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NON-CONSTANT POSITIVE STEADY STATE OF A DIFFUSIVE LESLIE-GOWER TYPE FOOD WEB SYSTEM

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Abstract A three species food web comprising of two preys and one predator in an isolated homogeneous habitat is considered. The preys are assumed to grow logistically. The predator follows modified Leslie-Gower dynamics and feeds upon the prey species according to Holling Type II functional response. The local stability of the constant positive steady state of the corresponding temporal system and the spatio-temporal system are discussed. The existence and non-existence of non- constant positive steady states are investigated.

Keywords Food web, Reaction Diffusion Equations, Non-constant positive steady state, Leslie-Gower.

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1. Introduction

In the last decades the temporal modeling of species interactions have been extensively investigated for two or more species in food chains / food webs [1, 5, 6, 7, 8, 9, 14, 17] assuming uniform distribution of resources in the habitat under consideration. The nonlinear system may admit several steady states and the long term behavior is studied through local and global stability of these steady states. However, in reality resources are not uniformly distributed in the habitat and hence their densities are spatially inhomogeneous. In such cases, spatio-temporal dynamics of an ecological system is represented by a system of coupled nonlinear reaction diffusion equations.

The spatio-temporal dynamics of two species with variety of functional responses have been extensively investigated by many authors [2, 3, 13, 15, 16, 20, 21, 22, 24, 26, 27, 29, 30]. Although the spatio-temporal system and its corresponding temporal system admits same uniform equilibrium states, the dynamic behavior may be quite different. The presence of diffusion may destabilize the otherwise stable uniform equilibrium state of the temporal system. The diffusive instabilities may lead to the emergence of Turing patterns in the ecological system [3, 24]. The existence of non-constant positive steady state is possible and its stability is of interest [2, 3, 13, 21, 22, 26, 27]. The complex stable pattern formations in such systems have been reported in many ecological situations.

In [13, 22], the authors have considered a Leslie-Gower predator-prey system with Beddington-DeAngelis functional response and general functional response,

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respectively. The long time behavior of solutions of the corresponding systems have be investigated. Moreover, the existence of non-constant positive steady states of the systems have been discussed. W. Ko and K. Ryu [13] pointed out that a Leslie-Gower predator-prey system with general functional response may admit at least one non-constant positive steady state. However, the system with linear predator dependent functional response has no non-constant positive steady state.

In nature, the ecological communities exhibit a very complex dynamical behavior. Two species autonomous models are insufficient to produce realistic dynamics. The complex dynamics is possible for multi-species systems. Complex temporal dynamics in food chain/food web models are reported by many investigators [1, 6, 7, 8, 9, 14]. However few attempts have been made to investigate the spatio-temporal dynamics in food chain/food web models [4, 10, 12, 19, 23, 25, 28]. The spatio-temporal dynamics of three species diffusive systems with different types of functional responses under Dirichlet and zero flux boundary condition have been explored. The local as well as global stability of the constant positive steady state, the existence and non-existence of the non-constant steady state of such systems have been discussed.

In[10], the authors have considered a diffusive predator-prey system with two predators (consumers) competing for one prey(resource) with Beddington-DeAngelis and Holling type II functional response and studied the non-existence of non-constant positive steady state, the existence and bifurcation of non-constant positive steady state. Zheng [28] studied the existence and stability of semi-trivial steady states of a reaction diffusion system comprising of two competing preys and a modified Leslie-Gower type predator dynamic under both Dirichlet and homogeneous Neumann boundary conditions. However, W. Chen and M. Wang [4] studied the existence of positive solution of the corresponding elliptic system of the reaction diffusion system considered by Zheng [28] under Dirichlet boundary conditions only.

The investigation of the existence and non-existence of non-constant positive steady state for a three species food web system comprising of two preys and a predator with the modified Leslie- Gower type dynamics seems to be rare. So, in this paper, we are interested in a diffusive food web system comprising of two apparently competing logistic preys and a predator, where the predator follows modified Leslie - Gower type dynamics and feeds upon the prey species' according to Holling Type II functional response. The main objective of this paper is to study the existence of non-constant positive steady state solution by using the Leray-Schauder degree theory.

This paper is organized as follows: the model is described and preliminary analysis is carried out in section 2. Section 3 is devoted to the local stability of the positive steady state of the temporal, non-spatial, system. In section 4, the uniform asymptotical stability of the constant positive steady state of the spatio-temporal system is discussed. In section 5, a prior estimate for the positive steady state of the system is mentioned. In sections 6 and 7, the non-existence and existence of the non-constant positive steady state are investigated respectively.

2. Model

Consider a food web in a homogeneous bounded habitat $\Omega \in \mathbb{R}^n (n > 1)$ consisting of two prey species, U(X,T),V(X,T), and a predator species,W(X,T). In the absence of predation, the two prey species grow logistically with carrying capacities K_1 and K_2 and intrinsic growth rates r_1 and r_2 respectively. The prey predator interaction is assumed to be Holling type II. The predator grows logistically with intrinsic growth rate r_3 and carrying capacity proportional to the renewable resources, population size of prey. The modified Leslie-Gower type dynamics for the predator species is considered. The three species are assumed to diffuse at rates $D_i (i = 1, 2, 3)$. Thus, the spatio-temporal dynamics of the three species is given by:

$$\begin{cases} \dfrac{\partial U}{\partial T} - D_1 \Delta U = r_1 \left(1 - \dfrac{U}{K_1}\right) U - \dfrac{A_1 UW}{1 + A_1 h_1 U + A_2 h_2 V}, X \in \Omega, T > 0, \\ \dfrac{\partial V}{\partial T} - D_2 \Delta V = r_2 \left(1 - \dfrac{V}{K_2}\right) V - \dfrac{A_2 VW}{1 + A_1 h_1 U + A_2 h_2 V}, X \in \Omega, T > 0, \\ \dfrac{\partial W}{\partial T} - D_3 \Delta W = r_3 W - \dfrac{A_3 W^2}{S_3 + S_1 U + S_2 V}, \qquad X \in \Omega, T > 0, \\ \dfrac{\partial U}{\partial \mathbf{n}} = \dfrac{\partial V}{\partial \mathbf{n}} = \dfrac{\partial W}{\partial \mathbf{n}} = 0, \qquad X \in \partial\Omega, T > 0 \\ U(X, 0) = U_0(X) \geq 0, V(X, 0) = V_0(X) \geq 0, W(X, 0) = W_0(X) \geq 0, X \in \Omega. \end{cases} \tag{2.1}$$

All the parameters K_j,$h_j (j = 1, 2)$,r_i,A_i,S_i,$D_i (i = 1, 2, 3)$ in system (2.1) are assumed to be positive constants and have usual meaning. The initial data $U_0(X)$, $V_0(X)$ and $W_0(X)$ are non-negative continuous functions of position X in Ω. The vector $\mathbf{n}$ is an outward unit normal vector to the smooth boundary $\partial\Omega$ of the habitat Ω. The homogeneous Neumann boundary condition signifies that the system is self contained and there is no population flux across the boundary $\partial\Omega$.

