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STABILITY OF A CUBIC FUNCTIONAL EQUATION IN FUZZY NORMED SPACE

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Abstract In this paper, the authors investigate the general solution of a new cubic functional equation

$$3f(x+3y) - f(3x+y) = 12[f(x+y) + f(x-y)] + 80f(y) - 48f(x)$$

and discuss its generalized Hyers - Ulam - Rassias stability in Banach spaces and stability in fuzzy normed spaces.

Keywords Generalized Hyers - Ulam - Rassias stability, cubic functional equation, fuzzy normed spaces.

MSC(2000) 39B55, 39B52, 39B82.

1. Introduction

In 1940, S.M.Ulam [28] raised the following question. Under what conditions does there exist an additive mapping near an approximately addition mapping?. The case of approximately additive functions was solved by D.H.Hyers [11] under the assumption that for $\epsilon > 0$ and $f : E_1 \to E_2$ be such that $\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$ for all $x, y \in E_1$ then there exist a unique additive mapping $T : E_1 \to E_2$ such that $\|f(x) - T(x)\| \leq \varepsilon$ for all $x \in E_1$.

In 1978, a generalized version of the theorem of Hyers for approximately linear mapping was given by Th.M.Rassias [22]. He proved that for a mapping $f : E_1 \to E_2$ be such that $f(tx)$ is continuous in $t \in \mathbb{R}$ and for each fixed $x \in E_1$ assume that there exist constant $\varepsilon > 0$ and $p \in [0, 1)$ with

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E_1$ then there exist a unique R-Linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\varepsilon}{2 - 2^p} \|x\|^p \tag{1.2}$$

for all $x \in E_1$.

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A number of mathematicians were attracted by the result of Th.M.Rassias. The stability concept that was introduced and investigated by Rassias is called the Hyers-Ulam-Rassias stability.

During the last decades, the stability problems of several functional equations have been extensively investigated by a number of authors [1, 5, 10, 13, 23, 24, 25].

In 1982-1989, J.M.Rassias [20, 21] replaced the sum appeared in right hand side of the equation (1.1) by the product of powers of norms. Infact, he proved the following theorem.

Theorem 1.1. *Let $f : E_1 \to E_2$ be a mapping from a normed vector space E_1 into Banach space E_2 subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon \|x\|^p \|y\|^p \tag{1.3}$$

for all $x, y \in E_1$, where ε and p are constants with $\varepsilon > 0$ and $0 \leq p < \frac{1}{2}$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{1.4}$$

exist for all $x \in E_1$ and $L : E_1 \to E_2$ is the unique additive mapping which satisfies

$$||f(x) - L(x)|| \leq \frac{\epsilon}{2 - 2^{2p}} ||x||^{2p} \tag{1.5}$$

for all $x \in E_1$. If $p > \frac{1}{2}$ the inequality (1.3) holds for $x, y \in E_1$ and the limit

$$A(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) \tag{1.6}$$

exist for all $x \in E_1$ and $A : E_1 \to E_2$ is the unique additive mapping which satisfies

$$||f(x) - A(x)|| \leq \frac{\epsilon}{2^{2p} - 2} ||x||^{2p} \tag{1.7}$$

for all $x \in E_1$.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.8}$$

is called a quadratic functional equation. Infact, every solution of the quadratic equation (1.8) is said to be a quadratic mapping.

A Hyers-Ulam stability problem for the quadratic functional equation (1.8) was discussed by Skof [27], Chowlewa [7], Czerwik [8] in different settings. Jun and Kim [12], Park and Jung [19] introduced the following functional equations

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x) \tag{1.9}$$

and

$$f(3x+y) + f(3x-y) = 3f(x+y) + 3f(x-y) + 48f(x) \tag{1.10}$$

and investigated its general solution and the generalized Hyers-Ulam-Rassias stability respectively. The functional equations (1.9) and (1.10) are called cubic functional equation because the function $f(x) = cx^3$ is a solution of the above functional equation (1.9) and (1.10).

In modeling applied problems only partial information may be known (or) there may be a degree of uncertainty in the parameters used in the model or some measurments may be imprecise. Due to such features, we are tempted to consider the study of functional equations in the fuzzy setting.

For the last 40 years, fuzzy theory has become very active area of research and a lot of development has been made in the theory of fuzzy sets to find the fuzzy analogues of the classical set theory. This branch finds a wide range of applications in the field of science and engineering.

