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THE LONG TIME BEHAVIOR FOR PARTLY DISSIPATIVE STOCHASTIC SYSTEMS

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Abstract In this paper, we consider the long time behaviors for the partly dissipative stochastic reaction diffusion equations in $D \subset \mathbb{R}^n$. The main purpose of this paper is to establish the existence of a compact global random attractor. The existence of a random absorbing set is first discussed for the systems and then an estimate on the solutions is derived when the time is large enough, which ensures the asymptotic compactness of solutions. Finally, we establish the existence of the global attractor in $L^2(D) \times L^2(D)$.

Keywords partly dissipative; random attractor; stochastic reaction-diffusion equation; asymptotic compactness.

MSC(2000) Primary 35B40; Secondary 35B41.

1. Introduction

The partly dissipative reaction diffusion system of the form

$$\frac{\partial}{\partial t}u + (-\mu\Delta u + \lambda u + \alpha v) = h(u) + f(x), \tag{1.1}$$

$$\frac{\partial}{\partial t}v + \delta v - \beta u = g(x) \tag{1.2}$$

is often used to describe the signal transmission across axon and is a model of FitzHugh-Nagumo equation in neurobiology [3, 8, 16]. The asymptotic behavior of the partly dissipative system was studied by several authors [4, 14, 15, 18]. However, to the best of our knowledge, there is little study on the existence of the global attractors of the partly dissipative reaction diffusion system with stochastic disturbances. It is worth mentioning that, in the case of lattice systems, the existence of a random attractor was proved recently in [20] and the deterministic lattice case was treated in [11] and [12].

In this paper, we instigate the asymptotic behavior of solutions of the following partly dissipative stochastic reaction diffusion equations with additive white noise

$$\begin{cases} du + (-\mu\Delta u + \lambda u + \alpha v)dt = (h(u) + f(x))dt + \sum_{j=1}^m h_j dw_j, \\ dv + (\delta v - \beta u)dt = g(x)dt + \sum_{j=1}^m h_j^* dw_j, \end{cases} \tag{1.3}$$

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where $\mu, \lambda, \alpha, \delta, \beta$ are positive constants, f, g, h_j and h_j^* are given functions. The nonlinear function $h(u)$ satisfies certain dissipative conditions and $\{w_j\}_{j=1}^m$ are independent two-sided real-valued Wiener processes on a probability space which will be specified later.

Our purpose of this paper is to study the long time behaviors of the stochastic system (1.3). The notion of random attractors for a stochastic dynamical system is introduced in [6, 7, 9]. Random attractors are compact invariant sets and depend on chance, but they move with time. The notion of random attractor is a generalization of the classical concept of global attractors for deterministic dynamical systems and has been applied to many infinite dimensional stochastic dynamical systems (see [1, 2, 5, 6, 7, 9, 10, 13, 17]). In our approach, we introduce a stationary Ornstein-Ohlenbeck process to transform the stochastic system (1.3) into the deterministic system with random coefficient. The main difficulty to obtain a global random attractor for the random dynamical system is the lack of compactness of the semigroup generated by the partly dissipative stochastic reaction diffusion equations. We obtain the asymptotic compactness of semigroup $S(t, \omega)$ by applying the method of operator decomposition (see [13]).

This paper is arranged as follows. In section 2, some relevant concepts and theories are given. In section 3, we introduce the Ornstein-Ohlenbeck process, give some properties and provide some basic settings about (1.3). Our results generalize a random dynamical system to proper function space. In section 4, we prove results on the existence of a unique random attractor of the random dynamical system generated by (1.3).

2. Preliminaries on random dynamical systems

In this section, we introduce some basic concepts related to random attractors for stochastic dynamical systems. Let $(X, ||.||_X)$ be a separable Hilbert space with Borel $\sigma-$algebra $\mathcal{B}(X)$ and $(\Omega, \mathcal{F}, P, (\vartheta_t)_{t\in R})$ be the ergodic metric dynamical system.

Definition 2.1. A continuous random dynamical system over $(\Omega, \mathcal{F}, P, (\vartheta_t)_{t\in R})$ is a $(\mathcal{B}(R^+) \times \mathcal{F} \times \mathcal{B}(X))$-measurable mapping

$$S : R^+ \times \Omega \times X \to X \ (t, \omega, x) \to \varphi(t, \omega, x)$$

such that the following properties hold:

(1) $S(0, \omega, x) = x$ for all $\omega \in \Omega$ and $x \in X$;

(2) $S(t + s, \omega, \cdot) = S(t, \vartheta_s\omega, \cdot) \circ S(s, \omega, \cdot)$ for all $s, t \geq 0$ and $\omega \in \Omega$;

(3) S is continuous in t and x.

Definition 2.2. (1) A set-valued mapping $\omega \to D(\omega) : \Omega \to 2^X$ is said to be a random set if the mapping $\omega \to d(x, D(\omega))$ is measurable for any $x \in X$. If $D(\omega)$ is closed (compact) for each $\omega \in \Omega$, the mapping $\omega \to D(\omega)$ is called a random closed (compact) set. A random set $\omega \to D(\omega)$ is said to be bounded if there exist $x_0 \in X$ and a random variable $R(\omega) > 0$ such that

$$D(\omega) \subset \{x \in X : ||x - x_0|| \leq R(\omega)\} \ \text{ for all } \omega \in \Omega.$$

(2) A random set $\omega \to D(\omega)$ is called tempered if for P-a.s. $\omega \in \Omega$,

$$\lim_{t\to 0} e^{-\beta t} \sup\{||b||_X : b \in D(\vartheta_{-t}\omega)\} = 0 \text{ for all } \beta > 0.$$

(3) Let $\mathcal{D}$ be a collection of random subsets of X and $\{K(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$. Then $\{K(\omega)\}_{\omega\in\Omega}$ is called a random absorbing set for S in $\mathcal{D}$ if for every $B \in \mathcal{D}$ and P -a.e. $\omega \in \Omega$, there exists $t_B(\omega) > 0$ such that

$$S(t, \vartheta_{-t}\omega, B(\vartheta_{-t}\omega)) \subset K(\omega) \quad \text{for all } t \geq t_B(\omega).$$

Definition 2.3. Let $\mathcal{D}$ be a collection of random subsets of X. A random set $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ of X is called a $\mathcal{D}$-random attractor (or $\mathcal{D}$-pullback attractor) for S if the following conditions are satisfied for P -a.e. $\omega \in \Omega$.

(i) $\mathcal{A}(\omega)$ is compact and $\omega \to d(x, \mathcal{A}(\omega))$ is measurable for $x \in X$;

(ii) $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ is invariant, that is,

$$S(t, \omega, \mathcal{A}(\omega)) = \mathcal{A}(\vartheta_t\omega) \quad \text{for all } t \geq 0.$$

(iii) $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ attracts every set in $\mathcal{D}$, that is, for every $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$,

$$\lim_{t\to\infty} d_H(S(t, \vartheta_{-t}\omega, B(\vartheta_{-t}\omega)), \mathcal{A}(\omega)) = 0,$$

where d_H is the Hausdorff semi-distance.

Lemma 2.1. *Let $\mathcal{D}$ be a collection of random subsets of X and S a continuous RDS on X over $(\Omega, \mathcal{F}, P, (\vartheta_t)_{t\in R})$. Suppose that $\{K(\omega)\}_{\omega\in\Omega}$ is a closed random absorbing set for S in $\mathcal{D}$ and S is asymptotically compact in X. Then S has a unique random attractor $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ given by*

$$\mathcal{A}(\omega) = \bigcap_{t\geq 0} \overline{\bigcup_{\tau\geq t} S(\tau, \vartheta_{-\tau}\omega, K(\vartheta_{-\tau}\omega))}.$$

Let B be a bounded set in a Banach space X. The Kuratowski measure of non-compactness $\alpha(B)$ of B is defined by

$$\alpha(B) = \inf\{d > 0 : B \text{ admits a finite cover by sets of diameter} \leq d\}.$$

We define $\alpha(B) = \infty$, if B is unbounded.