Using the following scaling:

$$u = \frac{U}{K_1}, v = \frac{V}{K_2}, w = \frac{W}{K_1}, t = r_1 T, x = X\sqrt{\frac{r_1}{D_2}},$$

and the parameters

$$d_1 = \frac{D_1}{D_2}, d_3 = \frac{D_3}{D_2}, \delta_2 = \frac{r_2}{r_1}, \delta_3 = \frac{r_3}{r_1},$$
$$c_3 = \frac{A_3 K_1}{S_3 r_1}, c_i = \frac{A_i K_1}{r_1}, b_i = A_i h_i K_i, s_i = \frac{S_i K_i}{S_3}; i = 1, 2.$$

the system (2.1) takes the form

$$\begin{cases} \dfrac{\partial u}{\partial t} - d_1 \Delta u = (1 - u)u - \dfrac{c_1 uw}{1 + b_1 u + b_2 v}, \quad x \in \Omega, t > 0, \\ \dfrac{\partial v}{\partial t} - \quad \Delta v = \delta_2 (1 - v)v - \dfrac{c_2 vw}{1 + b_1 u + b_2 v}, \quad x \in \Omega, t > 0, \\ \dfrac{\partial w}{\partial t} - d_3 \Delta w = \delta_3 w - \dfrac{c_3 w^2}{1 + s_1 u + s_2 v}, \quad x \in \Omega, t > 0, \\ \dfrac{\partial u}{\partial \mathbf{n}} = \dfrac{\partial v}{\partial \mathbf{n}} = \dfrac{\partial w}{\partial \mathbf{n}} = 0, \quad x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x) \geq 0, v(x, 0) = v_0(x) \geq 0, w(x, 0) = w_0(x) \geq 0, \quad x \in \Omega. \end{cases} \tag{2.2}$$

It is necessary to investigate the temporal dynamics of the system before studying the spatio-temporal system (2.2). In the absence of population gradient the spatiotemporal system (2.2) is reduced to the system:

$$\begin{cases} \dfrac{du}{dt} = \left(1 - u - \dfrac{c_1 w}{1 + b_1 u + b_2 v}\right) u \triangleq G_1(u, v, w), \\ \dfrac{dv}{dt} = \left(\delta_2(1 - v) - \dfrac{c_2 w}{1 + b_1 u + b_2 v}\right) v \triangleq G_2(u, v, w), \\ \dfrac{dw}{dt} = \left(\delta_3 - \dfrac{c_3 w}{1 + s_1 u + s_2 v}\right) w \triangleq G_3(u, v, w), \\ u(0) = u_0 \geq 0, v(0) = v_0 \geq 0, w(0) = w_0 \geq 0. \end{cases} \tag{2.3}$$

Denoting $\mathbf{u} = (u, v, w)^T$ and $\mathbf{G}(\mathbf{u}) = (G_1(\mathbf{u}), G_2(\mathbf{u}), G_3(\mathbf{u}))$, the system (2.3) can be written as:

$$\begin{cases} \dfrac{d\mathbf{u}}{dt} = \mathbf{G}(\mathbf{u}), \\ \mathbf{u}(0) = \mathbf{u}_0 \geq 0. \end{cases} \tag{2.4}$$

3. Analysis of The Temporal Model

It can be easily proved that the system (2.4) has positive solutions.

Proposition 3.1. *All the solutions of system* (2.4) *are bounded.*

Proof. Observe that $G_1(\mathbf{u}) \leq u(1 - u)$ and $G_2(\mathbf{u}) \leq \delta_2(1 - v)v$. The comparison argument applied to the first two equations of system (2.4) gives $0 \leq u(t) \leq 1,\ 0 \leq v(t) \leq 1$. Further

$$G_3(\mathbf{u}) \leq \frac{c_3}{1 + s_1 + s_2}\left(\frac{\delta_3}{c_3}(1 + s_1 + s_2) - w\right) w.$$

Again, applying a comparison argument gives $0 \leq w(t) \leq \dfrac{\delta_3}{c_3}(1 + s_1 + s_2)$. This proves that the system (2.4) has bounded solutions. □

The system (2.4) has the following equilibrium points.

(i) Zero equilibrium point: $\mathbf{u}_0 = (0, 0, 0)$.

(ii) Axial equilibrium points: $\mathbf{u}_1 = (1, 0, 0)$,$\mathbf{u}_2 = (0, 1, 0)$, $\mathbf{u}_3 = (0, 0, \frac{\delta_3}{c_3})$.

(iii) Boundary equilibrium points: $\mathbf{u}_{11} = (1, 1, 0)$,$\mathbf{u}^{'} = (u^{'}, 0, w^{'})$, $\mathbf{u}^{"} = (0, v^{"}, w^{"})$, where $w^{'} = \frac{\delta_3}{c_3}(1 + s_1 u^{'})$ and the equilibrium density $u^{'}$ is the unique positive root of the quadratic polynomial

$$b_1 c_3 (u^{'})^2 + ((1 - b_1)c_3 + c_1\delta_3 s_1)u^{'} + c_1\delta_3 - c_3 = 0; \quad c_1\delta_3 < c_3$$

Similarly, $w^{"} = \frac{\delta_3}{c_3}(1 + s_2 v^{"})$ and $v^{"}$ is the unique positive root of the quadratic polynomial

$$b_2\delta_2 c_3 (v^{"})^2 + ((1 - b_2)\delta_2 c_3 + s_1\delta_3 c_2)v^{"} + (c_2\delta_3 - \delta_2 c_3) = 0; \quad c_2\delta_3 < \delta_2 c_3$$

(iv) Interior equilibrium point: $\mathbf{u}^* = (u^*, v^*, w^*)$; $w^* = \frac{\delta_3}{c_3}(1 + s_1 u^* + s_2 v^*)$, u^* and v^* are related as $\delta_2 c_1 v^* - c_2 u^* = \delta_2 c_1 - c_2$.
The following two cases may arise.
Case I: If $\delta_2 c_1 > c_2$ then u^* is the positive root of the quadratic equation

$$c_3(\delta_2 c_1 b_1 + b_2 c_2)(u^*)^2 + B_1^{'} u^* + C_1^{'} = 0;$$

$$B_1^{'} = c_3(b_2(\delta_2 c_1 - c_2) + (\delta_2 c_1 - b_2 c_2)) + c_1(\delta_3 c_2 s_2 + \delta_2(-b_1 c_3 + \delta_3 c_1 s_1)),$$
$$C_1^{'} = -c_3 b_2(\delta_2 c_1 - c_2) + c_1(\delta_3 c_2 s_2 + \delta_2(c_3 - c_1(1 + s_2)\delta_3).$$

It may be observed that if $c_3 > c_1(1 + s_2\delta_3)$ then $C_1^{'}$ becomes negative. The negativity of $C_1^{'}$ ensures the unique existence of u^*. Also, $v^* = \frac{c_2 u^* + \delta_2 c_1 - c_2}{\delta_2 c_1} > 0$, $v^* > u^*$ and $w^* > 0$. Thus, $\mathbf{u}^* = (u^*, v^*, w^*)$ exists uniquely if $\delta_2 c_1 > c_2$ and $c_3 > c_1(1 + s_2)\delta_3$.
Case II: If $\delta_2 c_1 < c_2$ then v^*, which is the positive root of the quadratic equation

$$c_3(\delta_2 c_1 b_1 + b_2 c_2)(v^*)^2 + B^{'} v^* + C^{'} = 0;$$

$$B^{'} = (c_2)^2 s_2 \delta_3 + c_2 \delta_2(c_1 s_1 \delta_3 + c_3(1 - b_1)) + \delta_2 b_1 c_3(c_2 - 2\delta_3 c_1)),$$
$$C^{'} = -c_3 b_1 \delta_2(c_2 - c_1\delta_2) + c_1 c_2 s_1 \delta_2 \delta_3 + c_2(c_3\delta_2 - c_2(1 + s_1)\delta_3),$$

exists uniquely if $C^{'} < 0$. i.e. if $c_3\delta_2 > c_2(1 + s_1)\delta_3$. This in turn implies the unique existence and positivity of the equilibrium densities u^*,given as $u^* = \frac{\delta_2 c_1 v^* - \delta_2 c_1 + c_2}{c_2} > 0$, and w^*. In this case $v^* < u^*$. Thus, the positive equilibrium point $\mathbf{u}^*$ exists uniquely if $\delta_2 c_1 < c_2$ and $c_3\delta_2 > c_2(1 + s_1)\delta_3$.
From the above two cases it can be observed that the unique existence of the constant positive steady state $\mathbf{u}^*$ does not depend on the sign of the expression $(\delta_2 c_1 - c_2)$. Thus, we can have the following sufficient condition which ensures the unique existence of $\mathbf{u}^*$.

$$c_3 > \delta_3 \max\left\{c_1(1 + s_2), \frac{c_2(1 + s_1)}{\delta_2}\right\}. \tag{3.1}$$

Remark 3.1. It may be observed from the above existence and uniqueness conditions of the three equilibrium points; the existence of the positive equilibrium point $\mathbf{u}^*$ guarantees the existence of boundary points $\mathbf{u}^{'}$ and $\mathbf{u}^{''}$. However, the boundary points may exist even though the coexistence equilibrium point does not exist.