A.K.Katsaras [14] introduced an idea of fuzzy norm on a linear space in 1984, in the same year Cpmgxin Wu and Jinxuan Fang [29] introduced a notion of fuzzy normed space to give a generalization of the Kolmogoroff normalized theorem for fuzzy topological linear spaces. In 1991, R.Biswas [4] defined and studied fuzzy inner product spaces in linear space. In 1992, C.Felbin [9] introduced an alternative definition of a fuzzy norm on a linear topological structures of a fuzzy normed linear spaces. In 1994, S.C.Cheng and J.N.Mordeson [6] introduced a definition of fuzzy norm on a linear space in such a manner that the corresponding induced fuzzy metric is of I.Kramosil and J.Michalek [15]. In 2003, T.Bag and S.K.Samanta [2] modified the definition of S.C.Cheng and J.N.Mordeson [6] by removing a regular condition. Recently many various result have been investigated by numerous authors one can refer to[3, 16, 17, 26].

Before we proceed to the main theorems, we will introduce a definition and an example to illustrate the idea of fuzzy norm.

Definition. Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0, 1]$ is said to be fuzzy norm on X if for all $x, y \in X$ and all $a, b \in \mathbb{R}$:

(N_1) $N(x, a) = 0$ for $a \leq 0$;

(N_2) $x = 0$ if and only if $N(x, a) = 1$ for all $a > 0$;

(N_3) $N(ax, b) = N\left(x, \frac{b}{|a|}\right)$ if $a \neq 0$;

(N_4) $N(x + y, a + b) \geq \min\{N(x, a), N(y, b)\}$;

(N_5) $N(x, .)$ is a non-decreasing function on $\mathbb{R}$ and $\lim\limits_{a\to\infty} N(x, a) = 1$;

(N_6) For $x \neq 0$, $N(x, .)$ is (upper semi) continuous on $\mathbb{R}$.

The pair (X, N) is called a fuzzy normed linear space. One may regard $N(x, a)$ as the truth value of the statement the norm of x is less than or equal to the real number a .

Example 1.1. Let $(X, \|\cdot\|)$ be a normed linear space. Then

$$N(x, a) = \begin{cases} \frac{a}{a+\|x\|} & , a > 0 \quad , x \in X \\ 0 & , a \leq 0 \quad , x \in X \end{cases}$$

is a fuzzy norm on X .

In the following we will suppose that $N(x, \cdot)$ is left continuous for every x. A fuzzy normed linear space is a pair (X, N), where X is a real linear space and N is a fuzzy norm on X. Let (X, N) be a fuzzy normed linear space. A sequence $\{x_n\}$ in X is said to be convergent if there exist $x \in X$ such that $\lim\limits_{n\to\infty} N(x_n - x, t) = 1 (t > 0)$. In that case, x is called the limit of the sequence $\{x_n\}$ and we write $N-\lim\limits_{n\to\infty} x_n = x$.

A sequence $\{x_n\}$ in fuzzy normed space (X, N) is called cauchy if for each $\epsilon > 0$ and $\delta > 0$, there exist $n_0 \in N$ such that

$$N(x_m - x_n, \delta) > 1 - \in (m, n \geq n_0).$$

If each cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.

In this paper, we introduce a new cubic functional equation of the form

$$3f(x+3y) - f(3x+y) = 12[f(x+y) + f(x-y)] + 80f(y) - 48f(x) \tag{1.11}$$

and discuss its general solution in Section 2. In Section 3, we investigate the generalized Hyers - Ulam - Rassias stability for the functional equation (1.11). We have also given the counter example to illustrate the non-stability of the functional equation (1.11) for some cases. In Section 4, we obtain the fuzzy stability for the functional equation (1.11).

Now we proceed to find the general solution of the functional equation (1.11).

2. The General Solution of the Functional Equation (1.11)

In this section, we obtain the general solution of the functional equation (1.11). Through out this section, let X and Y be real vector spaces.

Theorem 2.1. *[18] A function $f : X \to Y$ satisfies the functional equation (1.9) if and only if $f : X \to Y$ satisfies the functional equation*

$$2f(x+2y) + f(2x-y) = 5f(x+y) + 5f(x-y) + 15f(y) \tag{2.1}$$

for all $x, y \in X$.

Theorem 2.2. *A function $f : X \to Y$ satisfies the functional equation (1.11) if and only if $f : X \to Y$ satisfies the functional equation (2.1).*

Proof. Putting $x = y = 0$ in (1.11), we get $f(0) = 0$. Let $y = 0$ in (1.11), we obtain

$$f(3x) = 27f(x) \tag{2.2}$$

for all $x \in X$. Setting $x = 0$ in equation (1.11) and using (2.2), we get

$$f(-y) = -f(y) \tag{2.3}$$

for all $y \in X$. Replacing y by x (1.11), we obtain

$$f(4x) = 6f(2x) + 16f(x) \tag{2.4}$$

for all $x \in X$. Again replacing x by $3y$ in (1.11) and using equation (2.4), we get

$$f(10y) = -72f(2y) - 192f(y) - 12f(2y) - 80f(y) + 1296f(y) + 81f(2y) \tag{2.5}$$

for all $y \in X$. Substituting y by $3x$ in equation (1.11) and using (2.4) and (2.5), we obtain

