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BIFURCATION OF LIMIT CYCLES AND ISOCHRONOUS CENTER AT INFINITY FOR A CLASS OF DIFFERENTIAL SYSTEMS*

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Abstract In this paper, we study a seventh degree polynomial differential system with full linear terms and cubic terms. The conditions of infinity to be a center and to be a fine focus of the highest order are given and it is proved that this system has eight limit cycles in the neighborhood of infinity. Moreover, the conditions of infinity to be an isochronous center for a rational system associated the seventh degree polynomial differential system are obtained.

Keywords Infinity, Limit cycle, Isochronous center, Singular point value, Period constant.

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1. Introduction

In the last decades, being concerned with Hilbert's 16th problem and the monotonicity of period solutions, the bifurcation of limit cycles, center and the isochronous center problems for differential systems are researched actively. In the case of critical points on the finite plane, a lot of work has been done(see, for instance [3, 4, 5, 9, 16] and the references therein). For the case of infinity, several researches are concentrated on the following $2n+1$ degree system

$$\begin{aligned} \frac{dx}{dt} &= \sum_{k=0}^{2n} X_k(x,y) + (\delta x - y)(x^2+y^2)^n \\ \frac{dy}{dt} &= \sum_{k=0}^{2n} Y_k(x,y) + (x + \delta y)(x^2+y^2)^n \end{aligned} \tag{1.1}$$

where $X_k(x,y), Y_k(x,y)$ are homogeneous polynomials of degree k of x and y. For this system, the equator Γ_∞ on the Poincaré closed sphere is a trajectory of the system, having no real critical point. Γ_∞ is called the closed orbit at infinity or

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the equator in this point of view. As far as bifurcation of limit cycles and center conditions at infinity are concerned, several special systems have been studied for instance: cubic systems in [2, 10, 12]; fifth degree systems in [8, 17].

In the first part of this paper, we investigate problems of centers and bifurcation of limit cycles at infinity for the following seventh degree system with full linear terms and cubic terms

$$\begin{aligned}\frac{dx}{dt} &= A_{10}x + A_{01}y + A_{30}x^3 + A_{21}x^2y + A_{12}xy^2 + A_{03}y^3 + (\delta x - y)(x^2+y^2)^3,\\ \frac{dy}{dt} &= B_{10}x + B_{01}y + B_{30}x^3 + B_{21}x^2y + B_{12}xy^2 + B_{03}y^3 + (x + \delta y)(x^2+y^2)^3,\end{aligned} \tag{1.2}$$

where δ, A_{10}, A_{01}, A_{30}, A_{21}, A_{12}, A_{03}, B_{30}, B_{10}, B_{01}, B_{21}, B_{12}, B_{03} are real constants. We prove that there are eight limit cycles bifurcated from infinity.

In [13], we gave the conditions of infinity to be an isochronous center for a special case of $n = 2$ of the following real rational systems

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{(x^2+y^2)^n}[\sum_{k=0}^{2n} X_k(x,y) - y(x^2+y^2)^n],\\ \frac{dy}{dt} &= \frac{1}{(x^2+y^2)^n}[\sum_{k=0}^{2n} Y_k(x,y) + x(x^2+y^2)^n]\end{aligned} \tag{1.3}$$

and proved firstly that the real rational system has an isochronous at infinity. We also discussed the isochronous center problem for another case of $n = 2$ of (1.3) in the complex field in [6] and that $n = 1$ of (1.3) in the real field in [7]. In the second part of this paper, we consider the following rational system, which is a special case ($n = 3$) of system (1.3) if $\delta = 0$ and, has the same center conditions and bifurcation of limit cycles at infinity to system (1.2):

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{(x^2+y^2)^3}[A_{10}x + A_{01}y + A_{30}x^3 + A_{21}x^2y + A_{12}xy^2\\ &\quad + A_{03}y^3 + (\delta x - y)(x^2+y^2)^3],\\ \frac{dy}{dt} &= \frac{1}{(x^2+y^2)^3}[B_{10}x + B_{01}y + B_{30}x^3 + B_{21}x^2y + B_{12}xy^2\\ &\quad + B_{03}y^3 + (x + \delta y)(x^2+y^2)^3].\end{aligned} \tag{1.4}$$

By computation of period constants, the necessary conditions of infinity of system (1.4) to be an isochronous center are obtained. By using several methods and skills, we prove the conditions are also sufficient.

2. Preliminaries

Consider a complex polynomial differential system of the form

$$\begin{aligned}\frac{dz}{dT} &= z + \sum_{k=2}^{\infty} Z_k(z,w) = Z(z,w),\\ \frac{dw}{dT} &= -w - \sum_{k=2}^{\infty} W_k(z,w) = -W(z,w),\end{aligned} \tag{2.1}$$

where z, w, T are complex variables and

$$Z_k(z,w) = \sum_{\alpha+\beta=k} a_{\alpha\beta} z^\alpha w^\beta,\ W_k(z,w) = \sum_{\alpha+\beta=k} b_{\alpha\beta} w^\alpha z^\beta.$$

By means of transformation

$$z = x + yi, w = x - yi, T = it, i = \sqrt{-1} \tag{2.2}$$

system (2.1) can be transformed into the following system

$$\begin{aligned} \frac{dx}{dt} &= -y + \sum_{k=2}^{\infty} X_k(x,y) \\ \frac{dy}{dt} &= x + \sum_{k=2}^{\infty} Y_k(x,y). \end{aligned} \tag{2.3}$$

We say that system (2.1) is the associated system of system (2.3) and vise versa. It is obvious that system (2.3) is real if and only if the coefficients of system (2.1) satisfy conjugate conditions, i.e.,

$$\overline{a_{\alpha\beta}} = b_{\alpha\beta},\ \alpha \geq 0,\ \beta \geq 0,\ \alpha + \beta \geq 2. \tag{2.4}$$

By means of transformation

$$z = re^{i\theta},\ w = re^{-i\theta},\ T = it \tag{2.5}$$

system (2.1) is then transformed into

$$\begin{aligned} \frac{dr}{dt} &= \frac{ir}{2} \sum_{m=1}^{\infty} \sum_{\alpha+\beta=m+2} (a_{\alpha,\beta-1} - b_{\beta,\alpha-1}) e^{i(\alpha-\beta)\theta} r^m, \\ \frac{d\theta}{dt} &= 1 + \frac{1}{2} \sum_{m=1}^{\infty} \sum_{\alpha+\beta=m+2} (a_{\alpha,\beta-1} + b_{\beta,\alpha-1}) e^{i(\alpha-\beta)\theta} r^m. \end{aligned} \tag{2.6}$$