In the next two sections, the local stability of the coexistence steady state $\mathbf{u}^* = (u^*, v^*, w^*)$ with respect to the temporal system (2.4) and the spatio-temporal system (2.2) are discussed under condition (3.1).

Theorem 3.1. *The constant positive steady state $\mathbf{u}^* = (u^*, v^*, w^*)$ of system (2.4) is locally asymptotically stable provided*

$$u^* > \max\left\{\frac{1}{2}, \frac{c_2}{c_1}\right\} \text{ and } c_1\delta_2 > c_2. \tag{3.2}$$

Proof. The system (2.4) is linearized at $\mathbf{u}^*$ as

$$\frac{d\mathbf{u}}{dt} = \mathbf{G_u}(\mathbf{u}^*)\mathbf{u},\ \mathbf{G_u}(\mathbf{u}^*) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

The matrix coefficients are

$$\begin{aligned}
a_{11} &= -\frac{u^*(1+b_2v^*+(-1+2u^*)b_1)}{c_1(1+b_1u^*+b_2v^*)}, \quad a_{12} = \frac{b_2c_1u^*w^*}{(1+b_1u^*+b_2v^*)^2} > 0,\\
a_{13} &= -\frac{c_1u^*}{(1+b_1u^*+b_2v^*)} < 0, \quad a_{21} = \frac{b_1c_2v^*w^*}{(1+b_1u^*+b_2v^*)^2} > 0,\\
a_{22} &= -\frac{v^*(1+b_1u^*+(-1+2v^*)b_2)\delta_2}{c_1(1+b_1u^*+b_2v^*)}, \quad a_{23} = -\frac{c_2v^*}{(1+b_1u^*+b_2v^*)} < 0,\\
a_{31} &= \frac{s_1\delta_3^2}{c_3} > 0, a_{32} = \frac{s_2\delta_3^2}{c_3} > 0, a_{33} = -\delta_3 < 0.
\end{aligned}$$

The characteristics polynomial of $\mathbf{G_u}(\mathbf{u}^*)$ is given by

$$\phi(\lambda) = \lambda^3 + P_2\lambda^2 + P_1\lambda + P_0,$$

with

$$\begin{aligned}
P_2 =& -(a_{11}+a_{22}+a_{33}),\\
P_1 =& (a_{11}a_{22}-a_{12}a_{21}) + (a_{11}+a_{22})a_{33} - a_{13}a_{31} - a_{23}a_{32},\\
P_0 =& (a_{13}a_{22}-a_{12}a_{23})a_{31} + (a_{11}a_{23}-a_{13}a_{21})a_{32} - (a_{11}a_{22}-a_{12}a_{21})a_{33},\\
P_1P_2 - P_0 =& -(a_{11}+a_{22})[(a_{11}a_{22}-a_{12}a_{21}) - a_{33}P_2] + (a_{12}a_{23}+a_{13}a_{33})a_{31}\\
&+ a_{11}a_{13}a_{31} + (a_{23}a_{33}+a_{13}a_{21})a_{32} + a_{22}a_{23}a_{32}.
\end{aligned}$$

According to Routh Hurwitz criterion, the necessary and sufficient conditions for stability are

$$P_1 > 0, P_2 > 0, P_0 > 0, P_1P_2 - P_0 > 0. \tag{3.3}$$

Algebraic manipulations of the expressions $(a_{11}a_{22}-a_{12}a_{21}), (a_{23}a_{33}+a_{13}a_{21})$ and $(a_{12}a_{23}+a_{13}a_{11})$ gives:

$$\begin{aligned}
&a_{11}a_{22}-a_{12}a_{21}\\
=& \frac{u^*v^*((b_1c_1\delta_2+b_2c_2)(-1+2u^*)+c_1\delta_2+b_2(c_1\delta_2-c_2))}{c_1(1+b_1u^*+b_2v^*)},\\
&a_{23}a_{33}+a_{13}a_{21}\\
=& \frac{c_2v^*(b_1(-1+2u^*)u^*+\delta_3(1+b_2)v^*+(1+b_2v^*+b_1\delta_3)u^*)}{(1+b_1u^*+b_2v^*)^2},\\
&a_{12}a_{23}+a_{13}a_{11}\\
=& \frac{u^*\left(b_2c_1\left(-\frac{c_2}{c_1}+u^*\right)v^* + c_1b_1(-1+2u^*)u^* + (c_1+c_2b_2v^*)u^*\right)}{(1+b_1u^*+b_2v^*)^2}.
\end{aligned}$$

The following are observed under condition (3.2):

$$\begin{gathered}
a_{11} < 0,\ a_{22} < 0,\ a_{11}a_{22}-a_{12}a_{21} > 0,\\
a_{23}a_{33}+a_{13}a_{21} > 0,\ a_{12}a_{23}+a_{13}a_{11} > 0.
\end{gathered}$$

Accordingly, conditions in (3.3) hold. Thus, the homogeneous steady state $\mathbf{u}^*$ is locally asymptotically stable. □

From the analysis, it is clear that condition (3.2) is sufficient for local stability. That is, $\mathbf{u}^*$ may be stable even if the condition (3.2) fails.

Remark 3.2. Choose the following data.

$$c_1 = 1.1, c_2 = 0.4, c_3 = 1.5, b_1 = 1.2, b_2 = 0.6,$$
$$s_1 = 2, s_2 = 0.3, \delta_2 = 1, \delta_3 = 0.6. \tag{3.4}$$

It is easy to check that both the existence condition (3.1) and the local stability condition (3.2) hold. However, for the following choice of parameters, the existence condition (3.1) is satisfied but the stability condition (3.2) fails.

$$c_1 = 1.15, c_2 = 0.5, c_3 = 1.45, b_1 = 2.5, b_2 = 0.1, s_1 = 2.8125;$$
$$s_2 = 0.1125, \delta_2 = 1.5, \delta_3 = 1.121. \tag{3.5}$$

The dynamics of the system (2.4) shown in figure 1(a) and (b) is obtained by solving the system (2.4) numerically for the choice of data in (3.4) and (3.5) respectively. Figure 1(a) shows the local stability of the constant positive steady state (0.524505, 0.827093, 0.918855) when condition (3.2) is satisfied. However, figure 1(b) shows the local stability of the constant positive steady state $(0.0860405, 0.735084, 1.02412)$ when condition (3.2) is violated. The stability of the equilibrium state is possible as condition (3.2) is only sufficient condition. It is also verified that the necessary and sufficient conditions in (3.3) are satisfied for the choice of data (3.5), which ensures the local stability of $(0.0860405, 0.735084, 1.02412)$.

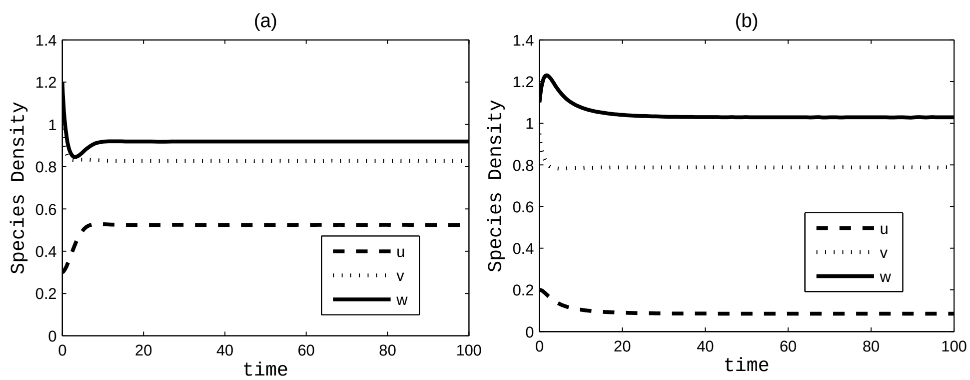

Figure 1. Local Stability of the constant positive steady state $\mathbf{u}^*$ (a) For the data set (3.4) when condition (3.2) is satisfied, (b) For the data set (3.5) when condition (3.2) is violated

4. Stability of the constant positive steady state $\mathbf{u}^*$ of the spatio-temporal system (2.2)

In this section, the local stability of the constant positive steady state $\mathbf{u}^*$of the spatio-temporal system (2.2) will be investigated. Let $0 = \mu_0 < \mu_1 < \mu_2 < \mu_3 <$ ... be the eigenvalues of the operator $-\Delta$ on Ω with the homogeneous Neumann boundary condition. Let $\mathbf{X}_i$ be the eigenspace corresponding to the eigenvalue μ_i, then the solution space $\mathbf{X} = \{\mathbf{u} \in [C^1(\overline{\Omega})]^3 | \frac{\partial \mathbf{u}}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega\}$ can be represented as $\mathbf{X} = \bigoplus_{i=1}^{\infty} \mathbf{X}_i$.

