center.

The paper is organized as follows. In Section 2 we give a brief summary on Lie's symmetries for a planar differential system. In Section 3 we recall the center problem and we establish its relations with the Lie symmetries and in Section 4 we give an illustrative example.

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2. Lie symmetries and planar differential systems

Let us consider a planar autonomous differential system

$$\dot{x} = \frac{dx}{dt} = P(x,y) \ , \quad \dot{y} = \frac{dy}{dt} = Q(x,y) \ , \tag{1}$$

with $P, Q \in C^1(\mathcal{U})$ and where $\mathcal{U}$ is an open subset of $\mathbb{R}^2$. Let $\mathcal{X}$ be the planar vector field associated to system (1), that is

$$\mathcal{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y} \ . \tag{2}$$

We denote by $\mathrm{div}\mathcal{X} = \partial P/\partial x + \partial Q/\partial y$ its divergence. We can also express differential system (1) like the vanishing of the 1-form $\omega = Q(x,y)\ dx - P(x,y)\ dy = 0$.

Let G be a *one-parameter Lie group* of transformations

$$x^*(x,y;\epsilon) = x + \epsilon\xi(x,y) + O(\epsilon^2) \ , \quad y^*(x,y;\epsilon) = y + \epsilon\eta(x,y) + O(\epsilon^2) \ , \tag{3}$$

acting on $\mathcal{U}$ with associated *infinitesimal generator* $\mathcal{Y}$ defined like

$$\mathcal{Y} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} \ , \tag{4}$$

with $\xi, \eta \in C^1(\mathcal{U})$. A *symmetry* of differential system (1) is defined to be a group of transformations (3) such that under the action of this group, a solution curve of system (1) is mapped into another solution curve of (1). The symmetry is called *non–trivial* in a neighborhood of a singular point if the infinitesimal generator $\mathcal{Y}$ is transversal to $\mathcal{X}$ at this point.

Let us define the *Lie bracket* of the C^1-vector fields $\mathcal{X}$ and $\mathcal{Y}$ as $[\mathcal{X},\mathcal{Y}] := \mathcal{X}\mathcal{Y} - \mathcal{Y}\mathcal{X}$. In other words we have

$$[\mathcal{X},\mathcal{Y}] = \left(P\frac{\partial \xi}{\partial x} - \xi\frac{\partial P}{\partial x} + Q\frac{\partial \xi}{\partial y} - \eta\frac{\partial P}{\partial y}\right)\frac{\partial}{\partial x} + \left(P\frac{\partial \eta}{\partial x} - \xi\frac{\partial Q}{\partial x} + Q\frac{\partial \eta}{\partial y} - \eta\frac{\partial Q}{\partial y}\right)\frac{\partial}{\partial y} \ . \tag{5}$$

The following proposition is well known, see for instance [23].

Proposition 1. *Let G be the one-parameter Lie group of transformations* (3). *Then G is a symmetry of system* (1) *if and only if the commutation relation*

$$[\mathcal{X},\mathcal{Y}] = \mu(x,y)\ \mathcal{X} \ , \tag{6}$$

is satisfied for some smooth scalar function $\mu(x,y)$.

The local dynamical effect of the Lie bracket in the phase plane is the following: Consider a point (x,y), and apply successively the flow after time t of the vector fields $\mathcal{X}$, $\mathcal{Y}$, $-\mathcal{X}$, $-\mathcal{Y}$ and the flow after time t^2 of the vector field $[\mathcal{X},\mathcal{Y}]$. Then, the reached point is precisely the starting point (x,y) (see Olver [19]). In particular when the Lie bracket vanishes, the flow of $\mathcal{Y}$ establishes a correspondence between the orbits $\mathcal{X}$, keeping the time unaltered, see [1] and Figure 1 (extracted also from [1]).