$$f(2x) = 8f(x) \tag{2.6}$$

for all $x \in X$. Replacing y by $y - x$ in (1.11) and using oddness of f, we arrive

$$3f(-2x + 3y) - f(2x + y) = 12[f(y) + f(2x - y)] + 80f(y - x) - 48f(x) \tag{2.7}$$

for all $x, y \in X$. Interchanging x and y in (2.7), we get

$$3f(3x - 2y) - f(2y + x) = 12[f(x) + f(2y - x)] + 80f(x - y) - 48f(y) \tag{2.8}$$

for all $x, y \in X$. Replacing x by $x - y$ in (1.11), we obtain

$$3f(x + 2y) - f(3x - 2y) = 12[f(x) + f(x - 2y)] + 80f(y) - 48f(x - y) \tag{2.9}$$

for all $x, y \in X$. Multiplying the above equation by 3 and using (2.8), we get

$$f(x + 2y) - 3f(x - 2y) = 6f(x) + 24f(y) - 8f(x - y) \tag{2.10}$$

for all $x, y \in X$. Replacing y by $-y$ in (2.10), we obtain

$$f(x - 2y) - 3f(x + 2y) = 6f(x) - 24f(y) - 8f(x + y) \tag{2.11}$$

for all $x, y \in X$. Using (2.10) in (2.11), we arrive

$$f(x + 2y) = 3f(x + y) + f(x - y) - 3f(x) + 6f(y) \tag{2.12}$$

for all $x, y \in X$. Replacing (x, y) by $(-y, x)$ in (2.12), we get

$$f(2x - y) = 3f(x - y) - f(x + y) + 3f(y) + 6f(x) \tag{2.13}$$

for all $x, y \in X$. Multiplying equation (2.13) by 2 and adding with (2.12), we arrive (2.1) for all $x, y \in X$.

Conversely, assume f satisfies the functional equation (2.1). Letting (x, y) by $(0, 0)$ in (2.1), we get $f(0) = 0$. Set $y = 0$ in (2.1), we obtain $f(2x) = 8f(x)$ for all $x \in X$. Putting $x = 0$ in (2.1), we obtain $f(-y) = -f(y)$ for all $y \in X$. Thus f is an odd function. Replacing x by $x + y$ in (2.1), we get

$$2f(x + 3y) + f(2x + y) = 5f(x + 2y) + 5f(x) + 15f(y) \tag{2.14}$$

for all $x, y \in X$. Again replacing y by $y - x$ in (2.1), we get

$$-2f(x - 2y) + f(3x - y) = 5f(y) + 5f(2x - y) + 15f(y - x) \tag{2.15}$$

for all $x, y \in X$. Substituting (x, y) by $(y, -x)$ in (2.15), we arrive

$$-2f(2x + y) + f(x + 3y) = -5f(x) + 5f(x + 2y) - 15f(x + y) \tag{2.16}$$

for all $x, y \in X$. Adding (2.14) and (2.16), we get

$$3f(x + 3y) - f(2x + y) = 10f(x + 2y) - 15f(x + y) + 15f(y) \tag{2.17}$$

for all $x, y \in X$. Again adding (2.17) and (2.15) and using (2.1), we arrive

$$3f(x + 3y) + f(3x - y) = 15f(x + y) + 15f(x - y) + 80f(y) \tag{2.18}$$

for all $x, y \in X$. Replacing y by $-y$ in (2.15), we arrive

$$f(3x + y) = 2f(x + 2y) - 5f(y) + 5f(2x + y) - 15f(x + y) \tag{2.19}$$

for all $x, y \in X$. Adding (2.18) and (2.19) and using (2.17) and (2.1), we get

$$f(3x+y)+f(3x-y)=-5f(x+y)-5f(x-y)+4f(2x+y)+4f(2x-y) \quad (2.20)$$

for all $x, y \in X$. Setting (x, y) by $(x-y, y)$ in(2.1), we obtain

$$2f(x+y)+f(2x-3y)=5f(x)+5f(x-2y)+15f(y) \quad (2.21)$$

for all $x, y \in X$. Setting (x, y) by $(x, x-y)$ in (2.1), we obtain

$$2f(3x-2y)+f(x+y)=5f(2x-y)+5f(y)+15f(x-y) \quad (2.22)$$

for all $x, y \in X$. Replacing x by y in (2.22) and using oddness of f, we arrive

$$-2f(2x-3y)+f(x+y)=-5f(x-2y)+5f(x)-15f(x-y) \quad (2.23)$$

for all $x, y \in X$. Multiplying equation (2.21) by 2

$$4f(x+y)+2f(2x-3y)=10f(x)+10f(x-2y)+30f(y) \quad (2.24)$$

for all $x, y \in X$. Adding (2.23) and (2.24),we obtain

$$5f(x-2y)=5f(x+y)-15f(x)-30f(y)+15f(x-y) \quad (2.25)$$

for all $x, y \in X$. Replacing x by y and using oddness of f in (2.25) and then dividing by 5, we arrive