Theorem 4.1. *Assume that the parameters in (2.4) satisfy (3.2). Then the constant positive steady state $\mathbf{u}^*$ of the spatio-temporal system (2.2) is uniformly asymptotically stable.*

Proof. The linearization of the system (2.2) about the constant positive steady state $\mathbf{u}^*$ is:

$$\mathbf{u}_t = \mathbf{L}\mathbf{u} = \mathbf{D}\Delta + \mathbf{G}_{\mathbf{u}}(\mathbf{u}^*); \mathbf{D} = diag(d_1, 1, d_3).$$

The eigenspace $\mathbf{X}_i, i \geq 0$, is invariant under the operator $\mathbf{L}$. λ is an eigenvalue of $\mathbf{L}$ on $\mathbf{X}_i$ if and only if it is an eigenvalue of the matrix $-\mu_i\mathbf{D} + \mathbf{G}_{\mathbf{u}}(\mathbf{u}^*)$. The characteristics polynomial of $-\mu_i\mathbf{D} + \mathbf{G}_{\mathbf{u}}(\mathbf{u}^*)$ is

$$\sigma_i(\lambda) = \lambda^3 + P_{2i}\lambda^2 + P_{1i}\lambda + P_{0i}, \tag{4.1}$$

with

$$\begin{aligned}
P_{2i} =& \mu_i(d_1 + 1 + d_3) + P_2,\\
P_{1i} =& \mu_i^2(d_1 + d_1d_3 + d_3) - \mu_i(d_1(a_{33} + a_{22}) + (a_{11} + a_{33}) + d_3(a_{11} + a_{22})) + P_1,\\
P_{0i} =& \mu_i^3(d_1d_3) + \mu_i^2(-d_1a_{33} - a_{22}d_1d_3 - a_{11}d_3) + \mu_i(d_1(a_{33}a_{22} - a_{23}a_{32})\\
&+ (a_{11}a_{33} - a_{13}a_{31}) + d_3(a_{11}a_{22} - a_{12}a_{21})) + P_0.
\end{aligned}$$

From the assumption, it follows that

$$P_{1i} > 0, P_{2i} > 0, P_{0i} > 0.$$

Algebraic manipulation of the expression $E_i = P_{1i}P_{2i} - P_{0i}$ yields

$$E_i = B_1\mu_i^3 + B_2\mu_i^2 + B_3\mu_i + P_1P_2 - P_0,$$

where

$$\begin{aligned}
B_1 =& (1 + d_1)(d_1 + d_3)(1 + d_3) > 0,\\
B_2 =& -a_{11}(1 + d_3)(1 + 2d_1 + d_3) - a_{22}(d_1 + d_3)(2 + d_1 + d_3)\\
&- a_{33}(1 + d_1)(1 + d_1 + 2d_3) > 0,\\
B_3 =& (-(a_{22} + a_{33})P_2 + (a_{11}a_{22} - a_{12}a_{21}) - a_{13}a_{31} + a_{33}a_{11})d_1\\
&+ (-(a_{11} + a_{33})P_2 + (a_{11}a_{22} - a_{12}a_{21}) - a_{23}a_{32} + a_{22}a_{33})\\
&+ (-(a_{11} + a_{22})P_2 + (a_{11} + a_{22})a_{33} - a_{23}a_{32} - a_{13}a_{31})d_3 > 0.
\end{aligned}$$

Clearly $E_i > 0; i \geq 0$, as $P_1P_2 - P_0 > 0$ under the given conditions. From the Routh-Hurwitz criterion it follows that, for each $i \geq 0$, all the three roots $\lambda_{i,1}, \lambda_{i,2}, \lambda_{i,3}$ of $\sigma_i(\lambda) = 0$ have negative real parts. Thus there exist some positive numbers κ_i such that

$$Re\{\lambda_{i,1}\}, Re\{\lambda_{i,2}\}, Re\{\lambda_{i,3}\} \leq -\kappa_i \quad \forall i.$$

Let $\widetilde{\kappa} = \min\{\kappa_i\}$. Then, $\widetilde{\kappa} > 0$ and $Re\{\lambda_{i,1}\}, Re\{\lambda_{i,2}\}, Re\{\lambda_{i,3}\} \leq -\widetilde{\kappa}\ \forall i$. Consequently, the spectrum of $\mathbf{L}$, which consists of eigenvalues, lies in $\{Re\lambda \leq -\widetilde{\kappa}\}$. Thus, theorem 5.1.1 of Dan Henry [11] concludes the uniform asymptotical stability of $\mathbf{u}^*$. $\square$

Remark 4.1. As a consequence of theorem 4.1, under condition (3.2), diffusion cannot destabilize the constant coexistence steady state $\mathbf{u}^*$ of the system (2.4) and Turing instability cannot occur in the vicinity of $\mathbf{u}^*$. Hence, system (2.2) will not have a non constant positive steady state in some neighborhood of $\mathbf{u}^*$ under condition (3.2). However,if condition (3.2) fails then there is a possibility for the occurrence of Turing instability. For example, for the data set (3.5), the condition (3.2) fails but condition (3.3) still holds and $\mathbf{u}^*$ is temporally stable. Further, P_{0i} changes its sign with wave number k^2 as shown in fig. 2. In this case, Turing instability is possible. Accordingly, the system may eventually go to non-constant positive steady state. Thus, existence of non-constant positive steady state may be possible when condition (3.2) fails.

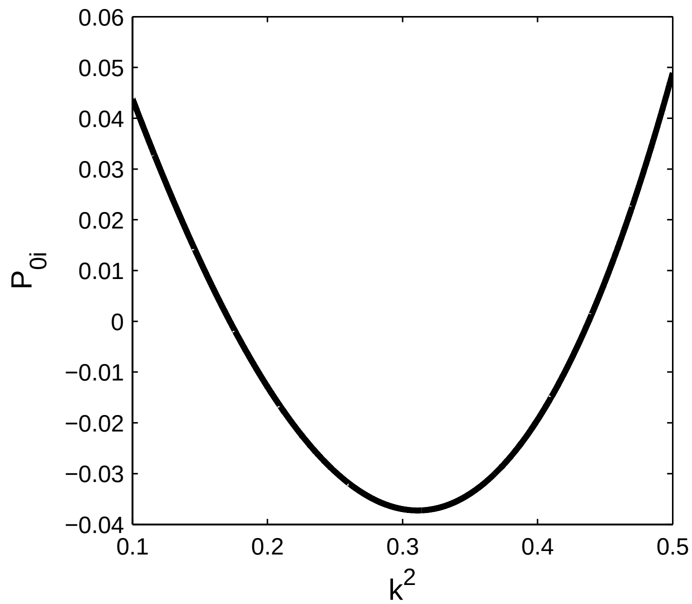

Figure 2. The occurrence of diffusion driven instability, Turing instability, as the coefficient P_{0i} of the dispersion relation (4.1) becomes negative for some range of the wave number k^2

In the next section, a priori positive upper and lower bounds for the positive steady state solutions of the system (2.2) are obtained.