Definition 2.4. [13] A random dynamical system S on a Polish space (X, d) is almost surely $\mathcal{D} - \alpha-$contracting if

$$\lim_{t\to\infty} \alpha(S(t, \vartheta_{-t}\omega, A(\vartheta_{-t}\omega))) = 0 \text{ for } A \in \mathcal{D}.$$

Lemma 2.2. *For a random dynamical system $S(t, \omega)$ on a separabal Banach space $(X, ||.||_X)$, if almost surely the following hold:*

(1) $S(t, \omega) = S_1(t, \omega) + S_2(t, \omega)$;

(2) For any tempered random variable $a \geq 0$, there exist $r(a)$ $(0 \leq r < \infty)$, a.s. such that for the closed ball B_a with radius a in X, $S_1(t, \vartheta_{-t}\omega, B_a(\vartheta_{-t}\omega))$ is precompact in X for all $t > r(a)$.

(3) $||S_2(t, \vartheta_{-t}\omega, u)||_X \leq K(t, \vartheta_{-t}\omega, a)$, $t > 0$, $u \in B_a(\omega)$ and $K(t, \omega, a)$ is a measurable function with respect to (t, ω, x) which satisfies

$$\lim_{t\to\infty} K(t, \vartheta_{-t}\omega, a) = 0.$$

Then $S(t, \omega)$ is almost surely $\mathcal{D} - \alpha-$contracting (see [13]).

Lemma 2.3. *Let $S(t, \omega)$ be a random dynamical system on a Polish space $(X, ||.||_X)$. Assume that*

(1) $S(t, \omega)$ has an absorbing set $B(\omega) \in \mathcal{D}$;

(2) $S(t, \omega)$ is almost surely $\mathcal{D} - \alpha-$contracting. Then $S(t, \omega)$ possesses a global random attractor in X.

3. Solutions of partly dissipative stochastic reaction diffusion equations

In this section, we present the existence and uniqueness of solutions of equation (1.3) and (1.2). Set $L^2(D)$, $H_0^1(D)$ and $E = L^2(D) \times H_0^1(D)$ with the following inner products and norms, respectively

$$(u, v) = \int_D uv dx,\ ||u|| = (u, v)^{\frac{1}{2}} \quad \forall u, v \in L^2(D),$$

$$((u, v)) = \int_D \nabla u \nabla v dx,\ ||u||_{H^1} = ((u, v))^{\frac{1}{2}} \quad \forall u, v \in H_0^1(D),$$

$$(y_1, y_2)_E = (u_1, u_2) + (v_1, v_2),\ ||y||_E = (y, y)_E^{\frac{1}{2}} \quad \forall y_i = (u_i, v_i)^T \in E \quad i = 1, 2.$$

In the sequel, we consider the probability space $(\Omega, \mathcal{F}, P)$ where

$$\Omega = \{\omega = (\omega_1, \omega_2, \cdots \omega_m) \in C(R, R^m) : \omega(0) = 0\}.$$

$\mathcal{F}$ is the Borel σ-algebra induced by the compact open topology of Ω and P the corresponding Wiener measure on $(\Omega, \mathcal{F})$. Then we identify ω with

$$w(t) \equiv (w_1(t), w_2(t), \cdots w_m(t)) = \omega(t) \text{ for } t \in \mathbb{R}.$$

Define the time shift by

$$\vartheta_t(\cdot) = \omega(\cdot + t) - \omega(t),\ \text{ for } \omega \in \Omega,\ t \in \mathbb{R}.$$

Then $(\Omega, \mathcal{F}, P, (\vartheta_t)_{t\in R})$ is a metric dynamical system.

Let $j = 1, 2, \cdots, m$. We consider the one-dimensional Ornstein-Uhlenbeck equation

$$dz_j + \lambda z_j dt = dw_j(t). \tag{3.1}$$

It is easy to check that a solution of (3.1) is given by

$$z_j(t) = z_j(\vartheta_t \omega_j) \equiv -\lambda \int_{-\infty}^0 e^{\lambda\tau} (\vartheta_t \omega_j)(\tau) d\tau,\ t \in \mathbb{R}. \tag{3.2}$$

Lemma 3.1. *For $\epsilon > 0$, there exists a tempered random variable $r : \Omega \to \mathbb{R}^+$ such that*

$$\sum_{j=1}^m (|z_j(\vartheta_t \omega_j)|^2 + |z_j(\vartheta_t \omega_j)|^q) \leq e^{\epsilon|t|} r(\omega) \text{ for all } t \in \mathbb{R} \text{ and } \omega \in \Omega,$$

where $q \geq 2$ and $r(\omega), \omega \in \Omega$ satisfies

$$r(\vartheta_t \omega) \leq e^{\epsilon|t|} r(\omega) \text{ for } t \in \mathbb{R}.$$

Proof. Let $j = 1, 2, \cdots, m$. Since $|z_j(\omega_j)|$ is a tempered random variable and the mapping $t \to \ln |z_j(\vartheta_t \omega_j)|$ is P-a.s.continuous, it follows from Proposition 4.3.3 in [17] that for $\epsilon_j > 0$, there is a tempered random variable $r_j(\omega_j) > 0$ such that

$$|z_j(\omega_j)| \leq r_j(\omega_j),$$

where $r_j(\omega_j)$ satisfies,

$$r_j(\vartheta_t\omega_j) \le e^{\epsilon_j|t|}r_j(\omega_j) \text{ for P-a.s. } \omega \in \Omega \text{ and } t \in \mathbb{R}.$$

Then taking

$$\epsilon_1 = \epsilon_2 = \cdots = \epsilon_m = \frac{\epsilon}{q},$$

we have

$$\begin{aligned}\sum_{j=1}^m (|z_j(\vartheta_t\omega_j)|^2 + |z_j(\vartheta_t\omega_j)|^q) &\le \sum_{j=1}^m (|r_j(\vartheta_t\omega_j)|^2 + |r_j(\vartheta_t\omega_j)|^q) \\ &\le \sum_{j=1}^m (e^{2\epsilon_j|t|}r_j^2(\omega_j) + e^{q\epsilon_j|t|}r_j^q(\omega_j)) \\ &\le e^{\epsilon|t|}\sum_{j=1}^m (r_j^2(\omega_j) + r_j^q(\omega_j)) \le e^{\epsilon|t|}r(\omega),\end{aligned}$$

where $r(\omega) = \sum_{j=1}^m (r_j^2(\omega_j) + r_j^q(\omega_j))$. □

Let $z(\vartheta_t\omega) = \sum_{j=1}^m h_j z_j(\vartheta_t\omega_j)$ and $z^*(\vartheta_t\omega) = \sum_{j=1}^m h_j^* z_j(\vartheta_t\omega_j)$, by (3.1) we have

$$dz + \lambda z dt = \sum_{j=1}^m h_j dw_j, \quad dz^* + \lambda z^* dt = \sum_{j=1}^m h_j^* dw_j. \tag{3.3}$$