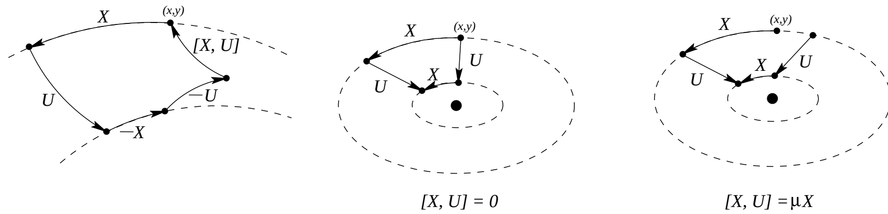

Figure 1. Geometrical interpretation of the Lie bracket $[\mathcal{X}, U]$, see [1].

Definition 1. Let $\mathcal{U}$ be the domain of definition of differential system (1), and let $\mathcal{V}$ be an open subset of $\mathcal{U}$. A C^1 function $V : \mathcal{V} \to \mathbb{R}$ such that $V \not\equiv 0$ and satisfying the linear partial differential equation

$$P\frac{\partial V}{\partial x} + Q\frac{\partial V}{\partial y} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) V \ , \tag{7}$$

is called an inverse integrating factor of system (1) on $\mathcal{V}$.

In other words, the inverse integrating factor is a C^1 function V such that ω/V is a closed form, i.e., its exterior derivative $d\omega$ is zero. We want to stress that the inverse integrating factor does not have to be defined in a neighborhood of a point for which P and Q is annulled, see [4]. Notice that the set $\{(x, y) \in \mathcal{V} \subset \mathbb{R}^2 : V(x, y) = 0\}$ is formed by orbits of system (1).

Remark 1. Recall that if we have a first integral $H \in C^1(\mathcal{V})$ then $VF(H)$ with $F \in C^1(\mathcal{V})$ is also an inverse integrating factor. Moreover, the first integral H associated to the inverse integrating factor V is given by

$$H(x, y) = \int \frac{P(x, y)}{V(x, y)} \ dy + f(x) \ ,$$

where the function f is calculated from the condition $\partial H/\partial x = -Q/V$.

On the other hand, it is known that a system (1) which admits a symmetry (3) has the following inverse integrating factor defined in $\mathcal{U}$

$$V(x, y) := \mathcal{X} \wedge \mathcal{Y} = P(x, y)\eta(x, y) - Q(x, y)\xi(x, y) \ , \tag{8}$$

provided $V(x, y) \not\equiv 0$. It is easy to prove this last result taking into account that $\mathcal{Y}H = 1$ where H is a C^1 first integral of system (1), see [19] or [23]. Conversely given an inverse integrating factor V we can get a Lie symmetry $\mathcal{Y}$ of $\mathcal{X}$ as

$$\mathcal{Y} = \frac{1}{\mathrm{div}\mathcal{X}} \ (-\frac{\partial V}{\partial y}\partial_x + \frac{\partial V}{\partial x}\partial_y), \tag{9}$$

defined in $\mathcal{U}\backslash\{(x, y) \in \mathcal{U} : \mathrm{div}\mathcal{X} = 0\}$.

Notice that if the components $P(x, y)$ and $Q(x, y)$ of system (1) are homogeneous polynomials then system (1) admits $\mathcal{Y} = x\ \partial/\partial x + y\ \partial/\partial y$ as infinitesimal generator of a Lie's symmetry. The following proposition is proved in [3].

Proposition 2. *Let $\mathcal{X} = P\ \partial/\partial x + Q\ \partial/\partial y$ and $\mathcal{Y} = \xi\ \partial/\partial x + \eta\ \partial/\partial y$ be two C^1 vector fields defined in an open subset $\mathcal{U} \subset \mathbb{R}^2$. Then the function $V := P\eta - Q\xi$ is an inverse integrating factor in $\mathcal{U}$ for both vector fields if and only if the local flows defined by the solutions of $\mathcal{X}$ and $\mathcal{Y}$ commute in the sense of Lie's bracket, that is $[\mathcal{X}, \mathcal{Y}] \equiv 0$ in $\mathcal{U}$.*