5. A priori estimates of non-constant positive steady state

The steady state problem corresponding to (2.2) is

$$\begin{cases} -d_1\Delta u = G_1(u,v,w), & x\in\Omega, \\ -\Delta v = G_2(u,v,w), & x\in\Omega, \\ -d_3\Delta w = G_3(u,v,w), & x\in\Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = \dfrac{\partial v}{\partial \mathbf{n}} = \dfrac{\partial w}{\partial \mathbf{n}} = 0, & x\in\partial\Omega. \end{cases} \tag{5.1}$$

The classical solutions of (5.1) are assumed to be in $C^2(\Omega)\bigcap C^1(\overline{\Omega})$. The following two results are needed for the estimates.

Lemma 5.1. *(Maximum Principle (see [15])) Let* $f(x,\phi)\in C(\Omega\times\mathbb{R})$

1. *If* $\phi\in C^2(\Omega)\bigcap C^1(\overline{\Omega})$ *satisfies* $-\Delta\phi(x)\le f(x,\phi(x))$ *in* Ω, $\frac{\partial\phi}{\partial \boldsymbol{n}}\le 0$ *on* $\partial\Omega$ *and* $\phi(x_0)=\max_{\overline{\Omega}}\phi$, *then* $f(x_0,\phi(x_0))\ge 0$.

2. *If* $\phi\in C^2(\Omega)\bigcap C^1(\overline{\Omega})$ *satisfies* $-\Delta\phi(x)\ge f(x,\phi(x))$ *in* Ω, $\frac{\partial\phi}{\partial \boldsymbol{n}}\ge 0$ *on* $\partial\Omega$ *and* $\phi(x_0)=\min_{\overline{\Omega}}\phi$, *then* $f(x_0,\phi(x_0))\le 0$.

Lemma 5.2. *(Harnack Inequality see [16]) Let* $\phi\in C^2(\Omega)\bigcap C^1(\overline{\Omega})$ *be a positive solution to* $-\Delta\phi(x)=c(x)\phi(x))$ *with* $c\in C(\overline{\Omega})$, *subject to homogeneous Neumann boundary condition. Then there exists a positive constant* $C=C(N,\Omega,\|c\|_\infty)$ *such that*

$$\max_{\overline{\Omega}}\phi\le C\min_{\overline{\Omega}}\phi.$$

The results of upper and lower bounds can be stated as follows:

Theorem 5.1. *(Upper bounds) For any classical solution* $\boldsymbol{u}=(u,v,w)^T$ *of (5.1),*

$$\max_{\overline{\Omega}} u\le 1,\ \max_{\overline{\Omega}} v\le 1,\ \max_{\overline{\Omega}} w\le \frac{\delta_3}{c_3}\left(1+s_1+s_2\right). \tag{5.2}$$

Proof. Observe that

$$(1-u)u-\frac{c_1uw}{1+b_1u+b_2v}\le u(1-u)$$
$$\text{and}\qquad \delta_2(1-v)v-\frac{c_2vw}{1+b_1u+b_2v}\le\delta_2(1-v)v.$$

Applying Lemma 5.1 gives

$$\max_{\overline{\Omega}} u\le 1,\ \max_{\overline{\Omega}} v\le 1.$$

Let $w(x_0)=\max_{\overline{\Omega}} w(x)$. Applying the maximum principle to the third equation of

(5.1) gives

$$\begin{aligned}
\delta_3 - &\frac{c_3 w(x_0)}{1+s_1u(x_0)+s_2v(x_0)} \geq 0,\\
\Rightarrow w(x_0) &\leq \frac{\delta_3}{c_3}(1+s_1u(x_0)+s_2v(x_0)),\\
&\leq \frac{\delta_3}{c_3}(1+s_1+s_2),\\
\Rightarrow \max_{\overline{\Omega}} w(x) &\leq \frac{\delta_3}{c_3}(1+s_1+s_2).
\end{aligned}$$

This completes the proof. □

Theorem 5.2. *(Lower bounds) Let $\Lambda = \Lambda(\Omega, c_3, b_i, s_i, \delta_i, c_i, i=1,2)$ and d be fixed positive constants such that $d \leq \min\{d_1, d_3, 1\}$. Assume that condition (3.1) holds. i.e.*

$$c_3 > \max\left\{\frac{c_2\delta_3(1+s_1)}{\delta_2}, c_1\delta_3(1+s_2)\right\}. \tag{5.3}$$

Then there exist positive constants C_i^*, $i=1,2,3$, which are dependent on Λ and d, such that any positive solution of (5.1) satisfies

$$\min_{\overline{\Omega}} u \geq C_1^*,\ \min_{\overline{\Omega}} v \geq C_2^*,\ \min_{\overline{\Omega}} w \geq C_3^*. \tag{5.4}$$

Proof. Let $u(x_0) = \min_{\overline{\Omega}} u(x), v(y_0) = \min_{\overline{\Omega}} v(x), w(z_0) = \min_{\overline{\Omega}} w(x)$ and $w(z_1) = \max_{\overline{\Omega}} w(x)$.
Applying the maximum principle gives

$$\left.\begin{aligned}
1-u(x_0) - \frac{c_1w(x_0)}{1+b_1u(x_0)+b_2v(x_0)} &\leq 0,\\
\delta_2(1-v(y_0)) - \frac{c_2w(y_0)}{1+b_1u(y_0)+b_2v(y_0)} &\leq 0,\\
\delta_3 - \frac{c_3w(z_0)}{1+s_1u(z_0)+s_2v(z_0)} &\leq 0,\\
\delta_3 - \frac{c_3w(z_1)}{1+s_1u(z_1)+s_2v(z_1)} &\geq 0,
\end{aligned}\right\} \tag{5.5}$$

and so

$$\left.\begin{aligned}
\frac{s_1\delta_3}{c_3}u(x_0) \leq \frac{s_1\delta_3}{c_3}u(z_0) \leq \frac{\delta_3}{c_3}\left(1+s_1u(z_0)+s_2v(z_0)\right) \leq w(z_0),\\
\frac{s_2\delta_3}{c_3}v(y_0) \leq \frac{s_2\delta_3}{c_3}v(z_0) \leq \frac{\delta_3}{c_3}\left(1+s_1u(z_0)+s_2v(z_0)\right) \leq w(z_0).
\end{aligned}\right\} \tag{5.6}$$

The fourth inequality in (5.5) gives

$$\left.\begin{aligned}
w(z_1) \leq \frac{\delta_3}{c_3}\left(1+s_1u(z_1)+s_2v(z_1)\right) \leq \frac{\delta_3}{c_3}\left(1+s_2\right) + \frac{\delta_3}{c_3}s_1\max_{\overline{\Omega}} u,\\
w(z_1) \leq \frac{\delta_3}{c_3}\left(1+s_1u(z_1)+s_2v(z_1)\right) \leq \frac{\delta_3}{c_3}\left(1+s_1\right) + \frac{\delta_3}{c_3}s_2\max_{\overline{\Omega}} v.
\end{aligned}\right\} \tag{5.7}$$

The first two inequalities in (5.5) together with (5.7) imply

$$\left.\begin{aligned} 1-u(x_0) &\le c_1 w(x_0) \le c_1 w(z_1) \le \frac{c_1\delta_3}{c_3}(1+s_2)+\frac{c_1\delta_3}{c_3}s_1\max_{\overline{\Omega}} u(x),\\ 1-v(y_0) &\le \frac{c_2}{\delta_2}w(y_0) \le \frac{c_2}{\delta_2}w(z_1) \le \frac{c_2\delta_3}{c_3\delta_2}(1+s_1)+\frac{c_2\delta_3}{c_3\delta_2}s_2\max_{\overline{\Omega}} v(x).\end{aligned}\right\} \tag{5.8}$$

