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NONLINEAR BOUNDARY CONDITIONS DERIVED BY SINGULAR PERTUBATION IN AGE STRUCTURED POPULATION DYNAMICS MODEL

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Abstract In this article, we derive Ricker's [22, 23] type nonlinear boundary condition for an age structured population dynamic model by using a singular perturbation. The question addressed in this paper is the convergence of the singularly perturbed system. We first obtain a finite time convergence for a fixed initial distribution. Then we focus on the convergence uniformly of the singularly perturbed system with respect to the initial distribution in bounded sets.

Keywords Population dynamics, age structured models, density dependent models, singular perturbations.

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1. Introduction

Age structured models have been extensively used in the context of population dynamics mainly to take care of the history of individuals. We refer to the books [1, 9, 13, 20, 30, 32] for a nice overview and more results on this topics. In this article, we consider a singular perturbation problem for a class of nonlinear age structured model. As far as we know, the only work going in that direction is due to Arino et al. [2]. In [2] they investigate the singular limit of patch and age structured model, and by assuming that the spatial motion is fast, and they derive a limit model without space. Here we will focus on another modelling issue. Our goal is to derive (to understand) Ricker's [22, 23] type nonlinear boundary condition as a singular limit. The main question adressed in this article is to understand in which sense the limit exists.

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The class of age structured model considered in this article is the following

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,a) + \dfrac{\partial u}{\partial a}(t,a) = -\mu(a)u(t,a) \\ \qquad\qquad -\underbrace{m\left(\int_0^{+\infty} \gamma_1(a)u(t,a)da\right) h(a)}_{\text{intraspecific competition}}]u(t,a),\ a \geq 0, \\ u(t,0) = \underbrace{\exp\left(-\int_0^{+\infty} \gamma_2(a)u(t,a)da\right)}_{\text{limitation of births}} \int_0^{+\infty} \beta(a)u(t,a)da \\ u(0,.) = \varphi \in L^p_+\left((0,+\infty);\mathbb{R}\right), \text{ with } 1 \leq p < +\infty. \end{cases} \tag{1.1}$$

In order to define properly this semiflow generated by (1.1) we make the following assumption.

Assumption 1.1. *We assume that* $m : [0,+\infty) \to [0,+\infty)$ *is locally Lipschitz continuous, and*

$$\mu, h \in L^\infty_+\left((0,+\infty);\mathbb{R}\right), \tag{1.2}$$

$$\beta, \gamma_1, \gamma_2 \in L^q_+\left((0,+\infty);\mathbb{R}\right), \tag{1.3}$$

where $1 < q \leq +\infty$ *with* $\dfrac{1}{p} + \dfrac{1}{q} = 1.$

In this context of ecology this class of model corresponds to density dependent age structured population dynamics models. The model (1.1) has been considered previously by Liu and Cohen [14] for

$$m(x) = x, \ \ \forall x \geq 0.$$

We also refer to [3, 4, 5, 6, 12, 11, 15, 21, 24, 25, 26, 31] for more information and results on this topic in the context of the ecology.

Here the distribution $a \to u(t,a)$ is the density of population at time $t \geq 0$. This means that

$$\int_{a_1}^{a_2} u(t,a)da$$

is the number of individuals at time t with an age in between a_1 and a_2 (with $0 \leq a_1 < a_2 \leq +\infty$). Therefore the total number of individuals in the population is given by

$$\int_0^{+\infty} u(t,a)da.$$

The term $-\mu(a)u(t,a)$ describes the mortality of individuals, and $\mu(a)$ is the age-specific mortality rate of individuals. The term $\int_0^{+\infty} \beta(a)u(t,a)da$ is the flux of new born individuals while $\beta(a)$ is the age-specific fertility rate. The term $-m\left(\int_0^{+\infty} \gamma_1(a)u(t,a)da\right) h(a)u(t,a)$ describes an intra-specific competition between individuals. Namely this term is introduced to describe the limitations for resources (food, space, etc...). Here we focus on the last term arising in the boundary condition of (1.1) that reads as

$$\exp\left(-\int_0^{+\infty} \gamma_2(a)u(t,a)da\right),$$

and that describes a birth limitation. In the context of fishieries, Ricker [22, 23] introduced this term to describe a canibalism phenomenon of larvea by adults. In the context forests, this term can be regarded as a term of competition for light between small trees with large trees.