3. Lie symmetries and the center problem

Consider now that system (1) is an analytic differential systems defined in a neighborhood $U \subset \mathbb{R}^2$ of the origin such that $P(0,0) = Q(0,0) = 0$. The origin of system (1) is called *monodromic* if there are no orbits tending to or leaving the origin with a certain angle. For analytic systems, a monodromic singular point is always a center or a focus, see [16]. To distinguish between a center and a focus at the origin of system (1) is the so-called *center problem*, see for instance [7, 8]. The center problem goes back to Poincaré [20] at the end of 19th century. Throughout the 20th century, various kinds of methods to approach the problem have been developed and an extensive literature has been consequently produced, see for instance [6, 9, 14, 15, 21] and a wide range of references therein. Let us write system (1), under the conditions given in this section, into the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A_i \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}, \tag{10}$$

with A_i a real 2×2 matrix. The system is defined in an open set $U \subset \mathbb{R}^2$ where f and g are analytic functions in U starting in at least second order terms. We suppose that system (10) has a center at the origin. Doing a linear change of coordinates and a rescaling of time (if necessary), the system can be written with its linear part into the Jordan form, that is, A_i must be of the form:

$$(\mathbf{i})\ A_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ (\mathbf{ii})\ A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ (\mathbf{iii})\ A_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{11}$$

The class (i) is called *non-degenerate center*, the class (ii) *nilpotent center* and (iii) *degenerate center*.

3.1. Non-degenerate centers

According to Poincaré, system (10) with linear part A_1 has a center at the origin if, and only if, there exists a near–identity analytic change of coordinates

$$(u,v) = \phi(x,y) = (x + o(|(x,y)|), y + o(|(x,y)|))\ ,$$

transforming system (10) with linear part A_1 into the normal form

$$\dot{u} = -v[1 + \psi(u^2+v^2)]\ ,\ \dot{v} = u[1 + \psi(u^2+v^2)]\ , \tag{12}$$

with ψ an analytic function near the origin such that $\psi(0) = 0$. The transformed system (12) is known as the Poincaré normal form of a non-degenerate center. Attending to the form of the transformed system (12) it is clear that the original system (10) is analytically integrable or analytically orbitally linearizable in the sense of the definitions given in [10, 12]. Moreover, the transformed system (12) admits $\mathcal{Y} = u\,\partial/\partial u + v\,\partial/\partial v$ as infinitesimal generator of a Lie's symmetry. Hence, it follows that system (10) with linear part A_1 having a center at the origin has an analytic Lie symmetry. We have the following result, proved in [1] see also [10], that characterizes non-degenerate centers of any analytic vector fields in terms of Lie symmetries.

Theorem 1. *The smooth (resp. analytic) system* (10) *with linear part A_1 has a center at the origin if, and only if, there exists a smooth (resp. analytic) vector field $\mathcal{Y}$ of the form $\mathcal{Y} = (x + o(x,y))\partial/\partial x + (y + o(x,y))\partial/\partial x$ and a smooth (resp. analytic) scalar function $\mu(x,y)$ with $\mu(0,0) = 0$ such that $[\mathcal{X}, \mathcal{Y}] = \mu\mathcal{X}$.*

If the origin is a center, then the problem that arises is to determine when the period of the solutions near the origin is constant. A center with such property is called an *isochronous center*. The following theorem characterizes the isochronicity of a center in terms of a Lie symmetry, see [1, 10].

Theorem 2. *A center of an analytic system is isochronous if, and only if, there exists an analytic vector field $\mathcal{Y}$ of the form $\mathcal{Y} = (x+o(x,y))\partial/\partial x+(y+o(x,y))\partial/\partial x$, such that $[\mathcal{X}, \mathcal{Y}] \equiv 0$.*

3.2. Nilpotent centers with analytic first integral

Strózyna and Żołądek have proved in [22] that there exists an analytic change of coordinates near the origin transforming system (10) with linear part A_2 into a generalized Liénard system $\dot{x} = y$, $\dot{y} = a(x) + y\bar{b}(x)$ with $a(x) = a_s x^s + \cdots$, $s \geq 2$, and $\bar{b}(0) = 0$. In fact, following [22], if the nilpotent singularity is monodromic then there is a change of variables and a time rescaling leading to $\dot{x} = y$, $\dot{y} = -x^{2n-1} + yb(x)$ with $n \geq 2$. Hence, the center problem for nilpotent singularities reduce to the study of the parity of the function $b(x)$ according with the center conditions for the Liénard systems, see also [2]. Thus the following theorem was established.