Let

$$M_1=\frac{c_1\delta_3}{c_3}s_1,\ M_2=\frac{c_2\delta_3}{c_3\delta_2}s_2,\ \overline{M}_1=1-\frac{c_1\delta_3}{c_3}(1+s_2)>0$$

and

$$\overline{M}_2=1-\frac{c_2\delta_3}{c_3\delta_2}(1+s_1)>0.$$

Simplifying (5.8) gives

$$\left.\begin{aligned} \overline{M}_1 &\le u(x_0)+M_1\max_{\overline{\Omega}} u(x)=\min_{\overline{\Omega}} u(x)+M_1\max_{\overline{\Omega}} u(x),\\ \overline{M}_2 &\le v(y_0)+M_2\max_{\overline{\Omega}} v(x)=\min_{\overline{\Omega}} v(x)+M_1\max_{\overline{\Omega}} v(x).\end{aligned}\right\} \tag{5.9}$$

Define,

$$\begin{aligned} Q_1(x)&=\left(1-u(x)-\frac{c_1 w(x)}{1+b_1u(x)+b_2v(x)}\right)d_1^{-1},\\ Q_2(x)&=\left(\delta_2(1-v(x))-\frac{c_2 w(x)}{1+b_1u(x)+b_2v(x)}\right).\end{aligned}$$

Then $Q_1(x)$ and $Q_2(x)$ satisfy

$$\begin{aligned} &\Delta u+Q_1(x)u=0 \text{ in } \Omega, \frac{\partial u}{\partial \mathbf{n}}=0 \text{ on } \partial\Omega\\ \text{and}\quad &\Delta v+Q_2(x)v=0 \text{ in } \Omega, \frac{\partial v}{\partial \mathbf{n}}=0 \text{ on } \partial\Omega.\end{aligned}$$

From (5.2), it follows that there exists a positive constant $\overline{C}(d,\Lambda)$ such that $\| Q_i \|_\infty \le \overline{C}, i=1,2$, if $d_1,d_3,1\ge d$. Thus, by Lemma 5.2, there exists a positive constant C_* such that

$$\max_{\overline{\Omega}} u(x)\le C_*\min_{\overline{\Omega}} u(x), \max_{\overline{\Omega}} v(x)\le C_*\min_{\overline{\Omega}} v(x). \tag{5.10}$$

Combining (5.9) with (5.10) gives

$$\begin{aligned} \min_{\overline{\Omega}} u(x) &\ge \frac{1}{\overline{M}_1+M_1C_*}=C_1^*,\\ \min_{\overline{\Omega}} v(x) &\ge \frac{1}{\overline{M}_2+M_2C_*}=C_2^*.\end{aligned}$$

From (5.6), it follows that

$$\min w(x)\ge\left(\frac{s_1\delta_3}{c_3}\right)\left(\frac{1}{\overline{M}_1+M_1C_*}\right)=C_3^*.$$

Hence, the proof is complete. □

The next sections explore the non-existence and existence of non-constant positive steady state.

6. Non-Existence of non-constant positive steady state

Theorem 6.1. *Let μ_1 be the smallest eigenvalue of the operator $-\Delta$ on Ω with homogeneous Neumann boundary condition and d_3^* be a fixed positive constant satisfying $\mu_1 d_3^* > \delta_3$. Then there exists a positive constant $D^* = D^*(d_1, \Lambda)$ such that (5.1) has no non-constant positive solution provided* $\min\{\mu_1 d_1, \mu_1\} \geq D^*$ *and $d_3 \geq d_3^*$.*

Proof. Let $\mathbf{u} = (u, v, w)^T$ be a positive solution of (5.1) and for any $\psi \in L^1(\Omega)$, denote $\overline{\psi} = |\Omega|^{-1} \int_\Omega \psi dx$. Multiplying the first, second and third equations in (5.1) by $(u - \overline{u}), (v - \overline{v}), (w - \overline{w})$, respectively, and integrating over Ω by parts yields

$$
\begin{aligned}
I = &\int_\Omega \left\{ d_1 |\nabla u|^2 + |\nabla v|^2 + d_3 |\nabla w|^2 \right\} dx, \\
= &\int_\Omega \Big\{ (u - \overline{u})(G_1(u, v, w) - G_1(\overline{u}, \overline{v}, \overline{w})) + (v - \overline{v})(G_2(u, v, w) - G_2(\overline{u}, \overline{v}, \overline{w})) \\
&+ (w - \overline{w})(G_3(u, v, w) - G_3(\overline{u}, \overline{v}, \overline{w})) \Big\} dx, \\
= &\int_\Omega \Big\{ (u - \overline{u})^2 \left(1 - (u + \overline{u}) - \frac{c_1(1 + b_2\overline{v})w}{\zeta_1\zeta_2} \right) \\
&+ 2(u - \overline{u})(v - \overline{v}) \left(\frac{(c_1 b_2 \overline{u} + b_1 c_2 \overline{v})}{\zeta_1\zeta_2} \right) \\
&+ (v - \overline{v})^2 \left(\delta_2(1 - (v + \overline{v})) - \frac{c_2(1 + b_1\overline{u})w}{\zeta_1\zeta_2} \right) \\
&+ 2(w - \overline{w})(v - \overline{v}) \left(\frac{s_2 c_3 w \overline{w}}{\chi_1\chi_2} - \frac{c_2 \overline{v}}{\chi_2} \right) + (w - \overline{w})^2 \left(\delta_3 - \frac{c_3 w}{\chi_1} - \frac{c_3 \overline{w}}{\chi_2} \right) \\
&+ 2(u - \overline{u})(w - \overline{w}) \left(\frac{s_1 c_3 w \overline{w}}{\zeta_1\zeta_2} - \frac{c_1 \overline{u}}{\zeta_1} \right) \Big\} dx.
\end{aligned}
$$

Here, $\zeta_1 = 2(1 + b_1 u + b_2 v), \zeta_2 = (1 + b_1\overline{u} + b_2\overline{v}), \chi_1 = (1 + s_1 u + s_2 v)$ and $\chi_2 = 2(1 + s_1\overline{u} + s_2\overline{v})$.

For some positive constants $\alpha_1, \alpha_2, \alpha_3$ and arbitrary positive constants $\epsilon_1, \epsilon_2, \epsilon_3$, from the Young's inequality, observe that

$$
2\alpha_1 |u - \overline{u}||v - \overline{v}| = 2\sqrt{\frac{\alpha_1}{\epsilon_1}} |u - \overline{u}| \sqrt{\alpha_1 \epsilon_1} |v - \overline{v}| \leq \frac{\alpha_1}{\epsilon_1}(u - \overline{u})^2 + \alpha_1 \epsilon_1 (v - \overline{v})^2.
$$

Similarly,

$$
\begin{aligned}
2\alpha_2 |v - \overline{v}||w - \overline{w}| &\leq \frac{\alpha_2}{\epsilon_2}(v - \overline{v})^2 + \alpha_2 \epsilon_2 (w - \overline{w})^2, \\
2\alpha_3 |u - \overline{u}||w - \overline{w}| &\leq \frac{\alpha_3}{\epsilon_3}(u - \overline{u})^2 + \alpha_3 \epsilon_3 (w - \overline{w})^2.
\end{aligned}
$$

Accordingly,

$$I \leq \int_{\Omega} \left\{ \left(1 + \frac{\alpha_1}{\epsilon_1} + \frac{\alpha_3}{\epsilon_3}\right)(u - \overline{u})^2 + \left(\delta_2 + \alpha_1\epsilon_1 + \frac{\alpha_2}{\epsilon_2}\right)(v - \overline{v})^2 \right.$$
$$\left. + (\delta_3 + \alpha_2\epsilon_2 + \alpha_3\epsilon_3)(w - \overline{w})^2 \right\} dx \triangleq I_1. \tag{6.1}$$

Further, due to Poincare inequality

$$I \geq \int_{\Omega} \left\{ \mu_1 d_1 (u - \overline{u})^2 + \mu_1 (v - \overline{v})^2 + \mu_1 d_3 (w - \overline{w})^2 \right\} dx \triangleq I_2. \tag{6.2}$$