For the complex constant $h, |h| \ll 1$, we write the solution of (2.6) satisfying the initial-value condition $r|_{\theta=0} = h$ as

$$r = \tilde{r}(\theta, h) = h + \sum_{k=2}^{\infty} v_k(\theta) h^k \tag{2.7}$$

and denote

$$\begin{aligned} \tau(\varphi, h) &= \int_0^{\varphi} \frac{dt}{d\theta} d\theta \\ &= \int_0^{\varphi} [1 + \frac{1}{2} \sum_{m=1}^{\infty} \sum_{\alpha+\beta=m+2} (a_{\alpha,\beta-1} + b_{\beta,\alpha-1}) e^{i(\alpha-\beta)\theta} \tilde{r}(\theta, h)^m]^{-1} d\theta. \end{aligned} \tag{2.8}$$

Definition 2.1 (see [14])**.** For any complex constant $h, |h| \ll 1$, the origin of system (2.1) or (2.3) is called a complex center if $\tilde{r}(2\pi, h) \equiv h$. The origin is called a complex isochronous center if $\tilde{r}(2\pi, h) \equiv h$, $\tau(2\pi, h) \equiv 2\pi$.

Obviously, the complex center (complex isochronous center) and the center (isochronous center) is equivalent when the system is real.

Lemma 2.1 (see [1])**.** *For system (2.1), we can derive uniquely the following formal series:*

$$\xi = z + \sum_{k+j=2}^{\infty} c_{kj} z^k w^j, \eta = w + \sum_{k+j=2}^{\infty} d_{k,j} w^k z^j, \tag{2.9}$$

where $c_{k+1,k} = d_{k+1,k} = 0,\ k = 1, 2, \cdots$, *such that*

$$\frac{d\xi}{dT} = \xi + \sum_{j=1}^{\infty} p_j\, \xi^{j+1} \eta^j, \frac{d\eta}{dT} = -\eta - \sum_{j=1}^{\infty} q_j\, \eta^{j+1} \xi^j. \tag{2.10}$$

Definition 2.2 (see [11, 14]). Let $\mu_0 = 0, \mu_k = p_k - q_k$, $\tau_k = p_k + q_k$ $k = 1, 2, \cdots$. μ_k and τ_k are called k-th singular point value and k-th period constant of the origin of system(2.1) and (2.3) respectively.

The methods to calculate μ_k is give in [11, 15] and that of τ_k is given in [14].

Lemma 2.2 (see [11, 14]). *For system (2.1) or (2.3), the origin is a complex center if and only if $\mu_k = 0, k = 1, 2, 3, \cdots$, and it is a complex isochronous center if and only if $\mu_k = \tau_k = 0, k = 1, 2, 3, \cdots$.*

Consider the real perturbed system of (2.3) of the form

$$\begin{aligned} \tfrac{dx}{d\tau} &= -\delta x - y + \sum_{k=2}^{\infty} X_k(x, y), \\ \tfrac{dy}{d\tau} &= x - \delta y + \sum_{k=2}^{\infty} Y_k(x, y). \end{aligned} \tag{2.11}$$

Under the polar coordinates $x = r\cos\theta, y = r\sin\theta$, system (2.11) is reduced to

$$\frac{dr}{d\theta} = r\frac{-\delta + \sum\limits_{k=2}^{\infty} r^{k-1}\varphi_{k+1}(\theta)}{1 + \sum\limits_{k=2}^{\infty} r^{k-1}\psi_{k+1}(\theta)}, \tag{2.12}$$

where

$$\begin{aligned} \varphi_{k+1}(\theta) &= \cos\theta X_k(\cos\theta, \sin\theta) + \sin\theta Y_k(\cos\theta, \sin\theta), \\ \psi_{k+1}(\theta) &= \cos\theta Y_k(\cos\theta, \sin\theta) - \sin\theta X_k(\cos\theta, \sin\theta), \end{aligned}$$

$k = 1, 2, \ldots$. For sufficiently small h, let

$$d(h) = r(2\pi, h) - h, \tag{2.13}$$

be the Poincaré succession function, where $r = r(\theta, h) = \sum\limits_{m=1}^{\infty} v_m(\theta, \delta)h^m$ is the solution of (2.12) associated with the initial-value condition $r(0, h) = h$. It is evident that

$$v_1(\theta, \delta) = e^{-\delta\theta}, \quad v_m(0, \delta) = 0, \quad m = 2, 3, \cdots \tag{2.14}$$

As we know, if $v_1(2\pi, \delta) \neq 1$ in the expression (2.13) then the origin is called a *strong focus* of system (2.11); if $v_1(2\pi, \delta) = 1$, and $v_2(2\pi, \delta) = v_3(2\pi, \delta) = \cdots = v_{2k}(2\pi, \delta) = 0, v_{2k+1}(2\pi, \delta) \neq 0$, then the origin is called a *fine focus* of order k and the quantity of $v_{2k+1}(2\pi, \delta)$ is called the k-th *focal value* at the origin $(k = 1, 2, \cdots)$; if $v_1(2\pi, \delta) = 1$ and $v_{2k+1}(2\pi, \delta) = 0$ for any positive integer k, then the origin is called a *center*.

Lemma 2.3. ([8]) *For system* $(2.11)|_{\delta=0}$ *and (2.1), the first non-vanishing focal value and the first non-vanishing singular point value of the origin are related by*

$$v_{2m+1}(2\pi, 0) = i\pi\mu_m. \tag{2.15}$$

3. Center conditions and bifurcation of limit cycles

By transformations

$$x = \frac{\xi}{(\xi^2+\eta^2)^4},\ y = \frac{\eta}{(\xi^2+\eta^2)^4}, \tag{3.1}$$

$$t = (x^2+y^2)^{-3}\tau. \tag{3.2}$$

system (1.2) becomes a system of the form

$$\begin{aligned}
\frac{d\xi}{d\tau} = \ & -\tfrac{\delta}{7}\xi - \eta + (\xi^2+\eta^2)^{12}[(\tfrac{-\xi^2}{7}+\eta^2)(A_{30}\xi^3 + A_{21}\xi^2\eta + A_{12}\xi\eta^2 + A_{03}\eta^3) \\
& -\tfrac{8}{7}\xi\eta(B_{30}\xi^3 + B_{21}\xi^2\eta + B_{12}\xi\eta^2 + B_{03}\eta^3)] \\
& +[(\tfrac{-\xi^2}{7}+\eta^2)(A_{10}\xi + A_{01}\eta) - \tfrac{8}{7}\xi\eta(B_{10}\xi + B_{01}\eta)](\xi^2+\eta^2)^{20} \\
\frac{d\eta}{d\tau} = \ & \xi - \tfrac{\delta}{7}\eta + (\xi^2+\eta^2)^{12}[(\xi^2 - \tfrac{\eta^2}{7})(B_{30}\xi^3 + B_{21}\xi^2\eta + B_{12}\xi\eta^2 + B_{03}\eta^3) \\
& -\tfrac{8}{7}\xi\eta(A_{30}\xi^3 + A_{21}\xi^2\eta + A_{12}\xi\eta^2 + A_{03}\eta^3)] \\
& +[(\tfrac{-\eta^2}{7}+\xi^2)(B_{10}\xi + B_{01}\eta) - \tfrac{8}{7}\xi\eta(A_{10}\xi + A_{01}\eta)](\xi^2+\eta^2)^{20}.
\end{aligned} \tag{3.3}$$

Since infinity of system (1.2) correspond to the origin of system (3.3), only need to discuss the properties at the origin of system (3.3) can we obtain the relative properties at infinity of system (1.2). Moreover, it is easy to see that the origin of system (3.3) is a strong focus if $\delta \neq 0$.