$$f(2x-y)=-f(x+y)+3f(y)+6f(x)+3f(x-y) \quad (2.26)$$

for all $x, y \in X$. Replacing y by $-y$ in equation (2.26), we obtain

$$f(2x+y)=-f(x-y)-3f(y)+6f(x)+3f(x+y) \quad (2.27)$$

for all $x, y \in X$. Using equation (2.26) and (2.27) in equation (2.20), we obtain

$$f(3x+y)+f(3x-y)=3f(x+y)+3f(x-y)+48f(x) \quad (2.28)$$

for all $x, y \in X$. Subtracting (2.28) from (2.18), we obtain (1.11). □

3. Generalized Hyers - Ulam - Rassias Stability of (1.11)

In this section, we consider X to be a real vector space and Y to be a Banach space, we present the Hyers - Ulam - Rassias stability of the functional equation (1.11) involving sum of powers of norms. Let us denote

$$\begin{aligned} D_f(x,y) = 3f(x+3y) - f(3x+y) - 12f(x+y) - 12f(x-y) \\ - 80f(y) + 48f(x) \end{aligned}$$

for all $x, y \in X$.

Theorem 3.1. *Let $\phi : X^2 \to [0, \infty)$ be a function such that*

$$\sum_{i=0}^{\infty} \frac{\phi(3^i x, 0)}{27^i} \quad \text{converges and} \quad \lim_{n\to\infty} \frac{\phi(3^n x, 3^n y)}{27^n} = 0 \tag{3.1}$$

for all $x, y \in X$. If a function $f : X \to Y$ satisfies

$$\|D_f(x, y)\| \leq \phi(x, y) \tag{3.2}$$

for all $x, y \in X$, then there exist a unique cubic function $C : X \to Y$ which satisfies (1.11) and

$$\|f(x) - C(x)\| \leq \frac{1}{27} \sum_{i=0}^{\infty} \frac{\phi(3^i x, 0)}{27^i}, \quad \forall\ x \in X. \tag{3.3}$$

The function C is given by

$$C(x) = \lim_{n\to\infty} \frac{f(3^n x)}{27^n}, \quad \forall\ x \in X. \tag{3.4}$$

Proof. Setting $y = 0$ in (3.2) and dividing by 27, we get

$$\left\| \frac{f(3x)}{27} - f(x) \right\| \leq \frac{1}{27} \phi(x, 0) \tag{3.5}$$

for all $x \in X$. Replacing x by $3x$ and dividing by 27 in equation (3.5), we obtain

$$\left\| \frac{f\left(3^2 x\right)}{27^2} - \frac{f(3x)}{27} \right\| \leq \frac{1}{27^2} \phi(3x, 0) \tag{3.6}$$

for all $x \in X$. Adding (3.5) and (3.6), we get

$$\left\| \frac{f\left(3^2 x\right)}{27^2} - f(x) \right\| \leq \sum_{i=0}^{1} \frac{\phi\left(3^i x, 0\right)}{27^{i+1}} \tag{3.7}$$

for all $x \in X$. Generalizing, we get

$$\begin{aligned} \left\| \frac{f(3^n x)}{27^n} - f(x) \right\| &\leq \frac{1}{27} \sum_{i=0}^{n-1} \frac{\phi\left(3^i x, 0\right)}{27^i} \\ &\leq \frac{1}{27} \sum_{i=0}^{\infty} \frac{\phi\left(3^i x, 0\right)}{27^i} \end{aligned} \tag{3.8}$$

for all $x \in X$. Now we have to prove that the sequence $\{f(3^n x)/27^n\}$ is a cauchy sequence for all $x \in X$. For every positive integer n, m and for all $x \in X$, consider

$$\begin{aligned} \left\| \frac{f(3^{n+m} x)}{27^{n+m}} - \frac{f(3^n x)}{27^n} \right\| &= \frac{1}{27^n} \left\| \frac{f(3^n \cdot 3^m x)}{27^m} - f(3^n x) \right\| \\ &\leq \frac{1}{27 \cdot 27^n} \sum_{i=0}^{m-1} \frac{\phi\left(3^i \cdot 3^n x, 0\right)}{27^i} \\ &\leq \frac{1}{27} \sum_{i=0}^{\infty} \frac{\phi\left(3^{i+n} x, 0\right)}{27^{i+n}} \end{aligned}$$

for all $x \in X$. By condition (3.1), the right-hand side approaches 0 as $n \to \infty$. Thus, the sequence is a cauchy sequence. Due to the completeness of the Banach space Y

$$C(x) = \lim_{n\to\infty} \frac{f(3^n x)}{27^n} \ \ \forall\, x \in X$$

is well-defined. We can see that (3.4) holds.