Corollary 3.1. *Suppose $h_j, h_j^* \in H^2 \cap W^{2,q}(D)$ for $j = 1, 2, \cdots, m$. For $\epsilon > 0$, there is a constant $c > 0$ such that for all $t \in R$, $\omega \in \Omega$,*

$$||z(\vartheta_t\omega)||_q^q + ||z(\vartheta_t\omega)||^2 + ||z^*(\vartheta_t\omega)||^2 + ||\nabla z(\vartheta_t\omega)||^2 \le k_1 e^{\epsilon|t|}r(\omega), \tag{3.4}$$

$$||\Delta z(\vartheta_t\omega)||_q^q + ||\Delta z(\vartheta_t\omega)||^2 + ||z^*(\vartheta_t\omega)||^2 \le k_2 e^{\epsilon|t|}r(\omega), \tag{3.5}$$

$$||\nabla z(\vartheta_t\omega)||^2 + ||\nabla z^*(\vartheta_t\omega)||^2 \le k_3 e^{\epsilon|t|}r(\omega), \tag{3.6}$$

where

$$\begin{aligned}k_1 &= (\sum_{j=1}^m ||h_j||_q^{\frac{q}{q-1}})^{q-1} + \sum_{j=1}^m (||h_j||^2 + ||\nabla h_j||^2 + ||h_j^*||^2), \\ k_2 &= (\sum_{j=1}^m ||\Delta h_j||_q^{\frac{q}{q-1}})^{q-1} + \sum_{j=1}^m (||h_j||^2 + ||h_j^*||^2) \quad \textit{and} \\ k_3 &= \sum_{j=1}^m (||\nabla h_j||^2 + ||\nabla h_j^*||^2).\end{aligned}$$

Proof. Since $z(\vartheta_t\omega) = \sum_{j=1}^m h_j z_j(\vartheta_t\omega_j)$, we get

$$||z(\vartheta_t\omega)||_q \le \sum_{j=1}^m ||h_j||_q \, |z_j(\vartheta_t\omega_j)| \le (\sum_{j=1}^m ||h_j||_q^{\frac{q}{q-1}})^{\frac{q-1}{q}} (\sum_{j=1}^m |z_j(\vartheta_t\omega_j)|^q)^{\frac{1}{q}}.$$

It follows from Lemma 3.1 that

$$||z(\vartheta_t\omega)||_q^q \le (\sum_{j=1}^m ||h_j||_q^{\frac{q}{q-1}})^{q-1} e^{\epsilon|t|}r(\omega).$$

Similarly,

$$||z(\vartheta_t\omega)||^2 \le (\sum_{j=1}^{m}||h_j||^2)e^{\epsilon|t|}r(\omega),$$

$$||z^*(\vartheta_t\omega)||^2 \le (\sum_{j=1}^{m}||h_j^*||^2)e^{\epsilon|t|}r(\omega),$$

and

$$||\nabla z(\vartheta_t\omega)||^2 \le (\sum_{j=1}^{m}||\nabla h_j||^2)e^{\epsilon|t|}r(\omega).$$

Adding the above three inequalities implies that (3.4) holds. The proofs of the two inequalities (3.5) and (3.6) are similar and omitted. □

Now we show that there is a continuous random dynamical system generated by the partly dissipative stochastic reaction-diffusion equations with additive noise:

$$du + (-\mu\Delta u + \lambda u + \alpha v)dt = (h(u) + f(x))dt + \sum_{j=1}^{m} h_j dw_j, \tag{3.7}$$

$$dv + (\delta v - \beta u)dt = g(x)dt + \sum_{j=1}^{m} h_j^* dw_j, \tag{3.8}$$

with the boundary condition

$$u(x,t)|_{x\in\partial D} = 0, \quad v(x,t)|_{x\in\partial D} = 0, \tag{3.9}$$

with the initial condition

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x). \tag{3.10}$$

where $D \subset \mathbb{R}^n$ with smooth boundary ∂D, $\mu, \lambda, \alpha, \delta, \beta$ are positive constants, f, g, h_j and h_j^* are given functions.The nonlinear function $h(u)$ satisfies the following condition:

$$h(s)s \le -\alpha_1|s|^q, \quad |h(s)| \le \alpha_2|s|^{q-1}, \quad \frac{\partial h(s)}{\partial s} \le \alpha_3. \tag{3.11}$$

where α_1, α_2 and α_3 are positive constants. To show that problem (3.7)-(3.10) generates a random dynamical system, we let $n(t) = u(t) - z(\vartheta_t\omega)$, $m(t) = v(t) - z^*(\vartheta_t\omega)$, where (u, v) is a solution of problem (3.7)-(3.10). Then $n(t),\ m(t)$ satisfy

$$\frac{\partial n}{\partial t} - \mu\Delta n + \lambda n + \alpha v = \mu\Delta z(\vartheta_t\omega) - \alpha z^*(\vartheta_t\omega) + h(u) + f(x), \tag{3.12}$$

$$\frac{\partial m}{\partial t} + \delta m - \beta n = \beta z(\vartheta_t\omega) + (\lambda - \delta)z^*(\vartheta_t\omega) + g(x), \tag{3.13}$$

with the initial data $(n_0, m_0) = (u_0 - z(\omega), v_0 - z^*(\omega))$ and homogeneous boundary conditions.

For each fixed $\omega \in \Omega$, (3.12)-(3.13) is a deterministic differential equations. By a Galerkin method, one can show that if h satisfies (3.11), then (3.12)-(3.13) have a unique solution $(n, m) \in C([0, \infty); L^2 \times L^2) \cap L^2((0, T); H^1 \times L^2)$ with (n_0, m_0) for every $T \ge 0$. Let $\varphi_0 = (n_0, m_0) = (u_0 - z(\omega), v_0 - z^*(\omega))$ and $\varphi(t, \omega, \varphi_0) = (n(t, \omega, \varphi_0(\omega)), m(t, \omega, \varphi_0(\omega)))$. Then the process $\phi = \varphi + (z(\vartheta_t\omega), z^*(\vartheta_t\omega))$ is the solution of problem (3.7)-(3.10). Therefore, ϕ is a continuous random dynamical system associated with the partly dissipative stochastic reaction-diffusion equations.

4. Uniform estimates of solutions

Let $\varphi = (n, m)$ be the solution of (3.12)-(3.13). For $\omega \in \Omega$, we need the priori estimates of the solution $\varphi = (n, m)$ in $E = L^2(D) \times L^2(D)$.

Lemma 4.1. *Assume that $f, g \in L^2$ and (3.11) holds. Let $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$, the collection of all tempered subsets of E and $\phi_0(\omega) = (u_0(\omega), v_0(\omega)) \in B(\omega)$. Then for P.a.e. $\omega \in \Omega$, there is $T_B(\omega) > 0$ such that for all $t \geq T_B(\omega)$,*

$$||\phi(t, \vartheta_{-t}\omega, \phi_0(\vartheta_{-t}\omega))||_E \leq c(1 + r(\omega)),$$

where c is a positive deterministic constant independent of $T_B(\omega)$ and $r(\omega)$ is a tempered function in Lemma 3.1.

Proof. Taking the inner product of both sides of (3.12) with βn, we find that

$$\begin{aligned}&\frac{1}{2}\beta\frac{d}{dt}||n||^2 + \mu\beta||\nabla n||^2 + \lambda\beta||n||^2 + \alpha\beta(v, n)\\ =&\beta\mu(\Delta z(\vartheta_t\omega), n) - \alpha\beta(z^*(\vartheta_t\omega), n) + \beta(h(u), n) + \beta(f, n).\end{aligned} \tag{4.1}$$

Similarly, taking the inner product of both sides of (3.13) with αv, we obtain

$$\begin{aligned}&\frac{1}{2}\alpha\frac{d}{dt}||m||^2 + \delta\alpha||m||^2 - \alpha\beta(n, m)\\ =&\alpha\beta(z(\vartheta_t\omega), m) + (\lambda - \delta)(z^*(\vartheta_t\omega), m) + \alpha(g, m).\end{aligned} \tag{4.2}$$

Summing up (4.1) and (4.2), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\beta||n||^2 + \alpha||m||^2) + \mu\beta||\nabla n||^2 + \lambda\beta||n||^2 + \delta\alpha||m||^2\\ =&\beta(h(u), n) + \beta\mu(\Delta z(\vartheta_t\omega), n) - \alpha\beta(z^*(\vartheta_t\omega), n) + \alpha\beta(z(\vartheta_t\omega), m)\\ &+ (\lambda - \delta)(z^*(\vartheta_t\omega), m) + \beta(f, n) + +\alpha(g, m).\end{aligned} \tag{4.3}$$