In order to make use of method of singular point values ([11]) to consider the conditions of the origin to be a center, we perform the transformation $z = \xi + i\,\eta$, $w = \xi - i\,\eta$ to reduce system $(3.3)|_{\delta=0}$ into its associated complex system

$$\begin{aligned}
\frac{dz}{dT} = \ & z + \tfrac{1}{7}w^{20}z^{21}(3w(a_{01}w + a_{10}z) + 4z(b_{10}w + b_{01}z)) \\
& +\tfrac{1}{7}w^{12}z^{13}(3w(a_{03}w^3 + a_{12}w^2z + a_{21}wz^2 + a_{30}z^3) \\
& +4z(b_{30}w^3 + b_{21}w^2z + b_{12}wz^2 + b_{03}z^3)) \\
\frac{dw}{dT} = \ & -[w + \tfrac{1}{7}w^{21}z^{20}(4w(a_{01}w + a_{10}z) + 3z(b_{10}w + b_{01}z)) \\
& +\tfrac{1}{7}w^{13}z^{12}(4w(a_{03}w^3 + a_{12}w^2z + a_{21}wz^2 + a_{30}z^3) \\
& +3z(b_{30}w^3 + b_{21}w^2z + b_{12}wz^2 + b_{03}z^3))]
\end{aligned} \tag{3.4}$$

where the relations of coefficients of the two systems are as follows:

$$\begin{aligned}
& a_{10} = \tfrac{1}{2}(-A_{01} - iA_{10} - iB_{01} + B_{10}), \quad b_{10} = \overline{a_{10}}, \\
& a_{01} = \tfrac{1}{2}(A_{01} - iA_{10} + iB_{01} + B_{10}), \quad b_{01} = \overline{a_{01}}, \\
& a_{30} = \tfrac{1}{8}(A_{03} + iA_{12} - A_{21} - iA_{30} + iB_{03} - B_{12} - iB_{21} + B_{30}), \quad b_{30} = \overline{a_{30}}, \\
& a_{03} = \tfrac{1}{8}(-A_{03} + iA_{12} + A_{21} - iA_{30} - iB_{03} - B_{12} + iB_{21} + B_{30}), \quad b_{03} = \overline{a_{03}}, \\
& a_{21} = \tfrac{1}{8}(-3A_{03} - iA_{12} - A_{21} - 3iA_{30} - 3iB_{03} + B_{12} - iB_{21} + 3B_{30}), \quad b_{21} = \overline{a_{21}}, \\
& a_{12} = \tfrac{1}{8}(3A_{03} - iA_{12} + A_{21} - 3iA_{30} + 3iB_{03} + B_{12} + iB_{21} + 3B_{30}), \quad b_{12} = \overline{a_{12}}.
\end{aligned} \tag{3.5}$$

According to [15], we obtain the following recursive formula to compute the singular point values of the origin of system (3.4).

Lemma 3.1. *For system (3.4), the singular point values μ_m ($m = 1, 2\cdots$) are determined by following recursive formula:*

$c[0,0] = 1$
when$(u = v > 0)$ *or* $u < 0$, *or* $v < 0$,
$c[u,v] = 0$

else
$c[u,v] = \frac{1}{-u+v}[(b_{01}(1+4u-3v)c[-22+u,-20+v] + (a_{10}(-1+3u-4v)+b_{10}(1+4u-3v))c[-21+u,-21+v] + a_{01}(-1+3u-4v)c[-20+u,-22+v] + b_{03}(1+4u-3v)c[-16+u,-12+v] + (a_{30}(-1+3u-4v)+b_{12}(1+4u-3v))c[-15+u,-13+v] + (a_{21}(-1+3u-4v)+b_{21}(1+4u-3v))c[-14+u,-14+v] + (a_{12}(-1+3u-4v)+b_{30}(1+4u-3v))c[-13+u,-15+v] + a_{03}(-1+3u-4v)c[-12+u,-16+v])]$.

$\mu_m = \frac{1}{7}[(b_{01}c[-22+m,-20+m] - a_{10}c[-21+m,-21+m] + b_{10}c[-21+m,-21+m] - a_{01}c[-20+m,-22+m] + b_{03}c[-16+m,-12+m] - a_{30}c[-15+m,-13+m] + b_{12}c[-15+m,-13+m] - a_{21}c[-14+m,-14+m] + b_{21}c[-14+m,-14+m] - a_{12}c[-13+m,-15+m] + b_{30}c[-13+m,-15+m] - a_{03}c[-12+m,-16+m])]$.

Using the recursive formula of Lemma 3.1 and computing with the computer algebra system-Mathematica, we calculate the first 98 singular point values at the origin of system (3.4) and simplify them, then we get the theorem below.

Theorem 3.1. *The first 98 singular point values of the origin of system (3.4) are given as follows:*

$$
\begin{aligned}
&\mu_7 = 0\\
&\mu_{14} = \tfrac{1}{7}(-a_{21}+b_{21})\\
&\mu_{21} = \tfrac{1}{7}(-a_{10}+b_{10})\\
&\mu_{28} = \tfrac{1}{7}(a_{12}a_{30}-b_{12}b_{30})\\
&case1: a_{30}b_{30} \neq 0,\\
&\mu_{35} = -\tfrac{1}{14}(a_{01}a_{30}-b_{01}b_{30})(-3+p)\\
&\mu_{42} = \tfrac{1}{14}(a_{03}a_{30}^2-b_{03}b_{30}^2)(-3+p)\\
&\mu_{49} = \mu_{56} = \cdots = \mu_{98} = 0\\
&case2: a_{30} = b_{30} = 0\\
&\mu_{35} = \tfrac{1}{14}(a_{12}b_{01}-a_{01}b_{12})\\
&\mu_{42} = 0\\
&\mu_{49} = \tfrac{1}{56}(a_{01}a_{12}b_{03}-a_{03}b_{01}b_{12})\\
&\mu_{56} = -\tfrac{1}{56}(a_{12}^2b_{03}-a_{03}b_{12}^2)(a_{21}+b_{21})\\
&\mu_{63} = -\tfrac{3}{112}(a_{12}^2b_{03}-a_{03}b_{12}^2)(a_{10}+b_{10})\\
&\mu_{70} = -\tfrac{1}{336}(a_{03}b_{03}-4a_{12}b_{12})(-a_{12}^2b_{03}+a_{03}b_{12}^2)\\
&\mu_{77} = \mu_{84} = \mu_{91} = 0\\
&\mu_{98} = \tfrac{3}{80}a_{12}^2b_{12}^2(a_{12}^2b_{03}-a_{03}b_{12}^2)
\end{aligned}
$$