Theorem 3 (Strózyna, Żołądek)**.** *Suppose that the analytic system* (10) *with linear part A_2 has a center at the origin. Then, there exists an analytic change of variables and a unity time rescaling such that it can be written as*

$$\dot{x} = y, \quad \dot{y} = -x^{2n-1} + yb(x)\ , \tag{13}$$

with $n \geq 2$ an integer and $b(x)$ an analytic odd function.

The following result given also by Strózyna and Żołądek in [22] is for the nilpotent centers with an analytic first integral.

Theorem 4 (Strózyna, Żołądek)**.** *The analytic system* (10) *with linear part A_2 and with a center at the origin has a local analytic first integral if, and only if, it is analytically orbitally equivalent to the Hamiltonian system*

$$\dot{x} = y, \quad \dot{y} = -x^{2k-1}\,. \tag{14}$$

It is straightforward to check that the Hamiltonian system (14) (which has the associated vector field $\bar{\mathcal{X}} = y\partial_x - x^{2k-1}\partial_y$) admits $\bar{\mathcal{Y}} = x\,\partial_x + k\,y\,\partial_y$ as infinitesimal generator of a Lie's symmetry and we have that $[\bar{\mathcal{X}}, \bar{\mathcal{Y}}] = (1-k)\bar{\mathcal{X}}$. Hence, the vector field $h\bar{\mathcal{X}}$ where $h = h(x,y) = 1 + f(x,y)$ with $f(x,y) = O(x,y)$ admits also

$\bar{\mathcal{Y}} = x\,\partial_x + k\,y\,\partial_y$ as infinitesimal generator of a Lie's symmetry because $[h\bar{\mathcal{X}}, \bar{\mathcal{Y}}] = ((1-k)h - \bar{\mathcal{Y}}(h))\bar{\mathcal{X}}$. Then, using the inverse of the near-identity analytic change of coordinates $\phi(x,y) = (x + \dots, y + \dots)$ from Theorem 3 we obtain the infinitesimal generator of a Lie's symmetry $\mathcal{Y} = \phi_* \bar{\mathcal{Y}}$ of the original system $\mathcal{X} = \phi_* \bar{\mathcal{X}}$ since the Lie bracket is coordinates free. Hence, we can establish the following result.

Theorem 5. *The analytic system* (10) *with linear part* A_2 *satisfying the monodromy conditions has a center at the origin with a local analytic first integral if, and only if, there exists a vector field* $\mathcal{Y}$ *of the form* $\mathcal{Y} = (x + o(x,y))\partial/\partial x + (y + o(x,y))\partial/\partial x$ *and an analytic scalar function* $\mu(x,y)$ *with* $\mu(0,0) = 0$ *such that* $[\mathcal{X}, \mathcal{Y}] = \mu\mathcal{X}$.

The problem that remains open is to characterize when a system has a nilpotent center without an analytic first integral and a degenerate center in terms of Lie's symmetries. This is the objective of the following section.

3.3. Nilpotent and degenerate centers

In the previous subsection, we have seen that all analytic non-degenerate and nilpotent centers with analytic first integral admit an analytic Lie symmetry. In the following we investigate the existence of a $\mathcal{C}^\infty$ Lie symmetry for any nilpotent center without analytic first integral and for any degenerate center. First we give some known results for any analytic center (non-degenerate, nilpotent or degenerate).

Assume that system (10) has a nilpotent or a degenerate center at the origin. In [17] the authors proved the following result.

Theorem 6 (Mazzi, Sabatini)**.** *System* (10) *has a center at the origin if, and only if, there exits a first integral of class* $\mathcal{C}^\infty$ *with an isolated minimum at the origin in a neighborhood of it.*

Mattei and Moussu [18] proved the next result for all isolated singularities.

Theorem 7 (Mattei, Moussu)**.** *Assume that system* (10) *with an isolated singularity at the origin has a formal first integral* $H \in \mathbb{R}[[x,y]]$ *around it. Then, there exists an analytic first integral* $H \neq 0$ *around the singularity.*

In the light of the former results we can conclude that a center of the analytic system (10) has either an analytic first integral or a $\mathcal{C}^\infty$ flat first integral. Moreover, in [17] the authors prove the following result for system (10) having a nilpotent or a degenerate center at the origin.