We now majorize the right-hand side of (4.3) as follows.

$$\begin{aligned}&\beta\int_D h(u)ndx = \beta\int_D h(u)udx - \beta\int_D h(u)z(\vartheta_t\omega)dx\\ &\leq -\beta\alpha_1\int |u|^q dx + \beta\alpha_2\int |u|^{q-1}|z(\vartheta_t\omega)|dx\\ &\leq -\frac{1}{2}\beta\alpha_1||u||_q^q + c_1||z(\vartheta_t\omega)||_q^q,\end{aligned} \tag{4.4}$$

$$|\beta\mu(\Delta z(\vartheta_t\omega), n)| \leq \frac{1}{2}\beta\mu||\nabla n||^2 + \frac{1}{2}\beta\mu||\nabla z(\vartheta_t\omega)||^2, \tag{4.5}$$

$$| - \alpha\beta(z^*(\vartheta_t\omega), n)| \leq \frac{\lambda\beta}{4}||n||^2 + \frac{1}{\lambda}\beta\alpha^2||z^*(\vartheta_t\omega)||^2, \tag{4.6}$$

$$|\beta(f, n)| \leq \frac{1}{4}\beta\lambda||n||^2 + \frac{1}{\lambda}\beta||f||^2, \tag{4.7}$$

$$|\alpha\beta(z(\vartheta_t\omega), m)| \leq \frac{1}{8}\alpha\delta||m||^2 + \frac{2}{\delta}\alpha\beta^2||z(\vartheta_t\omega)||^2, \tag{4.8}$$

$$|(\lambda - \delta)(z^*(\vartheta_t\omega), m)| \leq \frac{1}{8}\alpha\delta||m||^2 + \frac{2}{\delta\alpha}(\lambda - \delta)^2||z^*(\vartheta_t\omega)||^2, \tag{4.9}$$

and

$$|\alpha(g,n)| \leq \frac{1}{8}\alpha\delta||v||^2 + \frac{2}{\delta}\alpha||g||^2. \tag{4.10}$$

By (4.3)-(4.10), we obtain

$$\begin{aligned} &\frac{d}{dt}(\beta||n||^2 + \alpha||m||^2) + \mu\beta||\nabla n||^2 + \lambda\beta||n||^2 + \delta\alpha||m||^2 + \beta\alpha_1||u||_q^q \\ \leq& c_2(||z(\vartheta_t\omega)||_q^q + ||z(\vartheta_t\omega)||^2 + ||z^*(\vartheta_t\omega)||^2 + ||\nabla z(\vartheta_t\omega)||^2) + c_3 \\ \leq& p_0(\vartheta_t\omega) + c_3, \end{aligned} \tag{4.11}$$

where $p_0(\vartheta_t\omega) = c_2(||z(\vartheta_t\omega)||_q^q + ||z(\vartheta_t\omega)||^2 + ||\nabla z(\vartheta_t\omega)||^2)$ and $c_3 = \frac{2}{\lambda}\beta||f||^2 + \frac{4}{\delta}\alpha||g||^2$. Let $\nu = \min\{\delta, \lambda\}$, $\sigma = \min\{\alpha, \beta\}$ and $\gamma = \max\{\alpha, \beta\}$. Then we find

$$\frac{d}{dt}(\beta||n||^2 + \alpha||m||^2) + \nu(\beta||n||^2 + \alpha||m||^2) \leq p_0(\vartheta_t\omega) + c_3. \tag{4.12}$$

Applying Gronwall's lemma, we find that, for all $t \geq 0$

$$||\varphi(t,\omega,\varphi_0(\omega))||_E^2 \leq \frac{1}{\sigma}(\gamma e^{-\nu t}||\varphi_0(\omega)||_E^2 + \int_0^t e^{\nu(\tau-t)}p_0(\vartheta_\tau\omega)d\tau + \frac{c_3}{\nu}), \tag{4.13}$$

where $||\varphi||_E^2 = ||n||^2 + ||m||^2$, $\varphi_0 = (n_0, m_0)$. By replacing ω by $\vartheta_{-t}\omega$ in (4.13) and by Corollary 3.2 with $\epsilon = \frac{\nu}{2}$, we obtain, for all $t \geq 0$,

$$\begin{aligned} ||\varphi(t,\vartheta_{-t}\omega,\varphi_0(\vartheta_{-t}\omega))||_E^2 &\leq \frac{1}{\sigma}(\gamma e^{-\nu t}||\varphi_0(\vartheta_{-t}\omega)||_E^2 + \int_0^t e^{\nu(\tau-t)}p_0(\vartheta_{\tau-t}\omega)d\tau + \frac{c_3}{\nu}) \\ &\leq \frac{1}{\sigma}(\gamma e^{-\nu t}||\varphi_0(\vartheta_{-t}\omega)||_E^2 + \int_{-t}^0 e^{\nu\tau}p_0(\vartheta_\tau\omega)d\tau + \frac{c_3}{\nu}) \\ &\leq \frac{1}{\sigma}(\gamma e^{-\nu t}||\varphi_0(\vartheta_{-t}\omega)||_E^2 + c_2k_1\int_{-t}^0 e^{\frac{1}{2}\nu\tau}r(\omega)d\tau + \frac{c_3}{\nu}) \\ &\leq \frac{1}{\sigma}(\gamma e^{-\nu t}||\varphi_0(\vartheta_{-t}\omega)||_E^2 + \frac{2c_2k_1}{\nu}r(\omega) + \frac{c_3}{\nu}). \end{aligned} \tag{4.14}$$

By assumption $\{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ is tempered and $||z(\omega)||^2$, $||z^*(\omega)||^2$ is also tempered. Therefore, if $\phi_0(\vartheta_{-t}\omega) = (u_0(\vartheta_{-t}\omega), v_0(\vartheta_{-t}\omega)) \in B(\vartheta_{-t}\omega)$, then there exists $T_B(\omega) > 0$ such that for $t \geq T_B(\omega)$,

$$\begin{aligned} \gamma e^{-\nu t}||\varphi_0(\vartheta_{-t}\omega)||_E^2 \leq& \gamma e^{-\nu t}(||n_0(\vartheta_{-t}\omega)||^2 + ||m_0(\vartheta_{-t}\omega)||^2) \\ \leq& \gamma e^{-\nu t}(||u_0(\vartheta_{-t}\omega)||^2 + ||z(\vartheta_{-t}\omega)||^2 + ||m_0(\vartheta_{-t}\omega)||^2 \\ &+ ||z^*(\vartheta_{-t}\omega)||^2) \\ \leq& \frac{2c_2k_1}{\nu}r(\omega) + \frac{c_3}{\nu} \end{aligned} \tag{4.15}$$

It follows from (4.14) and (4.15) that

$$||\varphi(t,\vartheta_{-t}\omega,\varphi_0(\vartheta_{-t}\omega))||_E^2 \leq \frac{1}{\nu\sigma}(2c_2k_1 + c_3)(1 + r(\omega)). \tag{4.16}$$

Note that $\varphi_0 = (n_0, m_0) = (u_0 - z(\omega), v_0 - z^*(\omega))$ and $\phi = \varphi + (z(\vartheta_t\omega), z^*(\vartheta_t\omega))$, we have, for $t \geq T_B(\omega)$,

$$\begin{aligned} ||\phi(t, \vartheta_{-t}\omega, \phi_0(\vartheta_{-t}\omega))||_E^2 =& ||\varphi(t, \vartheta_{-t}\omega, \varphi_0(\vartheta_{-t}\omega)) + (z(\omega), z^*(\omega))||_E^2 \\ \leq& 2||\varphi(t, \vartheta_{-t}\omega, \varphi_0(\vartheta_{-t}\omega))||_E^2 + 2||z(\omega)||^2 + 2||z^*(\omega)||^2. \end{aligned} \tag{4.17}$$

By (3.4) with $t = 0$ in Corollary 3.1, we have

$$||z(\omega)||^2 + ||z^*(\omega)||^2 \leq k_1 r(\omega). \tag{4.18}$$

The result holds from (4.16)-(4.18). □

Denote by

$$S(\omega) = \{(u, v) \in L^2(D) \times L^2(D) : ||u||^2 + ||v||^2 \leq \frac{2}{\nu\sigma}(2c_2k_1 + k_1 + c_3)(1 + r(\omega))\}.$$

Then $\{S(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ is a random absorbing set.