others $\mu_k\ (k<98)$ *are zero, where* p *is a constant satisfying* $a_{12} = p\,b_{30}, b_{12} = p\,a_{30}$ *when* $a_{30}b_{30} \neq 0$. *In the above expression of* μ_k *, we have already let* $\mu_1 = \mu_2 = \cdots = \mu_{k-1} = 0, k = 2,3,\cdots,98.$

Since the expression of μ_k of Theorem 3.1 is brief, it is easy to get the following result.

Theorem 3.2. *The first 98 singular point values at the origin of system (3.4) are zero if and only if one of the following four conditions holds:*

$$(I)\ a_{21} = b_{21},\ a_{10} = b_{10},\ a_{12} = 3b_{30},\ b_{12} = 3a_{30},\ a_{30}b_{30} \neq 0; \tag{3.6}$$

$$(II)\ \ a_{21} = b_{21},\ a_{10} = b_{10},\ a_{12}a_{30} = b_{12}b_{30},\ a_{03}a_{30}^2 = b_{03}b_{30}^2,\ a_{30}\,b_{30} \neq 0,\ a_{12} \neq 3b_{30},\ b_{12} \neq 3a_{30}; \tag{3.7}$$

$$(III)\ \ a_{21} = b_{21},\ a_{10} = b_{10},\ a_{30} = b_{30} = 0,\ a_{12}b_{01} = b_{12}a_{01},\ a_{03}a_{30}^2 = b_{03}b_{30}^2\ a_{12}b_{12} \neq 0; \tag{3.8}$$

$$(IV)\ \ a_{21} = b_{21},\ a_{10} = b_{10},\ a_{30} = b_{30} = a_{12} = b_{12} = 0; \tag{3.9}$$

By using the method of [11] we have

Lemma 3.2. *All the elementary Lie-invariants of system (3.4) are given as follows:*

$$\begin{array}{l} a_{21}, b_{21}, a_{10}, b_{10} \\ a_{30}b_{30}, a_{12}b_{12}, a_{03}b_{03}, \\ a_{30}a_{12}, b_{30}b_{12}, a_{01}a_{30}, b_{01}b_{30}, a_{01}b_{12}, b_{01}a_{12}, \\ a_{30}^2a_{03}, b_{30}^2b_{03}, a_{30}b_{12}a_{03}, b_{30}a_{12}b_{03}, b_{12}^2a_{03}, a_{12}^2b_{03}. \end{array}$$

In Theorem 3.2, if condition (I) holds, system (3.4) has a first integral with the form

$$F(z,w) = 4w^{28}z^{28}/(2a_{01}z^{20}w^{22} + 4b_{10}z^{21}w^{21} + 2b_{01}z^{22}w^{20} + a_{03}z^{12}w^{16} + 4b_{30}z^{13}w^{15} + 2b_{21}z^{14}w^{14} + 4a_{30}z^{15}w^{13} + b_{03}z^{16}w^{12} + 1);$$

If condition (II) or (III) holds, from Lemma 3.2 and [11, Theorem 2.6], then we have all the singular point values are zero, therefore the origin of system (3.4) is a center;

If condition (IV) holds, then system (3.4) has a first integral with the form

$$\begin{aligned} F(z,w) \;=\; & 4w^{28}z^{28}/(1 + a_{03}w^{16}z^{12} + 2b_{21}w^{14}z^{14} + b_{03}w^{12}z^{16} + 2a_{01}w^{22}z^{20} \\ & + 4b_{10}w^{21}z^{21} + 2b_{01}w^{20}z^{22}). \end{aligned}$$

From Theorem 3.2 and the above discussion, then we obtain the following.

Theorem 3.3. *For system (3.4), all the singular point values at the origin are zero if and only if the first 98 singular point values at the origin are zero, i.e., one of the four conditions of Theorem 3.2 holds. Correspondingly, for system (3.4) or system (3.3)$|_{\delta=0}$, the origin to be a complex center if and only if one of the four conditions given by Theorem 3.2 holds.*

It is easy to obtain the following corollary.

Corollary 3.1. *The origin of system (3.3) is a center, correspondingly, the infinity of system (1.2)is a center, i.e., the trajectories in the sufficiently small neighborhood of Γ_∞ are all closed orbits if and only if $\delta = 0$ and one of the four conditions given by Theorem 3.2 holds.*

From Theorem 3.1, Theorem 3.3 and expression (2.15), we have the following.

Theorem 3.4. *The highest order of weak focus at the origin of system* $(3.3)|_{\delta=0}$ *is 98. The origin of system* $(3.3)|_{\delta=0}$ *is a weak focus of order 98, i.e.,* $v_1(2\pi,0) = 1, v_3(2\pi,0) = v_5(2\pi,0)\cdots = v_{89}(2\pi,0) = 0, v_{197}(2\pi,0) \neq 0$ *if and only if*

$$\begin{array}{l} a_{21} = b_{21} = a_{10} = b_{10} = a_{01} = b_{01} = a_{30} = b_{30} = 0, \\ a_{03}b_{03} = 4a_{12}b_{12},\ \ a_{12}b_{12} \neq 0. \end{array} \tag{3.10}$$

Proof. $v_{197}(2\pi,0) = i\pi\mu_{98} \neq 0$ implies $a_{12}b_{12} \neq 0$ and $a_{12}^2b_{03} - a_{03}b_{12}^2 \neq 0$. If $a_{12}^2b_{03} - a_{03}b_{12}^2 \neq 0$, then from Theorem 3.1, the origin of system $(3.3)|_{\delta=0}$ is a weak focus of order at most 49. By $a_{12}b_{12} \neq 0$, and consider $\mu_{35} = 0$ of the case 2 of Theorem 3.1, we can let $a_{01} = sa_{12}$, $b_{01} = sb_{12}$. Substituting $a_{01} = sa_{12}$, $b_{01} = sb_{12}$ into μ_{49} of the case 2 of Theorem 3.1, it is not difficult to obtain the conclusion of the theorem. □

From Theorem 3.4, then through structuring and computing carefully, the theorem about bifurcation of limit cycles at infinity is obtained as follow.