Proposition 3. *System* (10) *has a center at the origin if and only if there exists an invariant measure with density* R *of class* $\mathcal{C}^\infty$ *defined in a neighborhood of it.*

From a classical Liouville result, it is known that a measure with density R is invariant for the flow of $\mathcal{X}$ if and only if $\mathcal{X}(R) + R\,\mathrm{div}\mathcal{X} = 0$. Hence, Proposition 3 proves the existence of an integrating factor R of class $\mathcal{C}^\infty$ in a neighborhood U of the center located at the origin of system (10). Moreover, given an integrating factor R we can always construct another $\mathcal{C}^\infty$ integrating factor $\bar{R}$ of the form $\bar{R} = RH$ with H a $\mathcal{C}^\infty$ first integral of Theorem 6. This fact will be useful in what follows. Now we can establish the following proposition.

Proposition 4. *Assume that the analytic system* (10) *has a nilpotent or a degenerate center at the origin then, there exists a smooth* (*of class* $\mathcal{C}^\infty$) *vector field* $\mathcal{Y}$ *defined in a neighborhood of the origin and a smooth scalar function* $\mu(x,y)$ *with* $\mu(0,0)=0$ *such that* $[\mathcal{X},\mathcal{Y}]=\mu\mathcal{X}$.

Proof. The proof of is based in the existence of a $\mathcal{C}^\infty$ first integral of system (10) assertion given in Theorem 6. Let H be this $\mathcal{C}^\infty$ first integral. We now construct the following $\mathcal{C}^\infty$ vector field

$$\mathcal{Y}=\frac{\partial H}{\partial y}\,\partial_x-\frac{\partial H}{\partial x}\,\partial y.$$

It is straightforward to see that the Lie bracket of the vector fields $\mathcal{X}$ and $\mathcal{Y}$, where $\mathcal{X}$ is the vector associated to system (10), is

$$[\mathcal{X},\mathcal{Y}]=(\mathrm{div}\mathcal{X})\,R\,\mathcal{X},$$

where R is the $\mathcal{C}^\infty$ integrating factor associated to the first integral H, i.e.

$$R=\frac{\partial H}{\partial y}\frac{1}{P}=-\frac{\partial H}{\partial x}\frac{1}{Q}.$$

The existence of this $\mathcal{C}^\infty$ integrating factor is insured by Proposition 3. Hence, system (20) admits $\mathcal{Y}$ as $\mathcal{C}^\infty$ infinitesimal generator of a Lie's symmetry. However, this $\mathcal{C}^\infty$ Lie symmetry is trivial (i.e. $\mathcal{X}$ and $\mathcal{Y}$ are not transversal vector fields) because $\mathcal{X}\wedge\mathcal{Y}:=P\eta-Q\xi\equiv 0$. □

Open problem. Is it true that there is always a non–trivial symmetry for any nilpotent and degenerate center? If it exist, is it smooth (of class $\mathcal{C}^\infty$) or analytic?

This non-trivial Lie symmetry can not be of the form $\mathcal{Y}=(x+o(x,y))\partial_x+(y+o(x,y))\partial_y$ taking into account Proposition 16 given in [12]. From the illustrative example presented in Section 4 it seems that the answer to the open problem is positive taking into account that, in this case, the example has a non–trivial symmetry which in fact, is analytic. It is straightforward to see from the proof Proposition 4 the following corollary.

Corollary 1. *Assume that the analytic system* (10) *has a nilpotent or a degenerate center at the origin with an analytic first integral. Then there exists an analytic vector field* $\mathcal{Y}$ *defined in a neighborhood of the origin and an analytic scalar function* $\mu(x,y)$ *with* $\mu(0,0)=0$ *such that* $[\mathcal{X},\mathcal{Y}]=\mu\mathcal{X}$.

The proof of Corollary 1 is obvious from the fact that H is analytic. With the aim to compare the results we give the following theorem proved in [11].