Lemma 4.2. *Assume that $f, g \in L^2$ and (3.11) holds. Let $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$, the collection of all tempered subsets of E and $\phi_0(\omega) = (u_0(\omega), v_0(\omega)) \in B(\omega)$. Then for every $T \geq 0$ and for P-a.e. $\omega \in \Omega$, such that the solutions $(u, , v)$ of problem (3.7)-(3.10) and (n, m) of (3.12)-(3.13) satisfy, for $t \geq T$,*

$$\int_T^t e^{\nu(s-t)}||\varphi(s, \vartheta_{-t}\omega, \varphi_0(\vartheta_{-t}\omega))||_E^2 ds \leq c(1 + r(\omega) + ||\varphi_0(\vartheta_{-t}\omega)||_E^2 e^{-\nu t})(t - T), \tag{4.19}$$

$$\int_T^t e^{\nu(s-t)}||\nabla n(s, \vartheta_{-t}\omega, \varphi_0(\vartheta_{-t}\omega))||^2 ds \leq c(1 + r(\omega) + ||\varphi_0(\vartheta_{-t}\omega)||_E^2 e^{-\nu t}), \tag{4.20}$$

$$\int_T^t e^{\nu(s-t)}||u(s, \vartheta_{-t}\omega, \phi_0(\vartheta_{-t}\omega))||_q^q ds \leq c(1 + r(\omega) + ||\varphi_0(\vartheta_{-t}\omega)||_E^2 e^{-\nu t}), \tag{4.21}$$

$$\int_T^t e^{\nu(s-t)}||\nabla u(s, \vartheta_{-t}\omega, \phi_0(\vartheta_{-t}\omega))||^2 ds \leq c(1 + r(\omega) + ||\varphi_0(\vartheta_{-t}\omega)||_E^2 e^{-\nu t}), \tag{4.22}$$

where $\varphi_0(\omega) = \phi_0(\omega) - (z(\omega), z^*(\omega))$, c is a positive deterministic constant and $r(\omega)$ is a tempered function in Lemma 3.1.

Proof. First, replacing t by T and then replacing ω by $\vartheta_{-t}\omega$ in (4.13), we obtain

$$\begin{aligned} &||\varphi(T, \vartheta_{-t}\omega, \varphi_0(\vartheta_{-t}\omega))||_E^2 \\ \leq& \frac{1}{\sigma}(\gamma e^{-\nu T}||\varphi_0(\vartheta_{-t}\omega)||_E^2 + \int_0^T e^{\nu(\tau-T)}p_0(\vartheta_{\tau-t}\omega)d\tau + \frac{c_3}{\nu}). \end{aligned}$$

Multiplying the above by $e^{\nu(T-t)}$ and by Corollary 3.2 with $\epsilon=\frac{\nu}{2}$, we get

$$\begin{aligned}
&e^{\nu(T-t)}||\varphi(T,\vartheta_{-t}\omega,\varphi_0(\vartheta_{-t}\omega))||_E^2\\
\leq&\frac{1}{\sigma}(\gamma e^{-\nu t}||\varphi_0(\vartheta_{-t}\omega)||_E^2+\int_0^T e^{\nu(\tau-t)}p_0(\vartheta_{\tau-t}\omega)d\tau+\frac{c_3}{\nu}e^{\nu(T-t)})\\
\leq&\frac{1}{\sigma}(\gamma e^{-\nu t}||\varphi_0(\vartheta_{-t}\omega)||_E^2+\int_{-t}^{T-t} e^{\nu\tau}p_0(\vartheta_{\tau}\omega)d\tau+\frac{c_3}{\nu}e^{\nu(T-t)})\\
\leq&\frac{1}{\sigma}(\gamma e^{-\nu t}||\varphi_0(\vartheta_{-t}\omega)||_E^2+k_1\int_{-t}^{T-t} e^{\frac{1}{2}\nu\tau}r(\omega)d\tau+\frac{c_3}{\nu})\\
\leq&\frac{1}{\sigma}(\gamma e^{-\nu t}||\varphi_0(\vartheta_{-t}\omega)||_E^2+\frac{2}{\nu}k_1r(\omega)+\frac{c_3}{\nu}).
\end{aligned}\tag{4.23}$$

It follows from (4.23) that

$$\begin{aligned}
&\int_T^t e^{\nu(s-t)}||\varphi(s,\vartheta_{-t}\omega,\varphi_0(\vartheta_{-t}\omega))||_e^2ds\\
\leq&\frac{1}{\sigma}(\gamma e^{-\nu t}||\varphi_0(\vartheta_{-t}\omega)||_E^2+\frac{2}{\nu}k_1r(\omega)+\frac{c_3}{\nu})(t-T).
\end{aligned}\tag{4.24}$$

By (4.24), (4.19) holds. By (4.11) and (4.12), we obtain for $t\geq T$,

$$\begin{aligned}
&\mu\beta\int_T^t e^{\nu(s-t)}||\nabla n(s,\omega,\varphi_0(\omega))||^2ds+\beta\alpha_1\int_T^t e^{\nu(s-t)}||u(s,\omega,\phi_0(\omega))||_q^qds\\
\leq&\sigma e^{\nu(T-t)}||\varphi(T,\omega,\varphi_0(\omega))||_E^2+\int_T^t e^{\nu(s-t)}p_0(\vartheta_s\omega)ds+c_3\int_T^t e^{\nu(s-t)}ds.
\end{aligned}\tag{4.25}$$

Replacing ω by $\vartheta_{-t}\omega$ in (4.25), we have that, for $t\geq T$,

$$\begin{aligned}
&\mu\beta\int_T^t e^{\nu(s-t)}||\nabla n(s,\vartheta_{-t}\omega,\varphi_0(\vartheta_{-t}\omega))||^2ds\\
&+\beta\alpha_1\int_T^t e^{\nu(s-t)}||u(s,\vartheta_{-t}\omega,\phi_0(\vartheta_{-t}\omega))||_q^qds\\
\leq&\sigma e^{\nu(T-t)}||\varphi(T,\vartheta_{-t}\omega,\varphi_0(\vartheta_{-t}\omega))||_E^2\\
&+\int_T^t e^{\nu(s-t)}p_0(\vartheta_{s-t}\omega)ds+c_3\int_T^t e^{\nu(s-t)}ds\\
\leq&\sigma e^{\nu(T-t)}||\varphi(T,\vartheta_{-t}\omega,\varphi_0(\vartheta_{-t}\omega))||_E^2+\int_{T-t}^0 e^{\nu s}p_0(\vartheta_s\omega)ds+\frac{c_3}{\nu}\\
\leq&\sigma e^{\nu(T-t)}||\varphi(T,\vartheta_{-t}\omega,\varphi_0(\vartheta_{-t}\omega))||_E^2+\frac{2}{\nu}k_1r(\omega)+\frac{c_3}{\nu},
\end{aligned}\tag{4.26}$$

where we have used Corollary 3.1. It follows from (4.23) and (4.26) that, we have

$$\mu\beta\int_T^t e^{\nu(s-t)}||\nabla n(s,\vartheta_{-t}\omega,\varphi_0(\vartheta_{-t}\omega))||^2ds\leq\gamma e^{-\nu t}||\varphi_0(\vartheta_{-t}\omega)||_E^2+\frac{3}{\nu}k_1r(\omega)+\frac{2c_3}{\nu},\tag{4.27}$$

and

$$\beta\alpha_1\int_T^t e^{\nu(s-t)}||u(s,\vartheta_{-t}\omega,\phi_0(\vartheta_{-t}\omega))||_q^qds\leq\gamma e^{-\nu t}||\varphi_0(\vartheta_{-t}\omega)||_E^2+\frac{3}{\nu}k_1r(\omega)+\frac{2c_3}{\nu},\tag{4.28}$$