Theorem 3.5. *If δ and the coefficients of system (3.4) satisfy (accordingly, the coefficients of system (3.3)are determined):*

$$\begin{array}{l}\delta=-\epsilon_9,\\ a_{12}=1+\sqrt{\epsilon_1}i,\ b_{12}=1-\sqrt{\epsilon_1}i,\\ a_{30}=b_{30}=0,\\ a_{03}=(-2+\epsilon_2)(-i+\sqrt{\epsilon_1}),\ b_{03}=(-2+\epsilon_2)(i+\sqrt{\epsilon_1}),\\ a_{21}=\epsilon_4-\epsilon_8 i,\ b_{21}=\epsilon_4+\epsilon_8 i,\\ a_{10}=-\epsilon_3+\epsilon_7 i,\ b_{10}=-\epsilon_3-\epsilon_7 i,\\ a_{01}=-\epsilon_5-(\epsilon_6+\epsilon_5\sqrt{\epsilon_1})i,\ b_{01}=-\epsilon_5+(\epsilon_6+\epsilon_5\sqrt{\epsilon_1})i,\end{array}$$

where ϵ_i $i=1,2,\cdots,9$ are small parameters satisfying

$$0<\epsilon_9\ll\epsilon_8\ll\cdots\ll\epsilon_2\ll\epsilon_1\ll 1, \tag{3.11}$$

then system (3.3) has 8 limit cycles in the neighborhood of the origin. Correspondingly, in the neighborhood of infinity, system (1.2) has 8 limit cycles.

Proof. According to (2.14), the first focal value at the origin of perturbed system (3.3) is as follow

$$v_1(2\pi,\delta)-1=e^{-2\pi\delta/7}-1=\frac{2\pi}{7}\epsilon_9+o(\epsilon_9).$$

For the sake of convenience, other focal values of perturbed system are denoted by $v_{2k+1}(2\pi)$. Based on (2.15) and Theorem 3.1, we obtain the below focal values at the origin of system (3.3) after computing carefully:

$$\begin{array}{l}v_1(2\pi)-1=\frac{2\pi}{7}\epsilon_9+o(\epsilon_9),\\ v_{29}(2\pi)=-\frac{2\pi}{7}\epsilon_8,\\ v_{43}(2\pi)=\frac{2\pi}{7}\epsilon_7,\\ v_{71}(2\pi)=-\frac{\pi}{7}\epsilon_6,\\ v_{99}(2\pi)=\frac{\pi}{14}\epsilon_5+o(\epsilon_5),\\ v_{113}(2\pi)=-\frac{\pi}{7}\epsilon_4+o(\epsilon_4),\\ v_{127}(2\pi)=\frac{3\pi}{14}\epsilon_3+o(\epsilon_4),\\ v_{141}(2\pi)=-\frac{\pi}{21}\epsilon_2+o(\epsilon_2),\\ v_{197}(2\pi)=\frac{3\pi}{20}+o(1),\end{array} \tag{3.12}$$

From the above expressions we know that signs of any two successive nonzero focal values are alternate positive and negative. The absolute value of the former focal value is far less than the absolute value of the later. According to classical Multiple Hopf Bifurcation Theory, the conclusion of the theorem follows. □

4. Isochronous center at infinity of the system

We now discuss the conditions of infinity to be an isochronous center of system (1.4).

As reported before, system (1.4) has the same center conditions as that of system (1.2). Only under the condition $\delta=0$, infinity of system (1.2) may be a center, so we let conveniently $\delta=0$ in the following discussion. Namely, the following system is investigated:

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{(x^2+y^2)^3}[A_{10}x + A_{01}y + A_{30}x^3 + A_{21}x^2y + A_{12}xy^2 + A_{03}y^3 - y(x^2+y^2)^3],\\ \frac{dy}{dt} &= \frac{1}{(x^2+y^2)^3}[B_{10}x + B_{01}y + B_{30}x^3 + B_{21}x^2y + B_{12}xy^2 + B_{03}y^3 + x(x^2+y^2)^3].\end{aligned} \tag{4.1}$$

It is obvious that by transformation (3.1), system (4.1) is changed into system $(3.3)|_{\delta=0}$. Since transformation (3.1) is a homeomorphism without any time rescaling, we see that the conditions for center (isochronous center) at infinity of system (4.1) are the same as that at the origin of system $(3.3)|_{\delta=0}$. So we only need to investigate conditions of isochronous centers at the origin of system $(3.3)|_{\delta=0}$ or conditions of complex isochronous center at the origin of system (3.4).

From [14, Theorem 3.1] we have that

Lemma 4.1. *For system (3.4), the period constant τ_j can be determined by the following recursive formulas:*
$c(1,0) = d(1,0) = 1; c(0,1) = d(0,1) = 0;$
if $k < 0$ *or* $j < 0$ *or* $(j > 0$ *and* $k = j+1)$ *then* $c(k,j) = 0,\ d(k,j) = 0;$
else
$c(k,j) = -\frac{1}{7(j-k+1)}(28b_{01}c(k-22,j-20) + 3b_{01}jc(k-22,j-20) - 4b_{01}kc(k-22,j-20) - 21a_{10}c(k-21,j-21) + 21b_{10}c(k-21,j-21) + 4a_{10}jc(k-21,j-21) + 3b_{10}jc(k-21,j-21) - 3a_{10}kc(k-21,j-21) - 4b_{10}kc(k-21,j-21) - 28a_{01}c(k-20,j-22) + 4a_{01}jc(k-20,j-22) - 3a_{01}kc(k-20,j-22) + 28b_{03}c(k-16,j-12) + 3b_{03}jc(k-16,j-12) - 4b_{03}kc(k-16,j-12) - 7a_{30}c(k-15,j-13) + 21b_{12}c(k-15,j-13) + 4a_{30}jc(k-15,j-13) + 3b_{12}jc(k-15,j-13) - 3a_{30}kc(k-15,j-13) - 4b_{12}kc(k-15,j-13) - 14a_{21}c(k-14,j-14) + 14b_{21}c(k-14,j-14) + 4a_{21}jc(k-14,j-14) + 3b_{21}jc(k-14,j-14) - 3a_{21}kc(k-14,j-14) - 4b_{21}kc(k-14,j-14) - 21a_{12}c(k-13,j-15) + 7b_{30}c(k-13,j-15) + 4a_{12}jc(k-13,j-15) + 3b_{30}jc(k-13,j-15) - 3a_{12}kc(k-13,j-15) - 4b_{30}kc(k-13,j-15) - 28a_{03}c(k-12,j-16) + 4a_{03}jc(k-12,j-16) - 3a_{03}kc(k-12,j-16));$