To show that C satisfies (1.11), we set $(x,y) = (3^n x, 3^n y)$ in (3.2) and divide the resulting equation by 27^n, we obtain

$$\begin{aligned}\frac{1}{27^n}\Big\| & 3f(3^n(x+3y)) - f(3^n(3x+y)) - 12f(3^n(x+y)) - 12f(3^n(x-y)) \\ & - 80f(3^n(y)) + 48f(3^n(x))\Big\| \ \le \frac{\phi(3^n x, 3^n y)}{27^n}.\end{aligned}$$

Taking the limit as $n \to \infty$, using (3.1) and (3.4) the above equation becomes

$$3C(x+3y) - C(3x+y) = 12C(x+y) + 12C(x-y) + 80C(y) - 48C(x)$$

for all $x, y \in X$. Therefore C satisfies (1.11).

To prove the uniqueness of C, suppose that there exist another cubic function $S : X \to Y$ such that S satisfies (1.11) and (3.3), we have

$$\begin{aligned}\|S(x) - C(x)\| &\le \frac{1}{27^n}\left(\|S(3^n x) - C(3^n x)\|\right) \\ &\le \frac{1}{27^n}\left(\|S(3^n x) - f(3^n x)\| + \|f(3^n x) - C(3^n x)\|\right) \\ &\le \frac{2}{27}\sum_{i=0}^{\infty} \frac{\phi\left(3^{i+n}x, 0\right)}{27^{i+n}} \quad \forall\ x \in X.\end{aligned}$$

By condition (3.1), the right-hand side approaches 0 as $n \to \infty$, and it follows that $C(x) = S(x)$ for all $x \in X$. Hence, C is unique. This completes the proof of the theorem. □

Theorem 3.2. *Let $\phi : X \times X \to [0, \infty)$ be a function such that*

$$\sum_{i=1}^{\infty} 27^i \phi\left(\frac{x}{3^i}, 0\right) \ \ \textit{converges and} \ \ \lim_{n\to\infty} 27^n \phi\left(\frac{x}{3^n}, \frac{y}{3^n}\right) \tag{3.9}$$

for all $x, y \in X$. If a function $C : X \to Y$ satisfies

$$\|D_f(x,y)\| \le \phi(x,y) \tag{3.10}$$

for all $x, y \in X$, then there exist a unique cubic function $C : X \to Y$ which satisfies (1.11) and

$$\|f(x) - C(x)\| \le \frac{1}{27}\sum_{i=1}^{\infty} 27^i \phi\left(\frac{x}{3^i}, 0\right) \quad \forall\ x \in X. \tag{3.11}$$

The function C is given by

$$C(x) = \lim_{n\to\infty} 27^n f\left(\frac{x}{3^n}\right) \quad \forall\, x \in X. \tag{3.12}$$

Proof. The proof begins in the same manner as that of Theorem 3.1. The only difference starts with the replacement of inequality (3.5) by replacing x by $\frac{x}{3}$. $\square$

The following Corollary is the immediate consequences of Theorem 3.1 and 3.2 which gives the Hyers-Ulam and generalized Hyers-Ulam stability of the functional equation (1.11).

Corollary 3.1. *Let Y be a Banach space and let $\varepsilon \geq 0$ be a real number. If a function $f : X \to Y$ satisfies the functional equation*

$$\|D_f(x, y)\| \leq \varepsilon$$

for all $x, y \in X$, then there exist a unique cubic function $C : X \to Y$ defined by

$$C(x) = \lim_{n\to\infty} \frac{f(3^n x)}{27^n}$$

which satisfies the equation (1.11) and the inequality

$$\|f(x) - C(x)\| \leq \frac{\varepsilon}{26} \ \ \forall\, x \in X.$$

Moreover, if for each fixed $x \in X$ the mapping $t \to f(tx)$ from $\mathbb{R}$ to X is continuous, then $C(rx) = r^3 C(x)$ for all $r \in \mathbb{R}$.

Corollary 3.2. *Let X and Y be a real normed space and a Banach space respectively. If a function $f : X \to Y$ satisfies the functional equation*

$$\|D_f(x, y)\| \leq \varepsilon\left(\|x\|^p + \|y\|^p\right) \tag{3.13}$$

with $0 \leq p < 3$ or $p > 3$ for some $\varepsilon \geq 0$ and for all $x, y \in X$, then there exist a unique cubic function $C : X \to Y$ such that

$$\|f(x) - C(x)\| \leq \frac{\varepsilon}{|27 - 3^p|}\|x\|^p \ \ \forall\ x \in X.$$

Now we will provide an example to illustrate that the functional equation (1.11) is not stable for $p = 3$ in Corollary 3.4.