By (4.27) and (4.28), (4.20) and (4.21) holds. By (4.20), we obtain, for all $t \geq T$,

$$\begin{aligned}
&\int_T^t e^{\nu(s-t)}||\nabla u(s, \vartheta_{-t}\omega, \phi_0(\vartheta_{-t}\omega))||^2 ds \\
=&\int_T^t e^{\nu(s-t)}||\nabla n(s, \vartheta_{-t}\omega, \varphi_0(\vartheta_{-t}\omega)) + \nabla z(\vartheta_{s-t}\omega)||^2 ds \\
\leq&2\int_T^t e^{\nu(s-t)}||\nabla n(s, \vartheta_{-t}\omega, \varphi_0(\vartheta_{-t}\omega))||^2 ds + 2\int_T^t e^{\nu(s-t)}||\nabla z(\vartheta_{s-t}\omega)||^2 ds \\
\leq&2c(1 + r(\omega) + ||\varphi_0(\vartheta_{-t}\omega)e^{-\nu t}||_E^2) + 2\int_{T-t}^0 e^{\nu\tau}||\nabla z(\vartheta_\tau\omega)||^2 d\tau \\
\leq&2c(1 + r(\omega) + ||\varphi_0(\vartheta_{-t}\omega)e^{-\nu t}||_E^2) + 2\int_{T-t}^0 e^{\frac{1}{2}\nu\tau}k_3 r(\omega) d\tau \\
\leq&2c(1 + r(\omega) + ||\varphi_0(\vartheta_{-t}\omega)e^{-\nu t}||_E^2) + \frac{4}{\nu}k_3 r(\omega),
\end{aligned} \tag{4.29}$$

where we have used Corollary 3.1. (4.22) holds from (4.29). □

Lemma 4.3. *Assume that* $f, g \in L^2$ *and (3.11) holds. Let* $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$, *the collection of all tempered subsets of* E *and* $\phi_0(\omega) = (u_0(\omega), v_0(\omega)) \in B(\omega)$. *Then for P-a.e.* $\omega \in \Omega$, *there exists* $T_B(\omega) > 0$ *such that the solutions* $(u,, v)$ *of problem (3.7)-(3.10) and* (n, v) *of (3.12)-(3.13) satisfy, for* $t \geq T_B(\omega)$,

$$\int_t^{t+1} ||\varphi(s, \vartheta_{-t-1}\omega, \varphi_0(\vartheta_{-t-1}\omega))||^2 ds \leq c(1 + r(\omega)), \tag{4.30}$$

$$\int_t^{t+1} ||\nabla n(s, \vartheta_{-t-1}\omega, \varphi_0(\vartheta_{-t-1}\omega))||^2 ds \leq c(1 + r(\omega)), \tag{4.31}$$

$$\int_t^{t+1} ||u(s, \vartheta_{-t-1}\omega, \phi_0(\vartheta_{-t-1}\omega))||_q^q ds \leq c(1 + r(\omega)), \tag{4.32}$$

where $\varphi_0(\omega) = \phi_0(\omega) - (z(\omega), z^*(\omega))$, c *is a positive deterministic constant and* $r(\omega)$ *is a tempered function in Lemma 3.1.*

Proof. First, replacing t by $t + 1$ and then replacing T by t in (4.19), we obtain

$$\begin{aligned}
&\int_t^{t+1} e^{\nu(s-t-1)}||\varphi(s, \vartheta_{-t-1}\omega, \varphi_0(\vartheta_{-t-1}\omega))||_E^2 ds \\
\leq&c(1 + r(\omega) + ||\varphi_0(\vartheta_{-t-1}\omega)||_E^2 e^{-\nu(t+1)}).
\end{aligned} \tag{4.33}$$

Since

$$\begin{aligned}
||\varphi_0(\vartheta_{-t-1}\omega)||_E^2 =&||n_0(\vartheta_{-t-1}\omega)||^2 + ||m_0(\vartheta_{-t-1}\omega)||^2 \\
\leq&2||u_0(\vartheta_{-t-1}\omega)||^2 + 2||z(\vartheta_{-t-1}\omega)||^2 + 2||v_0(\vartheta_{-t-1}\omega)||^2 \\
&+ 2||z^*(\vartheta_{-t-1}\omega)||^2,
\end{aligned}$$

and $||u_0(\vartheta_{-t}\omega)||^2$, $||z(\vartheta_{-t}\omega)||^2$, $||v_0(\vartheta_{-t}\omega)||^2)$ and $||z^*(\vartheta_{-t}\omega)||^2$ are tempered, there is $T_B(\omega) > 0$ such that for $t \geq T_B(\omega)$,

$$||\varphi_0(\vartheta_{-t-1}\omega)||_E^2 e^{-\nu(t+1)} \leq c(1 + r(\omega)).$$

Hence, from (4.33) we have, for $t \geq T_B(\omega)$

$$e^{-\nu}\int_t^{t+1} ||\varphi(s,\vartheta_{-t-1}\omega,\varphi_0(\vartheta_{-t-1}\omega))||_E^2 ds \leq 2c(1+r(\omega)). \tag{4.34}$$

By (4.20) and (4.21), we can find that, for $t \geq T_B(\omega)$,

$$\begin{aligned}&\int_t^{t+1} ||\nabla n(s,\vartheta_{-t}\omega,\phi_0(\vartheta_{-t}\omega))||^2 ds \leq 2ce^{\nu}(1+r(\omega)).\\&\int_t^{t+1} ||u(s,\vartheta_{-t}\omega,\phi_0(\vartheta_{-t}\omega))||_q^q ds \leq 2ce^{\nu}(1+r(\omega)),\end{aligned} \tag{4.35}$$

The result follows from (4.34) and (4.35). □

Lemma 4.4. *Assume that $f, g \in L^2$ and (3.11) holds. Let $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ which is the collection of all tempered subsets of E and $\phi_0(\omega) = (u_0(\omega), v_0(\omega)) \in B(\omega)$. Then for P-a.e. $\omega \in \Omega$, there exists $T_B(\omega) > 0$ such that the solutions $(u,,v)$ of problem (3.7)-(3.10) satisfy, for $t \geq T_B(\omega)$,*

$$\int_t^{t+1} ||\nabla u(s,\vartheta_{-t-1}\omega,\phi_0(\vartheta_{-t-1}\omega))||^2 ds \leq c(1+r(\omega)), \tag{4.36}$$

where c is a positive deterministic constant and $r(\omega)$ is a tempered function in Lemma 3.1.