$d(k,j) = -\frac{1}{7(j-k+1)}(28a_{01}d(k-22,j-20) + 3a_{01}jd(k-22,j-20) - 4a_{01}kd(k-22,j-20) + 21a_{10}d(k-21,j-21) - 21b_{10}d(k-21,j-21) + 3a_{10}jd(k-21,j-21) + 4b_{10}jd(k-21,j-21) - 4a_{10}kd(k-21,j-21) - 3b_{10}kd(k-21,j-21) - 28b_{01}d(k-20,j-22) + 4b_{01}jd(k-20,j-22) - 3b_{01}kd(k-20,j-22) + 28a_{03}d(k-16,j-12) + 3a_{03}jd(k-16,j-12) - 4a_{03}kd(k-16,j-12) + 21a_{12}d(k-15,j-13) - 7b_{30}d(k-15,j-13) + 3a_{12}jd(k-15,j-13) + 4b_{30}jd(k-15,j-13) - 4a_{12}kd(k-15,j-13) - 3b_{30}kd(k-15,j-13) + 14a_{21}d(k-14,j-14) - 14b_{21}d(k-14,j-14) + 3a_{21}jd(k-14,j-14) + 4b_{21}jd(k-14,j-14) - 4a_{21}kd(k-14,j-14) - 3b_{21}kd(k-14,j-14) + 7a_{30}d(k-13,j-15) - 21b_{12}d(k-13,j-15) + 3a_{30}jd(k-13,j-15) + 4b_{12}jd(k-13,j-15) - 4a_{30}kd(k-13,j-15) - 3b_{12}kd(k-13,j-15) - 28b_{03}d(k-12,j-16) + 4b_{03}jd(k-12,j-16) - 3b_{03}kd(k-12,j-16));$

$p(j) = \frac{1}{7}(-24b_{01}c(j-21,j-20) + b_{01}jc(j-21,j-20) + 24a_{10}c(j-20,j-21) - 17b_{10}c(j-20,j-21) - a_{10}jc(j-20,j-21) + b_{10}jc(j-20,j-21) + 31a_{01}c(j-19,j-22) - a_{01}jc(j-19,j-22) - 24b_{03}c(j-15,j-12) + b_{03}jc(j-15,j-12) + 10a_{30}c(j-14,j-13) - 17b_{12}c(j-14,j-13) - a_{30}jc(j-14,j-13) + b_{12}jc(j-14,j-13) + 17a_{21}c(j-13,j-14) - 10b_{21}c(j-13,j-14) - a_{21}jc(j-13,j-14) + b_{21}jc(j-13,j-14) + 24a_{12}c(j-12,j-15) - 3b_{30}c(j-12,j-15) - a_{12}jc(j-12,j-15) + b_{30}jc(j-12,j-15) + 31a_{03}c(j-11,j-16) - a_{03}jc(j-11,j-16));$

$q(j) = \frac{1}{7}(-24a_{01}d(j-21,j-20) + a_{01}jd(j-21,j-20) - 17a_{10}d(j-20,j-21) + 24b_{10}d(j-20,j-21) + a_{10}jd(j-20,j-21) - b_{10}jd(j-20,j-21) + 31b_{01}d(j-19,j-22) - b_{01}jd(j-19,j-22) - 24a_{03}d(j-15,j-12) + a_{03}jd(j-15,j-12) -$

$17a_{12}d(j-14,j-13)+10b_{30}d(j-14,j-13)+a_{12}jd(j-14,j-13)-b_{30}jd(j-14,j-13)-10a_{21}d(j-13,j-14)+17b_{21}d(j-13,j-14)+a_{21}jd(j-13,j-14)-b_{21}jd(j-13,j-14)-3a_{30}d(j-12,j-15)+24b_{12}d(j-12,j-15)+a_{30}jd(j-12,j-15)-b_{12}jd(j-12,j-15)+31b_{03}d(j-11,j-16)-b_{03}jd(j-11,j-16))$;

$\tau_j = p(j)+q(j)$.

Based on the four center conditions of the origin of system (3.4), we investigate the complex isochronous center conditions of system (3.4) by the following four cases.

1. The center condition (3.6)

Since $a_{30}\,b_{30}\neq 0$, then from expression (3.6), we put expression $a_{12}=3\,b_{30}$, $b_{12}=3\,a_{30}$, $a_{21}=b_{21}=r_{21}, a_{10}=b_{10}=r_{10}$ into recursive formulas given by Lemma 4.1 and computing by the two cases $a_{01}=b_{01}=0$ and $a_{01}b_{01}\neq 0$ respectively, therefore we get the theorem below.

Theorem 4.1. *For system (3.4), the first 98 period constants of the origin are as follows:*

$$\begin{array}{l} \textit{Case1}\ a_{01}=b_{01}=0,\\ \tau_7=0,\ \ \tau_{14}=2r_{21}, \tau_{21}=2r_{10},\ \ \tau_{28}=\tau_{35}=0,\\ \tau_{42}=-6(a_{03}a_{30}^2+b_{03}b_{30}^2), \tau_{49}=\tau_{56}=\cdots=\tau_{98}=0,\\ \textit{Case2}\ \ a_{01}b_{01}\neq 0,\\ \tau_7=0,\ \ \tau_{14}=2r_{21}, \tau_{21}=2r_{10},\ \ \tau_{28}=0, \tau_{35}=3(a_{01}a_{30}+b_{01}b_{30}),\\ \tau_{42}=\frac{15}{4}(a_{03}a_{30}^2-b_{03}b_{30}^2)s,\ \ \tau_{56}=\tau_{63}=0,\\ \tau_{70}=-\frac{1}{48}a_{30}b_{30}s^2(192a_{30}b_{30}-s^4),\ \ \tau_{77}=0,\\ \tau_{84}=-\frac{800}{49}a_{30}^2b_{30}^2s^2(6a_{30}b_{30}g+s^2),\ \ \tau_{91}=0,\\ \tau_{98}=-\frac{205a_{30}b_{30}s^{10}}{82944},\\ \tau_k=0,\ k\neq 7i, i<14, i\in\mathbf{N}. \end{array} \tag{4.2}$$

where s and g are constants satisfying $a_{01}=sb_{30}$, $b_{01}=-sa_{30}$ and $a_{03}=gb_{30}^2$, $b_{03}=ga_{30}^2$ respectively. In the above expression of τ_k, we have already assumed that $\tau_1=\cdots=\tau_{k-1}=0, k=2,3,\cdots 98$.

For case 2 of the theorem, $a_{01}b_{01}\neq 0$ implies $s\neq 0$. It reduces that $\tau_{98}=-\frac{205a_{30}b_{30}s^{10}}{82944}\neq 0$. From expression (4.2) it is obtained the following result.