From (6.1) and (6.2), it follows that

$$I_1 \geq I_2. \tag{6.3}$$

Since $\mu_1 d_3 > \delta_3$ by the assumption, we can find a sufficiently small $\epsilon_1, \epsilon_2, \epsilon_3 > 0$ such that
$\mu_1 d_3 \geq (\delta_3 + \alpha_2\epsilon_2 + \alpha_3\epsilon_3)$. Finally, by taking $D_1^* := (1 + \alpha_1/\epsilon_1 + \alpha_3/\epsilon_3)$, $D_2^* := (\delta_2 + \alpha_1\epsilon_1 + \alpha_2/\epsilon_2)$ and setting $D^* = \max\{D_1^*, D_2^*\}$, one can conclude from (6.3) that $u = \overline{u} =$ constant, $v = \overline{v} =$ constant and $w = \overline{w} =$ constant provided $\min\{\mu_1 d_1, \mu_1\} \geq D^*$. This completes the proof. □

7. Existence of non-constant positive steady state

The main aim of this section is to discuss the existence of non-constant positive solutions to (5.1) by using Leray-Schauder Theorem. Theorem 6.1 implies that when the assumptions of the theorem holds then (5.1) will not have non-constant positive solution. In addition to this, as a consequence of theorem 4.1 the system (2.2) will not have a non-constant positive steady state in some neighborhood of $\mathbf{u}^*$ if the stability condition (3.2) holds. However, by properly choosing the parameters and contradicting the condition (3.2), it is possible to obtain a non-constant positive solution for (5.1) and hence a non-constant positive steady state of (2.2). This ensures that stationary patterns and more interesting Turing patterns can arise as a result of diffusion.

The linearization of (5.1) at $\mathbf{u}^*$ is presented as follows. Let $\mathbf{X}$ be as in section 4 and define $\mathbf{X}^+ = \{\mathbf{u} \in \mathbf{X} | u > 0, v > 0, w > 0\}$ on $\overline{\Omega}$, $B(C) = \{\mathbf{u} \in \mathbf{X} | C^{-1} < u, v, w < C\}$ on $\overline{\Omega}$ where C is a positive constant in which its existence is ensured by theorems 5.1 and 5.2. Thus, (5.1) is equivalent to

$$\begin{cases} -\mathbf{D}\Delta\mathbf{u} = \mathbf{G}(\mathbf{u}), & x \in \Omega, \\ \dfrac{\partial \mathbf{u}}{\partial \mathbf{n}} = 0, & x \in \partial\Omega. \end{cases} \tag{7.1}$$

Also, $\mathbf{u}$ is a positive solution of (7.1) if and only if

$$\varphi(\mathbf{u}) \triangleq \mathbf{u} - (\mathbf{I} - \Delta)^{-1} \left\{ \mathbf{D}^{-1}\mathbf{G}(\mathbf{u}) + \mathbf{u} \right\} = 0 \text{ in } \mathbf{X}^+,$$

where, $\mathbf{I}$ is the identity map from $C^1(\Omega)$ to itself and $(\mathbf{I} - \Delta)^{-1}$ is the inverse of $\mathbf{I} - \Delta$ in $\mathbf{X}$ subject to Neumann boundary condition. It can be noticed that the Leray -

Schauder degree $deg(\varphi(.), 0, B)$ is well defined if $\varphi(\mathbf{u}) \neq 0$ for any $\mathbf{u} \in \partial B(C)$. It is straight forward to see that

$$D_{\mathbf{u}}\varphi(\mathbf{u}^*) = \mathbf{I} - (\mathbf{I} - \Delta)^{-1}\left\{\mathbf{D}^{-1}\mathbf{G}_{\mathbf{u}}(\mathbf{u}^*) + \mathbf{I}\right\}$$

and recall that if $D_{\mathbf{u}}\varphi(\mathbf{u}^*)$ is invertible then the index of φ at $\mathbf{u}^*$ is defined as $index(\varphi(.), \mathbf{u}^*) = (-1)^\rho$, where ρ is the total number of eigenvalues with negative real parts(counting multiplicities) of $D_{\mathbf{u}}\varphi(\mathbf{u}^*)$ [18].

As in the proof of theorem (3.1), note that, λ is an eigenvalue of the matrix $D_{\mathbf{u}}\varphi(\mathbf{u}^*)$ on $\mathbf{X}_i$ if and only if it is an eigenvalue of the matrix $\frac{1}{1+\mu_i}\left\{\mu_i\mathbf{I} - \mathbf{D}^{-1}\mathbf{G}_{\mathbf{u}}(\mathbf{u}^*)\right\}$. So, $D_{\mathbf{u}}\varphi(\mathbf{u}^*)$ is invertible if and only if $\frac{1}{1+\mu_i}\left\{\mu_i\mathbf{I} - \mathbf{D}^{-1}\mathbf{G}_{\mathbf{u}}(\mathbf{u}^*)\right\}$ is non-singular for any $i \geq 1$. For the sake of convenience, denote

$$\Theta(\mu) \triangleq det\left(\mu\mathbf{I} - \mathbf{D}^{-1}\mathbf{G}_{\mathbf{u}}(\mathbf{u}^*)\right) = \frac{1}{d_1 d_3} det\left(\mu\mathbf{D} - \mathbf{G}_{\mathbf{u}}(\mathbf{u}^*)\right). \tag{7.2}$$

The number of negative eigenvalues μ of $D_{\mathbf{u}}\varphi(\mathbf{u}^*)$ on $\mathbf{X}_i$ is odd if and only if $\Theta(\mu_i, 0) < 0$.
Now,

$$\begin{aligned} & det\left(\mu\mathbf{D} - \mathbf{G}_{\mathbf{u}}(\mathbf{u}^*)\right) = l_1\mu^3 + l_2\mu^2 + l_3\mu - det\left(\mathbf{G}_{\mathbf{u}}(\mathbf{u}^*)\right) \triangleq l(\mu); \qquad (7.3)\\ & l_1 = d_1 d_3,\ l_2 = -a_{22}d_1 d_3 - a_{11}d_3 - a_{33}d_1,\\ & l_3 = (a_{22}a_{33} - a_{23}a_{32})d_1 + (a_{11}a_{33} - a_{13}a_{31}) + (a_{11}a_{22} - a_{12}a_{21})d_3. \end{aligned}$$

Let $\overline{\mu}_1, \overline{\mu}_2$ and $\overline{\mu}_3$ be the three roots of $l(\mu) = 0$. Then $\overline{\mu}_1\overline{\mu}_2\overline{\mu}_3 = det\left(\mathbf{G}_{\mathbf{u}}(\mathbf{u}^*)\right)$. Since $det\left(\mathbf{G}_{\mathbf{u}}(\mathbf{u}^*)\right) < 0$, i.e $\overline{\mu}_1\overline{\mu}_2\overline{\mu}_3 < 0$, and $l_1 > 0$, one of $\overline{\mu}_1$, $\overline{\mu}_2$, $\overline{\mu}_3$ is real and negative, and the product of the other two is positive. For a sufficiently large d_3, i.e $d_3 \to \infty$, we have

$$\lim_{d_3\to\infty}\left\{\frac{l(\mu)}{d_3}\right\} = \mu[\tau_1\mu^2 - \tau_2\mu + \tau_3],$$

where

$$\begin{aligned} & \tau_1 = \lim_{d_3\to\infty}\left\{\frac{l_1(\mu)}{d_3}\right\} = d_1,\ \tau_2 = \lim_{d_3\to\infty}\left\{\frac{l_2(\mu)}{d_3}\right\} = a_{11} + a_{22}d_1,\\ & \tau_3 = \lim_{d_3\to\infty}\left\{\frac{l_3(\mu)}{d_3}\right\} = (a_{11}a_{22} - a_{12}a_{21}). \end{aligned}$$

The next two propositions are used in the main result, theorem 7.1, of this section.

Proposition 7.1. *Assume that the matrix* $\mu_i\boldsymbol{I} - \boldsymbol{D}^{-1}\boldsymbol{G}_{\boldsymbol{u}}(\boldsymbol{u}^*)$; $i \geq 1$, *is non-singular. Let* $m(\mu_i)$ *be the multiplicity of the eigenvalue* μ_i *and* $\rho = \sum_{i\geq 1, \Theta(\mu_i)<0} m(\mu_i)$. *Then* $index(\varphi(.), \boldsymbol{u}^*) = (-1)^\rho$.