Example 3.1. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$\phi(x) = \begin{cases} a\ x^3, & \text{if } |x| < 1 \\ a\ , & \text{otherwise} \end{cases}$$

where $a > 0$ is a constant and a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} \frac{\phi(3^n x)}{27^n} \qquad \textit{for all} \quad x \in \mathbb{R}.$$

Then f satisfies the functional inequality

$$\begin{aligned} &\|3f(x+3y) - f(3x+y) - 12f(x+y) - 12f(x-y) \\ &\quad - 80f(y) + 48f(x)\| \leq 4374\ a\left(|x|^3 + |y|^3\right) \end{aligned} \tag{3.14}$$

for all $x, y \in \mathbb{R}$. Then there do not exist a cubic mapping $C : \mathbb{R} \to \mathbb{R}$ and a constant $\beta > 0$ such that

$$|f(x) - C(x)| \leq \beta |x|^3 \qquad \textit{for all} \quad x \in \mathbb{R}. \tag{3.15}$$

Proof. Now

$$|f(x)| \leq \sum_{n=0}^{\infty} \frac{|\phi(3^n x)|}{|27^n|} = \sum_{n=0}^{\infty} \frac{a}{27^n} = \frac{27\ a}{26}.$$

Therefore we see that f is bounded. We are going to prove that f satisfies (3.14).

If $x = y = 0$ then (3.14) is trivial. If $|x|^3 + |y|^3 \geq \frac{1}{27}$ then the left hand side of (3.14) is less than 162 a. Now suppose that $0 < |x|^3 + |y|^3 < \frac{1}{27}$. Then there exist a positive integer k such that

$$\frac{1}{27^{k+1}} \leq |x|^3 + |y|^3 < \frac{1}{27^k}, \tag{3.16}$$

so that $27^{k-1} x^3 < \frac{1}{27}$, $27^{k-1} y^3 < \frac{1}{27}$ and consequently

$$\begin{gathered} 3^{k-1}(x), 3^{k-1}(y), 3^{k-1}(x+y), 3^{k-1}(x-y), 3^{k-1}(3x+y), \\ 3^{k-1}(x+3y) \in (-1, 1). \end{gathered}$$

Therefore for each $n = 0, 1, \ldots, k-1$, we have

$$\begin{gathered} 3^n(x), 3^n(y), 3^n(x+y), 3^n(x-y), 3^n(3x+y), \\ 3^n(x+3y) \in (-1, 1). \end{gathered}$$

and

$$\begin{aligned} 3\phi(3^n(x+3y)) - \phi(3^n(3x+y)) - 12\phi(3^n(x+y)) - 12\phi(3^n(x-y)) \\ - 80\phi(3^n(y)) + 48\phi(3^n(x)) = 0 \end{aligned}$$

for $n = 0, 1, \ldots, k-1$. From the definition of f and (3.16), we obtain that

$$\begin{aligned} &\Big| 3f(x+3y) - f(3x+y) - 12f(x+y) - 12f(x-y) \\ &\qquad - 80f(y) + 48f(x) \Big| \\ &\leq \sum_{n=0}^{\infty} \frac{1}{27^n} \Big| 3\phi(3^n(x+3y)) - \phi(3^n(3x+y)) - 12\phi(3^n(x+y)) - 12\phi(3^n(x-y)) \\ &\qquad - 80\phi(3^n(y)) + 48\phi(3^n(x)) \Big| \\ &\leq \sum_{n=k}^{\infty} \frac{1}{27^n} \Big| 3\phi(3^n(x+3y)) + \phi(3^n(3x+y)) + 12\phi(3^n(x+y)) + 12\phi(3^n(x-y)) \\ &\qquad + 80\phi(3^n(y)) + 48\phi(3^n(x)) \Big| \\ &\leq \sum_{n=k}^{\infty} \frac{1}{27^n} 156\ a = \frac{4212a}{26} \times \frac{1}{27^k} = 4374\ a \left(|x|^3 + |y|^3 \right). \end{aligned}$$

Thus f satisfies (3.14) for all $x, y \in \mathbb{R}$ with $0 < |x|^3 + |y|^3 < \frac{1}{27}$. $\square$

We claim that the cubic functional equation (1.11) is not stable for $p = 3$ in Corollary 3.4. Suppose on the contrary, there exist a cubic mapping $C : \mathbb{R} \to \mathbb{R}$ and a constant $\beta > 0$ satisfying (3.15). Since f is bounded and continuous for all $x \in \mathbb{R}$, C is bounded on any open interval containing the origin and continuous at the origin. In view of Corollary 3.4, $C(x)$ must have the form $C(x) = kx^3$ for any x in $\mathbb{R}$. Thus we obtain that

$$|f(x)| \leq (\beta + |k|)\,|x|^3. \tag{3.17}$$

But we can choose a positive integer m with $ma > \beta + |k|$.