Theorem 4.2. *Under condition (3.6), the first 98 period constants of the origin of system (3.4) are zero if and only if*

$$a_{21}=b_{21}=a_{10}=b_{10}=a_{01}=b_{01}=0,\ a_{03}a_{30}^2+b_{03}b_{30}^2=0,\ a_{30}b_{30}\neq 0. \tag{4.3}$$

If expression (4.3) holds, system (3.4) becomes

$$\begin{array}{l} \frac{dz}{dT}=z+\frac{4}{7}b_{03}w^{12}z^{17}+\frac{15}{7}a_{30}w^{13}z^{16}+\frac{13}{7}b_{30}w^{15}z^{14}+\frac{3}{7}a_{03}w^{16}z^{13}\\ \frac{dw}{dT}=-(w+\frac{4}{7}a_{03}z^{12}w^{17}+\frac{15}{7}b_{30}z^{13}w^{16}+\frac{13}{7}a_{30}z^{15}w^{14}+\frac{3}{7}b_{03}z^{16}w^{13}) \end{array} \tag{4.4}$$

Considering $a_{03}a_{30}^2+b_{03}b_{30}^2=0$, $a_{30}b_{30}\neq 0$, we assume that $a_{03}=qb_{30}^2$, $b_{03}=-qa_{30}^2$, where q is a pure imaginary. Hence, system (4.4) becomes

$$\begin{array}{l} \frac{dz}{dT}=z-\frac{4}{7}{a_{30}}^2qw^{12}z^{17}+\frac{15}{7}a_{30}w^{13}z^{16}+\frac{13}{7}b_{30}w^{15}z^{14}+\frac{3}{7}{b_{30}}^2qw^{16}z^{13},\\ \frac{dw}{dT}=-(w+\frac{4}{7}{b_{30}}^2qz^{12}w^{17}+\frac{15}{7}b_{30}z^{13}w^{16}+\frac{13}{7}a_{30}z^{15}w^{14}-\frac{3}{7}{a_{30}}^2qz^{16}w^{13}). \end{array} \tag{4.5}$$

For the above system we assume that $a_{30} = b_{30} = 1$, otherwise we can choose a proper complex constant $\alpha + i\beta$ and perform transformations $z = (\alpha + i\beta)z_1, w = (\alpha - i\beta)w_1$ to bring system (4.5) to the case of $a_{30} = b_{30} = 1$. Hence, the system turns into

$$\begin{aligned} \frac{dz}{dT} &= z - \frac{4}{7}qw^{12}z^{17} + \frac{15w^{13}z^{16}}{7} + \frac{13w^{15}z^{14}}{7} + \frac{3}{7}qw^{16}z^{13}, \\ \frac{dw}{dT} &= -(w + \frac{4}{7}qz^{12}w^{17} + \frac{15z^{13}w^{16}}{7} + \frac{13z^{15}w^{14}}{7} - \frac{3}{7}qz^{16}w^{13}) \end{aligned} \tag{4.6}$$

Under the transformation $z = re^{i\theta}$, $w = re^{-i\theta}$, $t = -iT$, system (4.6) takes the form

$$\begin{aligned} \frac{dr}{dt} &= -\frac{1}{14}ie^{-4i\theta}\left(e^{8i\theta}q + q + 2e^{2i\theta} - 2e^{6i\theta}\right)r^{29} \\ \frac{d\theta}{dt} &= 1 + g(\theta)r^{28}, \end{aligned} \tag{4.7}$$

where $g(\theta) = \frac{1}{2}\left(e^{-2i\theta} + e^{2i\theta}\right)\left(e^{-2i\theta}q - e^{2i\theta}q + 4\right) = 2\cos(2\theta)(s\sin(2\theta) + 2)$, $s = \frac{q}{i} = -iq$. The first integral of system (4.7) is of the form

$$\frac{r^{56}}{2g(\theta)r^{28} + 1} = C \tag{4.8}$$

where C is a constant.

By (4.8), then for system (4.7), the solution satisfying initial condition $r|_{\theta=0} = h$ is

$$\begin{aligned} r^{28} &= \frac{g(\theta)h^{56} + \sqrt{g(\theta)^2h^{112} + 8h^{84} + h^{56}}}{8h^{28}+1} \\ &= \frac{2h^{56}\cos(2\theta)(s\sin(2\theta)+2)}{\sqrt{4(s\sin(2\theta)+2)^2(1-\sin^2(2\theta))h^{112} + 8h^{84} + h^{56}}} \end{aligned} \tag{4.9}$$

Substituting (4.9) into (4.7), we have that

$$\begin{aligned} \frac{dt}{d\theta} &= 1 - \frac{h^{56}g(\theta)}{\sqrt{g(\theta)^2h^{112} + 8h^{84} + h^{56}}} \\ &= 1 - \frac{2h^{56}\cos(2\theta)(s\sin(2\theta)+2)}{\sqrt{4(s\sin(2\theta)+2)^2(1-\sin^2(2\theta))h^{112} + 8h^{84} + h^{56}}} \end{aligned} \tag{4.10}$$

Since

$$\begin{aligned} &\int \frac{2h^{56}\cos(2\theta)(s\sin(2\theta)+2)}{\sqrt{4(s\sin(2\theta)+2)^2(1-\sin^2(2\theta))h^{112} + 8h^{84} + h^{56}}}d\theta \\ = &\int \frac{h^{56}(s\sin(2\theta)+2)}{\sqrt{4(s\sin(2\theta)+2)^2(1-\sin^2(2\theta))h^{112} + 8h^{84} + h^{56}}}d\sin(2\theta), \end{aligned}$$

so $\int_0^{2\pi} \frac{dt}{d\theta}dt = 2\pi$. So, the origin of system (4.4) is a complex isochronous center.