Theorem 8. *Consider a* C^1 *planar differential system (10) defined in an open subset* U *of* $\mathbb{R}^2$ *having a* C^1 *first integral* H *and a* C^1 *integrating factor* R *defined in open and dense subsets* V_H *and* V_R *of* U*, respectively. Assume that the Lebesgue measure of the set* $\{R(R_xH_y-R_yH_x)(P_x+Q_y)=0\}$ *in* $V_H\cap V_R$ *is zero. Then, the change of variables* $(x,y)\mapsto(u,v)$ *defined by*

$$u=R(x,y),\qquad v=R(x,y)H(x,y),\tag{15}$$

in the open and dense subset

$$(V_H\cap V_R)\setminus\{R(R_xH_y-R_yH_x)(P_x+Q_y)=0\}\tag{16}$$

of U*, transforms system (10) into the linear differential system* $\dot u=u$ *and* $\dot v=v$.

Remark 2. Proposition 4 can not be a consequence of Theorem 8 beause the change of variables (15) is not defined (in general) in a neighborhood of any nilpotent or degenerate center.

The following example was initially studied in [5]. This example has a nilpotent center without an analytic first integral (this fact was proved in [5]). We know, applying Theorem 4, that the system has a trivial $\mathcal{C}^\infty$ Lie symmetry, but as we will see it also has a non–trivial analytic Lie symmetry.

4. An illustrative example

Consider the nilpotent differential system

$$\dot{x} = y + x^2, \qquad \dot{y} = -x^3. \tag{17}$$

System (17) has a nilpotent center at the origin because it satisfies the monodromy conditions and it is time–reversible i.e. it is invariant under the transformation $(x, y, t) \to (-x, y, -t)$. In [5] it was proved that system (17) has not a formal first integral at the origin and consequently neither does analytic.

In [13] was studied the relationship between reversibility and the center problem and the integrability problem. Following the definitions given in [13] system (17) is time–reversible by means of the involution $R_0 = (-x, y)$, that is, it has the form

$$\dot{x} = y + P(x^2, y), \quad \dot{y} = x\,Q(x^2, y). \tag{18}$$

Hence, taking $z = x^2$ in system (18), we obtain (after ignoring a common factor) the reduced system

$$\dot{z} = 2(y + P(z, y)), \quad \dot{z} = Q(z, y). \tag{19}$$

The corresponding associated reduced system (19) of system (17) is

$$\dot{z} = 2(y + z), \qquad \dot{y} = -z. \tag{20}$$

System (20) is an homogeneous system which has a strong focus at the origin. Hence, we can obtain a non–continuous first integral of system (20) (because the origin is a focus it can not have a continuous one) and therefore a non–continuous first integral of system (17) is

$$H(x, y) = e^{2\arctan\left[1 + \frac{2y}{x^2}\right]}(x^4 + 2x^2 y + 2y^2)\,.$$

However, we know by Theorem 6 that system (17) has a $\mathcal{C}^\infty$ first integral. Probably, this $\mathcal{C}^\infty$ first integral can not be expressed by means of elementary or Liouvillian functions. Therefore, we know the existence of the $\mathcal{C}^\infty$ first integral and also by Theorem 4 the existence of a $\mathcal{C}^\infty$ Lie symmetry of system (17).

We recall that the origin of system (20) has a strong focus which is diagonalizable in $\mathbb{C}^2$ using a certain rotation. Moreover, the generator of dilations $\bar{\mathcal{Y}} = z\partial_z + y\partial y$ satisfies $[\bar{\mathcal{X}}, \bar{\mathcal{Y}}] = 0$ where $\bar{\mathcal{X}}$ is the vector field associated to system (20). Hence, we can apply the map $z = x^2$ to obtain, in this case, an analytic Lie symmetry of the original system (17) given by $\mathcal{Y} = x\partial_x + 2y\partial_y$, such that $[\mathcal{X}, \mathcal{Y}] = -\mathcal{X}$ where $\mathcal{X}$ is the vector field associated to system (17). Finally, using Proposition 2 we obtain the polynomial inverse integrating factor $V := P\eta - Q\xi = x^4 + 2x^2 y + 2y^2$ of system (17).

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