Proof. By Lemma 4.3, for $t \geq T_B(\omega)$, $s \in (t,t+1)$, we find that

$$\begin{aligned}&||\nabla u(s,\vartheta_{-t-1}\omega,\phi_0(\vartheta_{-t-1}\omega))||^2\\=&||\nabla n(s,\vartheta_{-t-1}\omega,\varphi_0(\vartheta_{-t-1}\omega)) + \nabla z(\vartheta_{s-t-1}\omega)||^2\\\leq&2(||\nabla n(s,\vartheta_{-t-1}\omega,\varphi_0(\vartheta_{-t-1}\omega))||^2 + ||\nabla z(\vartheta_{s-t-1}\omega)||^2).\end{aligned} \tag{4.37}$$

By Corollary 3.2 with $\epsilon = \frac{\nu}{2}$ we get

$$||\nabla z(\vartheta_{s-t-1}\omega)||^2 \leq k_3 e^{\frac{\nu}{2}(t+1-s)} r(\omega) \leq k_3 e^{\frac{\nu}{2}} r(\omega). \tag{4.38}$$

Integrating (4.37) with respect to s over $(t,t+1)$, by Lemma 4.3 and inequality (4.38), we have

$$\begin{aligned}&\int_t^{t+1} ||\nabla u(s,\vartheta_{-t-1}\omega,\phi_0(\vartheta_{-t-1}\omega))||^2 ds\\\leq&2(\int_t^{t+1} ||\nabla n(s,\vartheta_{-t-1}\omega,\varphi_0(\vartheta_{-t-1}\omega))||^2 ds + \int_t^{t+1} ||\nabla z(\vartheta_{s-t-1}\omega)||^2 ds)\\\leq&2c(1+r(\omega)) + 2k_3 e^{\frac{\nu}{2}} r(\omega).\end{aligned} \tag{4.39}$$

The results holds from (4.39). □

Lemma 4.5. *Assume that $f, g \in L^2$ and (3.11) holds. Let $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$, the collection of all tempered subsets of E and $\phi_0(\omega) = (u_0(\omega), v_0(\omega)) \in B(\omega)$. Then for P-a.e. $\omega \in \Omega$, there exists $T_B(\omega) > 0$ such that for $t \geq T_B(\omega)$,*

$$||\nabla u(t,\vartheta_{-t}\omega,\phi_0(\vartheta_{-t}\omega))||^2 \leq c(1+r(\omega)).$$

where c is a positive deterministic constant and $r(\omega)$ is a tempered function in Lemma 3.1.

Proof. Taking the inner product of (3.12) with $-\Delta n$ in L^2, we get that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}||\nabla n||^2+\mu||\Delta n||^2+\lambda||\nabla n||^2+\alpha(m,-\Delta n)\\=&(h(u),-\Delta n)+\mu(\Delta z,-\Delta n)+(f,-\Delta n)-\alpha(z^*,-\Delta n).\end{aligned}\tag{4.40}$$

Note that

$$\begin{aligned}-\int_D h(u)\Delta n dx=&-\int_D h(u)\Delta u dx+\int_D h(u)\Delta z(\vartheta_t\omega)dx\\ \leq&\int_D h^{'}(u)|\nabla u|^2dx+\int_D|h(u)\Delta z(\vartheta_t\omega)|dx\\ \leq&\,\alpha_3||\nabla u||^2+\alpha_2\int_D|u|^{q-1}|\Delta z(\vartheta_t\omega)|dx\\ \leq&\, c(||\nabla u||^2+||u||_q^q)+c||\Delta z(\vartheta_t\omega)||_q^q,\end{aligned}\tag{4.41}$$

and

$$\begin{aligned}&|-\alpha(m,-\Delta n)+\mu(\Delta z,-\Delta n)+(f,-\Delta n)-\alpha(z^*,-\Delta n)|\\ \leq&\frac{\mu}{2}||\Delta n||^2+2\frac{\alpha^2}{\mu}||m||^2+2\mu||\Delta Z(\vartheta_t\omega)||^2+2\frac{1}{\mu}||f||^2+2\frac{\alpha^2}{\mu}||z^*(\vartheta_t\omega)||^2.\end{aligned}\tag{4.42}$$

It follows from (4.40)-(4.42) that

$$\frac{d}{dt}||\nabla n||^2\leq c(||m||^2+||\nabla u||^2+||u||_q^q)+p_1(\vartheta_t\omega),\tag{4.43}$$

where $p_1(\vartheta_t\omega)=c(||\Delta z(\vartheta_t\omega)||^2+||\Delta z(\vartheta_t\omega)||_q^q+||z^*(\vartheta_t\omega)||^2+1)$. Let $T_B(\omega)$ is the positive constant in Lemma 4.1, take $t\geq T_B(\omega)$ and $s\in(t,t+1)$. Then integrate (4.43) over $(s,t+1)$ to get

$$\begin{aligned}&||\nabla n(t+1,\omega,\varphi_0(\omega))||^2-||\nabla n(s,\omega,\varphi_0(\omega))||^2\\ \leq&\int_s^{t+1}p_1(\vartheta_\tau\omega)d\tau+c\int_s^{t+1}||m(\tau,\omega,\varphi_0(\omega))||^2d\tau\\ &+c\int_s^{t+1}(||\nabla u(\tau,\omega,\phi_0(\omega))||^2+||u(\tau,\omega,\phi_0(\omega))||_q^q)d\tau\\ \leq&\int_t^{t+1}p_1(\vartheta_\tau\omega)d\tau+c\int_t^{t+1}||m(\tau,\omega,\varphi_0(\omega))||^2d\tau\\ &+c\int_t^{t+1}(||\nabla u(\tau,\omega,\phi_0(\omega))||^2+||u(\tau,\omega,\phi_0(\omega))||_q^q)d\tau.\end{aligned}\tag{4.44}$$

Now first integrating the above with respect to s over $(t,t+1)$ and then replacing ω by $\vartheta_{-t-1}\omega$ we find that

$$\begin{aligned}&||\nabla n(t+1,\vartheta_{-t-1}\omega,\varphi_0(\vartheta_{-t-1}\omega))||^2-\int_t^{t+1}||\nabla n(s,\vartheta_{-t-1}\omega,\varphi_0(\vartheta_{-t-1}\omega))||^2ds\\ \leq&\int_t^{t+1}p_1(\vartheta_{\tau-t-1}\omega)d\tau+c\int_t^{t+1}||m(\tau,\vartheta_{-t-1}\omega,\varphi_0(\vartheta_{-t-1}\omega))||^2d\tau\\ &+c\int_t^{t+1}(||\nabla u(\tau,\vartheta_{-t-1}\omega,\phi_0(\vartheta_{-t-1}\omega))||^2+||u(\tau,\vartheta_{-t-1}\omega,\phi_0(\vartheta_{-t-1}\omega))||_q^q)d\tau.\end{aligned}\tag{4.45}$$

Since $||m(\tau, \vartheta_{-t-1}\omega, \varphi_0(\vartheta_{-t-1}\omega))||^2 \leq ||\varphi(\tau, \vartheta_{-t-1}\omega, \varphi_0(\vartheta_{-t-1}\omega))||^2$, by Lemmas 4.3 and 4.4 and Corollary 3.2, it follows from (4.45) that for all $t \geq T_B(\omega)$,

$$\begin{aligned} ||\nabla n(t+1, \vartheta_{-t-1}\omega, \varphi_0(\vartheta_{-t-1}\omega))||^2 &\leq c(1+r(\omega)) + \int_t^{t+1} p_1(\vartheta_{\tau-t-1}\omega)d\tau \\ &\leq c(1+r(\omega)) + \int_{-1}^0 p_1(\vartheta_\tau\omega)d\tau \\ &\leq c(1+r(\omega)) + \int_{-1}^0 (ck_2 r(\omega)e^{-\frac{1}{2}\nu\tau} + c_4)d\tau \\ &\leq c(1+r(\omega)) + \frac{2}{\nu}ck_1 r(\omega)e^{\frac{1}{2}\nu} + c_4. \end{aligned} \tag{4.46}$$

The result follows from (4.46). □

In the following, we prove v is precompact in $L^2(D)$. For the solutions of (3.7)-(3.10), we decompose $v = m_1 + m_2 + z^*(\vartheta_t\omega) = m_1 + v^*$, where $v^*(t,\omega) = m_2(t,\omega) + z^*(\vartheta_t\omega)$, m_i $(i = 1, 2)$ solves respectively,

$$\frac{\partial}{\partial t}m_1 + \delta m_1 = 0, \tag{4.47}$$

$$m_{1,0}(0) = m_0 = v_0 - z^*(\omega) \tag{4.48}$$

and

$$\frac{\partial}{\partial t}m_2 + \delta m_2 = \beta u + g(x) + (\lambda - \delta)z^*(\vartheta_t\omega), \tag{4.49}$$

$$m_{2,0}(0) = 0. \tag{4.50}$$

For m_i $(i = 1, 2)$ we have the following Lemma.