If $x \in \left(0, \frac{1}{3^{m-1}}\right)$, then $3^n x \in (0,1)$ for all $n = 0, 1, \ldots, m-1$. For this x, we get

$$f(x) = \sum_{n=0}^{\infty} \frac{\phi(3^n x)}{27^n} \geq \sum_{n=0}^{m-1} \frac{a(3^n x)^3}{27^n} = max^3 > (\beta + |k|)\, x^3$$

which contradicts (3.17). Therefore the cubic functional equation (1.11) is not stable in sense of Ulam, Hyers and Rassias if $p = 3$, assumed in the inequality (3.13).

We obtain the following Corollary for Theorem 3.1 and 3.2, which gives J.M. Rassias stability of the functional equation (1.11)

Corollary 3.3. *Let X and Y be a real normed space and a Banach space respectively. If a function $f : X \to Y$ satisfies the functional equation*

$$\|D_f(x,y)\| \leq \varepsilon \left(\|x\|^p \|y\|^p + \left[\|x\|^{3p} + \|y\|^{3p} \right] \right)$$

with $0 < p < 1$ or $p > 1$ for some $\varepsilon > 0$ and for all $x, y \in X$, then there exist a unique cubic function $C : X \to Y$ such that

$$\|f(x) - C(x)\| \leq \frac{\varepsilon}{3\,|9 - 3^{3p-1}|} \|x\|^{3p} \quad \forall\, x \in X.$$

4. Fuzzy Stability of the Functional Equation (1.11)

Throughout this section, assume that X , (Z, N') and (Y, N) are linear space, fuzzy normed space and fuzzy Banach space respectively.

We now investigate the fuzzy stability of the functional equation (1.11).

Theorem 4.1. Let $\beta \in \{1, -1\}$ be fixed and let $\phi_1 : X \times X \to Z$ be a mapping such that for some $\alpha > 0$ with $\left(\frac{\alpha}{27}\right)^{\beta} < 1$

$$N'(\phi_1(3^{\beta} x, 0), a) \geq N'(\alpha^{\beta} \phi_1(x, 0), a) \tag{4.1}$$

for all $x \in X$ and all $a > 0$, and

$$\lim_{n \to \infty} N'(\phi_1(3^{\beta n} x, 3^{\beta n} y), 27^{\beta n} a) = 1$$

for all $x, y \in X$ and all $a > 0$. Suppose an odd mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$N(D_f(x,y), a) \geq N'(\phi_1(x,y), a) \tag{4.2}$$

for all $a > 0$ and all $x, y \in X$. Then the limit

$$C(x) = N - \lim_{n\to\infty} \frac{1}{27^{\beta n}} f(3^{\beta n}x)$$

exist for all $x \in X$ and the mapping $C : X \to Y$ is the unique cubic mapping satisfying

$$N(f(x) - C(x), a) \geq N'(\phi_1(x, 0), a\,|27 - \alpha|) \tag{4.3}$$

for all $x \in X$ and all $a > 0$.

Proof. Let $\beta = 1$. Letting $y = 0$ in (4.2), we get

$$N(f(3x) - 27f(x), a) \geq N'(\phi_1(x, 0), a) \tag{4.4}$$

for all $x \in X$ and all $a > 0$. Replacing x by $3^n x$ in (4.4), we obtain

$$N\left(\frac{f(3^{n+1}x)}{27} - f(3^n x), \frac{a}{27}\right) \geq N'(\phi_1(3^n x, 0), a) \tag{4.5}$$

for all $x \in X$ and all $a > 0$. Using (4.1), we get

$$N\left(\frac{f(3^{n+1}x)}{27} - f(3^n x), \frac{a}{27}\right) \geq N'\left(\phi_1(x, 0), \frac{a}{\alpha^n}\right) \tag{4.6}$$

for all $x \in X$ and all $a > 0$. Replacing a by $\alpha^n a$ in (4.6), we get

$$N\left(\frac{f(3^{n+1}x)}{27^{n+1}} - \frac{f(3^n x)}{27^n}, \frac{a\alpha^n}{27(27^n)}\right) \geq N'(\phi_1(x, 0), a) \tag{4.7}$$

for all $x \in X$ and all $a > 0$. It follows from

$$\frac{f(3^n x)}{27^n} - f(x) = \sum_{i=0}^{n-1} \frac{f(3^{i+1}x)}{27^{i+1}} - \frac{f(3^i x)}{27^i}$$

and (4.7)that

$$\begin{aligned} N&\left(\frac{f(3^n x)}{27^n} - f(x), \sum_{i=0}^{n-1} \frac{a\alpha^i}{27(27^i)}\right) \\ &\geq \min\left\{N\left(\frac{f(3^{i+1}x)}{27^{i+1}} - \frac{f(3^i x)}{27^i}, \frac{a\alpha^i}{27(27^i)}\right) : i = 0, 1, ..., n-1\right\} \\ &\geq N'(\phi_1(x, 0), a) \end{aligned} \tag{4.8}$$