Proposition 7.2. *Assume that* $a_{11} > 0, \tau_2 > 0$ *and (3.1) holds. Then there exists a positive constant* D_3 *such that when* $d_3 \geq D_3$, *the three roots* $\overline{\mu}_1, \overline{\mu}_2, \overline{\mu}_3$ *of* $l(\mu) = 0$

are all real and satisfy

$$\begin{cases} \lim\limits_{d_3\to\infty} \overline{\mu}_1 = \dfrac{(a_{11}+a_{22}d_1) - \sqrt{(a_{11}-a_{22}d_1)^2 + 4a_{12}a_{21}}}{2d_1} \equiv \widetilde{\mu}_1 < 0, \\ \lim\limits_{d_3\to\infty} \overline{\mu}_2 = \dfrac{(a_{11}+a_{22}d_1) + \sqrt{(a_{11}-a_{22}d_1)^2 + 4a_{12}a_{21}}}{2d_1} \equiv \widetilde{\mu}_2 > 0, \\ \lim\limits_{d_3\to\infty} \overline{\mu}_3 = 0. \end{cases} \tag{7.4}$$

Moreover, we have

$$\begin{cases} -\infty < \widetilde{\mu}_1 < 0 < \widetilde{\mu}_3 < \widetilde{\mu}_2, \\ l(\mu) < 0, \ \textit{when } \mu \in (-\infty, \widetilde{\mu}_1) \cup (\widetilde{\mu}_3, \widetilde{\mu}_2), \\ l(\mu) > 0, \ \textit{when } \mu \in (\widetilde{\mu}_1, \widetilde{\mu}_3) \cup (\widetilde{\mu}_2, \infty). \end{cases} \tag{7.5}$$

The following theorem proves the existence of non-constant positive solutions of (5.1) for some fixed positive constants Λ and d_1, for sufficiently large diffusion coefficient d_3.

Theorem 7.1. *Assume that the parameters Λ and d_1 are fixed, and satisfy (3.1), $a_{11} > 0$ and $\tau_2 > 0$. Let $\widetilde{\mu}_2$ be given by the limit (7.4). If $\widetilde{\mu}_2 \in (\mu_p, \mu_{p+1})$ for some $p \geq 1$ and the sum $\rho = \sum_{i=1}^{n} m(\mu_i)$ is odd then there exists a positive constant $\overline{D}_3$ such that,if $d_3 \geq \overline{D}_3$, (5.1) admits at least one non-constant positive solution.*

Proof. From proposition 7.2, it follows that there exists a positive constant D_3 such that,when $d_3 \geq D_3$, (7.5) holds and

$$0 = \mu_0 < \overline{\mu}_3 < \mu_1, \widetilde{\mu}_2 \in (\mu_p, \mu_{p+1}). \tag{7.6}$$

Now we prove that, for any $d_3 \geq \overline{D}_3$, (5.1) admits at least one non-constant positive solution. Assume that the assertion is not true for some $d_3 = \overline{d}_3 \geq \overline{D}_3$. In this proof, the homotopy invariance of the topological degree is used to derive a contradiction. Fix $d_3 = \overline{d}_3$. For $t \in [0,1]$, define $\mathbf{D}(t) = diag(d_1(t), 1, d_3(t))$ with $d_i(t) = td_i(t) + (1-t)\widehat{d}_i(t), i = 1, 3$, and consider the problem

$$\begin{cases} -\mathbf{D}(t)\Delta\mathbf{u} = \mathbf{G}(\mathbf{u}), & x \in \Omega, \\ \dfrac{\partial \mathbf{u}}{\partial \mathbf{n}} = 0, & x \in \partial\Omega. \end{cases} \tag{7.7}$$

Thus, $\mathbf{u}$ is a non-constant positive solution of (5.1) if and only if it is a positive solution of (7.7) for $t = 1$. Clearly, $\mathbf{u}^*$ is the unique constant positive solution of (7.7) for any $0 \leq t \leq 1$. For any $0 \leq t \leq 1$, $\mathbf{u}$ is a positive solution of (5.1) if and only if it is a solution of the following problem

$$\varphi(t;\mathbf{u}) \triangleq \mathbf{u} - (\mathbf{I} - \Delta)^{-1} \{\mathbf{D}^{-1}(t)\mathbf{G}(\mathbf{u}) + \mathbf{u}\} = 0 \text{ in } \mathbf{X}^+.$$

It is clear that $\varphi(1;\mathbf{u}) = \varphi(0;\mathbf{u})$. From theorem 6.1 it follows that $\varphi(0;\mathbf{u}) = 0$ has only the positive solution $\mathbf{u}^*$ in $\mathbf{X}^+$. It is easy to see that

$$D_{\mathbf{u}}\varphi(t;\mathbf{u}^*) = \mathbf{I} - (\mathbf{I} - \Delta)^{-1} \{\mathbf{D}^{-1}(t) G_{\mathbf{u}}(\mathbf{u}^*) + \mathbf{I}\}.$$

In particular,

$$D_{\mathbf{u}}\varphi(0;\mathbf{u}^*) = \mathbf{I} - (\mathbf{I}-\Delta)^{-1}\{\widehat{\mathbf{D}}^{-1}G_{\mathbf{u}}(\mathbf{u}^*) + \mathbf{I}\},$$
$$D_{\mathbf{u}}\varphi(1;\mathbf{u}^*) = \mathbf{I} - (\mathbf{I}-\Delta)^{-1}\{\mathbf{D}^{-1}G_{\mathbf{u}}(\mathbf{u}^*) + \mathbf{I}\} = D_{\mathbf{u}}\varphi(\mathbf{u}^*).$$

where $\widehat{\mathbf{D}} = diag(\widehat{d}_1, 1, \widehat{d}_3)$. From (7.2) and (7.3) we have

$$\Theta(\mu) = \frac{1}{d_1 d_3} l(\mu). \tag{7.8}$$

For $t = 1$, by (7.5), (7.6) and (7.8), we have

$$\begin{cases} \Theta(\mu_0) = \Theta(0) > 0, \\ \Theta(\mu_i) < 0, 1 \leq i \leq p, \\ \Theta(\mu_i) > 0, i \geq p+1. \end{cases}$$

Hence, zero is not the eigenvalue of the matrix $\mu_i\mathbf{I} - \mathbf{D}^{-1}\mathbf{G}_{\mathbf{u}}(\mathbf{u}^*)$ for all $p \geq 0$ and $\sum_{i\geq 1, \Theta(\mu_i)<0} m(\mu_i) = \sum_{i}^{p} m(\mu_i) = \theta_p$, which is odd. Then proposition 7.1 yields

$$index(\varphi(1;.), \mathbf{u}^*) = (-1)^{\rho} = (-1)^{\theta_p} = -1. \tag{7.9}$$

Similarly, it is possible to prove

$$index(\varphi(0;.), \mathbf{u}^*) = (-1)^{\rho} = (-1)^{0} = 1. \tag{7.10}$$

In view of theorems 5.1 and 5.2, there exists a positive constant C such that, for all $0 \leq t \leq 1$, the positive solutions of (7.7) satisfy $C^{-1} < u, v, w < C$ and hence $\varphi(t;\mathbf{u}) \neq 0$ on $\partial B(C)$. By the homotopy invariance of the topological degree, we have

$$deg(\varphi(1;.), 0, B(C)) = deg(\varphi(0;.), 0, B(C)). \tag{7.11}$$

Since both equations $\varphi(1;\mathbf{u}) = 0$ and $\varphi(0;\mathbf{u}) = 0$ have the unique positive solution $\mathbf{u}^*$ in $B(C)$, by (7.9) and (7.10), we have

$$deg(\varphi(0;.), 0, B(C)) = index(\varphi(0;.), \mathbf{u}^*) = 1,$$
$$deg(\varphi(1;.), 0, B(C)) = index(\varphi(1;.), \mathbf{u}^*) = -1.$$

This contradicts (7.11). Hence the proof is complete. $\square$

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