Lemma 4.6. *Assume that $f \in L^2$, $g \in H_0^1$ and* (3.11) *holds. Let $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$, the collection of all tempered subsets of E and $\phi_0(\omega) = (u_0(\omega), v_0(\omega)) \in B(\omega)$. Then for every $\varepsilon > 0$ and for P-a.e. $\omega \in \Omega$, there exists $T_B(\omega) > 0$ such that the solutions $(u, , v)$ of problem* (3.7)-(3.10) *and m_1, m_2 of* (4.47)-(4.50) *satisfy, for $t \geq T_B(\omega)$,*

$$||m_1(t, \vartheta_{-t}\omega, m_0(\vartheta_{-t}\omega))||^2 \leq c\varepsilon, \tag{4.51}$$

$$||\nabla m_2(t, \vartheta_{-t}\omega)||^2 \leq c(1 + r(\omega)), \tag{4.52}$$

$$||\nabla v^*(t, \vartheta_{-t}\omega, z^*(\vartheta_{-t}\omega))||^2 \leq c(1 + r(\omega)), \tag{4.53}$$

where c is a positive deterministic constant and $r(\omega)$ is a tempered function in Lemma 3.1.

Proof. Taking the inner product of (4.47) with m_1 in L^2, we obtain

$$\frac{d}{dt}||m_1|| + 2\delta||m_1|| = 0, \tag{4.54}$$

Applying Gronwall's Lemma, we find that, for all $t \geq 0$,

$$||m_1(t, \omega, m_0(\omega))||^2 = e^{-2\delta t}||m_0(\omega)||^2. \tag{4.55}$$

Replacing ω by $\vartheta_{-t}\omega$, we have

$$||m_1(t,\vartheta_{-t}\omega,m_0(\vartheta_{-t}\omega))||^2 = ||m_0(\vartheta_{-t}\omega)||^2 e^{-2\delta t}. \tag{4.56}$$

Multiplying (4.49) by $-\Delta m_2$ and integrating over $(0,t)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||\nabla m_2||^2 + \delta||\nabla m_2||^2 \\
=&\beta(u,-\Delta m_2) + (g,-\Delta m_2) + (\lambda-\delta)(z^*(\vartheta_t\omega),-\Delta m_2) \\
\le&\frac{\delta}{2}||\nabla m_2||^2 + \frac{2\beta^2}{\delta}||\nabla u||^2 + \frac{2}{\delta}||\nabla g||^2 + +\frac{2}{\delta}(\lambda-\delta)^2||z^*(\vartheta_t\omega)||^2.
\end{aligned} \tag{4.57}$$

Since $\nu=\min\{\delta,\mu\}$, it follows from (4.57) that

$$\frac{d}{dt}||\nabla m_2||^2 + \nu||\nabla m_2||^2 \le \frac{4}{\nu}(\beta^2||\nabla u||^2 + ||\nabla g||^2 + (\lambda-\delta)^2||z^*(\vartheta_t\omega)||^2). \tag{4.58}$$

By Gronwall's inequality, we obtain

$$\begin{aligned}
||\nabla m_2(t,\omega)||^2 \le&\frac{4}{\nu}(\beta^2\int_0^t e^{\nu(s-t)}||\nabla u(s,\omega,\phi_0(\omega))||^2 ds + \int_0^t e^{\nu(s-t)}||\nabla g||^2 ds) \\
&+\frac{4}{\nu}(\lambda-\delta)^2\int_0^t e^{\nu(s-t)}||z^*(\vartheta_s\omega)||^2)ds.
\end{aligned} \tag{4.59}$$

Replacing ω by $\vartheta_{-t}\omega$, we have

$$\begin{aligned}
||\nabla m_2(t,\vartheta_{-t}\omega)||^2 \le&\frac{4}{\nu}\beta^2\int_0^t e^{\nu(s-t)}||\nabla u(s,\vartheta_{-t}\omega,\phi_0(\vartheta_{-t}\omega))||^2 ds + \frac{4}{\nu^2}||\nabla g||^2 \\
&+\frac{4}{\nu}(\lambda-\delta)^2\int_0^t e^{\nu(s-t)}||z^*(\vartheta_{s-t}\omega)||^2)ds \\
\le&\frac{4}{\nu}\beta^2 c(1+r(\omega)+||\varphi_0(\vartheta_{-t}\omega)||_E^2 e^{-\nu t}) + \frac{4}{\nu^2}||\nabla g||^2 \\
&+\frac{4}{\nu}(\lambda-\delta)^2\int_{-t}^0 e^{\nu s}||z^*(\vartheta_s\omega)||^2)ds \\
\le&\frac{4}{\nu}\beta^2 c(1+r(\omega)+||\varphi_0(\vartheta_{-t}\omega)||_E^2 e^{-\nu t}) + \frac{4}{\nu^2}||\nabla g||^2 \\
&+\frac{4}{\nu}(\lambda-\delta)^2 k_1\int_{-t}^0 e^{\frac{1}{2}\nu s}r(\omega)||^2)ds \\
\le&\frac{4}{\nu}\beta^2 c(1+r(\omega)+||\varphi_0(\vartheta_{-t}\omega)||_E^2 e^{-\nu t}) + \frac{4}{\nu^2}||\nabla g||^2 \\
&+\frac{8}{\nu^2}(\lambda-\delta)^2 k_1 r(\omega),
\end{aligned} \tag{4.60}$$

where we have used (4.22) with $T=0$ and Corollary 3.2. Then there exists $T_B(\omega) > 0$, such that for $t\ge T_B(\omega)$, we obtain

$$||\nabla m_2(t,\vartheta_{-t}\omega)||^2 \le c_4 + c_5 r(\omega). \tag{4.61}$$

Since $v^*(t,\omega) = m_2(t,\omega) + z^*(\vartheta_t\omega)$, then we have, for $t \geq T_B(\omega)$

$$\begin{aligned} ||\nabla v^*(t,\vartheta_{-t}\omega, v_0^*(\vartheta_{-t}\omega))||^2 &= ||\nabla v^*(t,\vartheta_{-t}\omega, z^*(\vartheta_{-t}\omega))||^2 \\ &= ||\nabla m_2(t,\vartheta_{-t}\omega) + \nabla z^*(\omega)||^2 \\ &\leq 2||\nabla m_2(t,\vartheta_{-t}\omega)||^2 + 2||\nabla z^*(\omega)||^2 \\ &\leq c_4 + c_5 r(\omega)) + k_3 r(\omega). \end{aligned} \tag{4.62}$$

The results hold from (4.56), (4.61) and (4.62). □

Assume that $f \in L^2$, $g \in H_0^1$. Then $S(t)$ forms a random dynamical system. By Lemma 4.1, we have shown that $S(t)$ has a random absorbing set in $L^2(D) \times L^2(D)$. By Lemma 4.5, Lemma 4.6 and Lemma 2.6, we get $S(t)$ is almost surely $\mathcal{D} - \alpha-$contracting in $L^2(D) \times L^2(D)$. We are now in a position to state our main result.

Theorem 4.1. *Assume that $f \in L^2$, $g \in H_0^1$ and* (3.11) *holds. Then the problem of* (3.7)-(3.10) *has a global random attractor in $L^2(D) \times L^2(D)$, which is a compact invariant set and attracts every tempred set in $L^2(D) \times L^2(D)$.*

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