for all $x \in X$ and all $a > 0$. Replacing x by $3^m x$ in (4.8),we get

$$\begin{aligned} N\left(\frac{f(3^{n+m}x)}{27^{n+m}} - \frac{f(3^m x)}{27^m}, \sum_{i=0}^{n-1} \frac{a\alpha^i}{27(27^i)(27^m)}\right) &\geq N'(\phi_1(3^m x, 0), a) \\ &\geq N'\left(\phi_1(x, 0), \frac{a}{\alpha^m}\right) \end{aligned}$$

and so

$$N\left(\frac{f(3^{n+m}x)}{27^{n+m}} - \frac{f(3^m x)}{27^m}, \sum_{i=m}^{n+m-1} \frac{a\alpha^i}{27(27^i)}\right) \geq N'(\phi_1(x, 0), a) \tag{4.9}$$

for all $x \in X$, $a > 0$ and all $m, n \geq 0$. Replacing a by $\frac{a}{\sum_{i=m}^{n+m-1} \frac{\alpha^i}{27(27^i)}}$, we obtain

$$N\left(\frac{f(3^{n+m}x)}{27^{n+m}} - \frac{f(3^m x)}{27^m}, a\right) \geq N'\left(\phi_1(x,0), \frac{a}{\sum_{i=m}^{n+m-1} \frac{\alpha^i}{27(27^i)}}\right) \tag{4.10}$$

for all $x \in X$, $a > 0$ and all $m, n \geq 0$. Since $0 < \alpha < 27$ and $\sum_{i=0}^{\infty} \left(\frac{\alpha}{27}\right)^i < \infty$, the cauchy criterion for convergence and (N_5) imply that $\left\{\frac{f(3^n x)}{27^n}\right\}$ is a cauchy sequence in (Y, N) . Since (Y, N) is a fuzzy Banach space, this sequence converges to some point $C(x) \in Y$. Define the mapping $C : X \to Y$ by

$$C(x) := N - \lim_{n\to\infty} \frac{f(3^n x)}{27^n}$$

for all $x \in X$. Since f is odd, C is odd. Letting $m = 0$ in (4.10), we get

$$N\left(\frac{f(3^n x)}{27^n} - f(x), a\right) \geq N'\left(\phi_1(x,0), \frac{a}{\sum_{i=0}^{n-1} \frac{\alpha^i}{27(27^i)}}\right) \tag{4.11}$$

for all $x \in X$ and all $a > 0$. Taking the limit as $n \to \infty$ and using (N_6), we get

$$N(f(x) - C(x), a) \geq N'(\phi_1(x,0), a(27-\alpha))$$

for all $x \in X$ and all $a > 0$. Now we claim that C is cubic. Replacing x, y by $3^n x, 3^n y$ in (4.2) respectively, we get

$$N\left(\frac{1}{27^n} D_f(3^n x, 3^n y), a\right) \geq N'(\phi_1(3^n x, 3^n y), 27^n a)$$

for all $x, y \in X$ and all $a > 0$. Since

$$\lim_{n\to\infty} N'(\phi_1(3^n x, 3^n y), 27^n a) = 1$$

C satisfies the functional equation (1.11). Hence $C : X \to Y$ is Cubic. To prove the uniqueness of C, let $C' : X \to Y$ be another cubic mapping satisfying (4.3). Fix $x \in X$, clearly $C(3^n x) = 27^n C(x)$ and $C'(3^n x) = 27^n C'(x)$ for all $x \in X$and all $n \in N$. It follows from (4.3) that

$$\begin{aligned}
N(C(x) - C'(x), a) &= N\left(\frac{C(3^n x)}{27^n} - \frac{C'(3^n x)}{27^n}, a\right) \\
&\geq \min\left\{N\left(\frac{C(3^n x)}{27^n} - \frac{f(3^n x)}{27^n}, \frac{a}{2}\right), N\left(\frac{f(3^n x)}{27^n} - \frac{C'(3^n x)}{27^n}, \frac{a}{2}\right)\right\} \\
&\geq N'\left(\phi_1(3^n x, 0), \frac{27^n a(27-\alpha)}{2}\right) \\
&\geq N'\left(\phi_1(x, 0), \frac{27^n a(27-\alpha)}{2\alpha^n}\right)
\end{aligned}$$

for all $x \in X$ and all $a > 0$. Since $\lim_{x\to\infty} \frac{27^n a(27-\alpha)}{2\alpha^n} = \infty$, we obtain

$$\lim_{n\to\infty} N'\left(\phi_1(x,0), \frac{27^n a(27-\alpha)}{2\alpha^n}\right) = 1.$$

Thus $N(C(x) - C'(x), a) = 1$ for all $x \in X$ and all $a > 0$, and so $C(x) = C'(x)$. For $\beta = -1$, we can prove the result by similar method.

$\square$

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