Theorem 4.3. *Under the center condition (3.6), the origin of system (3.4) to be a complex isochronous center (correspondingly, infinity of system (4.1) is an isochronous center) if and only if condition (4.3) holds.*

2. The center condition (3.7)

Since $a_{30}b_{30} \neq 0$, from expression (3.7) we can let $a_{12} = hb_{30}$, $b_{12} = ha_{30}$, $a_{03} = kb_{30}^2$, $b_{03} = ka_{30}^2$, $a_{01} = sb_{30}$, $b_{01} = sa_{30}, a_{21} = b_{21} = r_{21}$, $a_{10} = b_{10} = r_{10}$. Putting the above expression into the recursive formulas given by Lemma 4.1, after computation carefully we find that the first 42 period constants are as follows:

$$\begin{aligned} &\tau_{14} = 2r_{21}, \\ &\tau_{21} = 2r_{10}, \\ &\tau_{28} = a_{30}b_{30}(-3 + h)(1 + h), \\ &\tau_{35} = -6a_{30}b_{30}s, \\ &\tau_{42} = 8a_{30}^2b_{30}^2k, \\ &\text{others } \tau_m = 0,\ m < 42, m \in \mathbf{N}. \end{aligned} \tag{4.11}$$

Considering that $a_{12} \neq 3b_{30}$, $b_{12} \neq 3a_{30}$, obviously, under condition (3.7), the first 42 period constants at the origin of system (3.4) are zero if and only if

$$a_{21} = b_{21} = a_{10} = b_{10} = a_{01} = b_{01} = a_{03} = b_{03} = 0,\ a_{12} = -b_{30},\ b_{12} = -a_{30}. \tag{4.12}$$

Theorem 4.4. *Under condition (3.7), the origin of system (3.4) is a comlex isochronous center (correspondingly, infinity of system* (4.1) *is an isochronous center) if and only if (4.12) holds.*

Proof. Necessity has already been explained, now we are proving the sufficiency.

If (4.12) holds, system (3.4) becomes

$$\begin{aligned} \frac{dz}{dT} &= z - \frac{1}{7}a_{30}w^{13}z^{16} + \frac{1}{7}b_{30}w^{15}z^{14} \\ \frac{dw}{dT} &= -(w - \frac{1}{7}b_{30}z^{13}w^{16} + \frac{1}{7}a_{30}z^{15}w^{14}), \end{aligned} \tag{4.13}$$

Let $z = re^{i\theta}, w = re^{-i\theta}$, then we have

$$\theta = \frac{1}{2i}(logz - logw). \tag{4.14}$$

Differentiating both sides of (4.14) with respect to T along the trajectories of system (4.13), we obtain

$$\frac{d\theta}{dT} = \frac{1}{2i}(\frac{1}{z}\frac{dz}{dT} + \frac{1}{w}\frac{dw}{dT}) = -i. \tag{4.15}$$

namely,

$$\frac{d\theta}{dt} = i\frac{d\theta}{dT} = 1. \tag{4.16}$$

Therefore, the origin of system (4.13) is a complex isochronous center. □

3. The center condition (3.8)

Since $a_{12}b_{12} \neq 0$, from expression (3.8) we can let $a_{30} = b_{30} = 0$, $a_{01} = sb_{12}$, $b_{01} = sa_{12}$, $a_{03} = gb_{12}^2$, $b_{03} = ga_{12}^2$, $a_{21} = b_{21} = r_{21}$, $a_{10} = b_{10} = r_{10}$. Putting the above expression into the recursive formulas given by Lemma 4.1, after computing we have $\tau_{28} = a_{12}b_{12} \neq 0$. Therefore, under the center condition (3.8), the origin of system (3.4) (corresponding to infinity of system (4.1)) is not an isochronous center.

4. The center condition (3.9)

Substituting condition $a_{30} = b_{30} = a_{12} = b_{12} = 0$, $a_{21} = b_{21} = r_{21}$, $a_{10} = b_{10} = r_{10}$ into the recursive formulas given by Lemma 4.1 and computing, then we obtain the first 42 period constants of system (3.4) as following:

$$\begin{aligned} \tau_{14} &= 2r_{21}, \\ \tau_{21} &= 2r_{10}, \\ \tau_{42} &= 2a_{01}b_{01}, \end{aligned} \tag{4.17}$$

and others are zero.

From $\tau_{14} = \tau_{21} = \tau_{42} = 0$ we get that $a_{21} = b_{21} = a_{10} = b_{10} = a_{01} = b_{01} = 0$.

Theorem 4.5. *Under condition (3.9), the origin of system (3.4) is a complex isochronous center (correspondingly, infinity of system* (4.1) *is an isochronous center) if and only if*

$$a_{21} = b_{21} = a_{30} = b_{30} = a_{03} = a_{12} = a_{10} = b_{10} = a_{01} = b_{01} = 0 \tag{4.18}$$

Proof. We only need to prove the sufficiency. Under the condition of the theorem, system (3.4) becomes

$$\begin{array}{l}\frac{dz}{dT} = z + \frac{4}{7}b_{03}w^{12}z^{17} + \frac{3}{7}a_{03}w^{16}z^{13}\\ \frac{dw}{dT} = -(w + \frac{4}{7}a_{03}z^{12}w^{17} + \frac{3}{7}b_{03}z^{16}w^{13}).\end{array} \tag{4.19}$$

Similar to the discussion of system (4.5), we can put that $a_{03} = b_{03} = 1$, then the system turns into

$$\begin{array}{l}\frac{dz}{dT} = z + \frac{4}{7}w^{12}z^{17} + \frac{3}{7}w^{16}z^{13}\\ \frac{dw}{dT} = -(w + \frac{4}{7}z^{12}w^{17} + \frac{3}{7}z^{16}w^{13}).\end{array} \tag{4.20}$$

By transformation $z = re^{i\theta}$, $w = re^{-i\theta}$, $t = -i\,T$, system (4.20) becomes:

$$\begin{array}{l}\frac{dr}{dt} = \frac{1}{14}e^{-4i\theta}\left(e^{8i\theta} - 1\right)r^{29}\\ \frac{d\theta}{dt} = 1 + cos(4\theta)r^{28}.\end{array} \tag{4.21}$$

The first integral of system (4.21) is

$$\frac{4r^{56}}{2\cos(4\theta)r^{28} + 1} = C, \tag{4.22}$$

and C is a constant.

By (4.22), the solution of system (4.21) satisfying initial condition $r|_{\theta=0} = h$ is

$$r^{28} = \frac{\cos(4\theta)h^{56} + \sqrt{\cos^2(4\theta)h^{112} + 2h^{84} + h^{56}}}{2h^{28} + 1}. \tag{4.23}$$

Substituting (4.23) into (4.21) and we get

$$\frac{dt}{d\theta} = 1 - \frac{h^{28}\cos(4\theta)}{\sqrt{\cos^2(4\theta)h^{56} + 2h^{28} + 1}}. \tag{4.24}$$

From $\int_0^{2\pi} \frac{h^{28}\cos(4\theta)}{\sqrt{\cos^2(4\theta)h^{56}+2h^{28}+1}}d\theta = 0$ we have that $\int_0^{2\pi} \frac{dt}{d\theta}dt = 2\pi$. Thus the origin of system (4.19) is a complex isochronous center and infinity of system (4.1) is an isochronous center accordingly. □

Summing up the above discussion, we get the following main theorem in this section.

Theorem 4.6. *Infinity of system (1.4) is an isochronous center if and only if $\delta = 0$ and one of conditions (4.3), (4.12),(4.18) holds.*

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