In order to understand this term, we consider the following system

$$\left\{\begin{array}{l}\dfrac{\partial u_\varepsilon}{\partial t}(t,a)+\dfrac{\partial u_\varepsilon}{\partial a}(t,a)=-\left[\mu(a)+\underbrace{m\left(\int_0^{+\infty}\gamma_1(a)u_\varepsilon(t,a)da\right)h(a)}_{\text{intraspecific competition}}\right]u_\varepsilon(t,a)\\ \qquad\qquad -\left[\underbrace{\dfrac{1}{\varepsilon}\mathbf{1}_{[0,\varepsilon]}(a)\int_0^{+\infty}\gamma_2(a)u_\varepsilon(t,a)da}_{\text{fast intraspecific competition}}\right]u_\varepsilon(t,a),\ a\geq 0,\\ u_\varepsilon(t,0)=\int_0^{+\infty}\beta(a)u_\varepsilon(t,a)da\\ u_\varepsilon(0,.)=\varphi\in L^p_+((0,+\infty);\mathbb{R}).\end{array}\right. \tag{1.4}$$

where

$$\mathbf{1}_{[0,\varepsilon]}(a):=\begin{cases}1, \text{ if } a\in[0,\varepsilon],\\ 0, \text{ otherwise.}\end{cases}$$

One may observe that system (1.4) has a linear boundary condition. From a modelling point of view the term $\int_0^{+\infty}\gamma_2(a)u(t,a)da$ appears in system (1.4) as a fast predation or competition process between young individuals and older individuals.

Since the population is usually assumed to have a finite number of individuals at each time. Hence, one usually imposes

$$\int_0^{+\infty}u(t,a)da<+\infty, \forall t\geq 0,$$

so that the natural state space for age structured model is L^1 (i.e. $p=1$). Here in order to derive some convergence results when $\varepsilon\to 0$ we will rather consider the general case $p\in[1,+\infty)$. We refer to Magal and Ruan [16, 17, 18] and Thieme [29] for results on this topic. A direct consequence of the results in [17, (see section 3)] combined together with the application part in [16, (section 6)] we obtain the following result.

Theorem 1.2. *Let Assumption 1.1 be satisfied. Let $p\in[1,\infty)$ and $\varepsilon>0$. There exists a unique continuous semiflow $\{\widehat{U}_\varepsilon(t)\}_{t\geq 0}$ on $L^p_+(0,\infty)$ (respectively $\{\widehat{U}(t)\}_{t\geq 0}$ on $L^p_+(0,\infty)$) such that for each $\varphi\in L^p_+(0,\infty)$ the map $t\to\widehat{U}_\varepsilon(t)\varphi=u^\varphi_\varepsilon(t)$ (respectively the map $t\to U(t)\varphi$) is an integrated (or mild) solution of (1.4) (respectively of (1.1)).*

In order to derive a convergence result of the solution of system (1.4) to the solution of system (1.1) when $\varepsilon\to 0$, we need to impose some extra conditions on the map γ_2.

Assumption 1.3. *We assume that*

$$\gamma_2\in W^{1,q}((0,+\infty),\mathbb{R}), \tag{1.5}$$

and

$$\widehat{\kappa} := \sup_{\varepsilon \in (0,1]} \frac{1}{\varepsilon} \|\gamma_2\|_{L^q([0,\varepsilon])} < +\infty. \tag{1.6}$$

wherein $q \in (1, \infty]$ *is defined by* $\frac{1}{p} + \frac{1}{q} = 1$.

The condition (1.6) reads as

$$\limsup_{\varepsilon \to 0} \frac{1}{\varepsilon^q} \int_0^{\varepsilon} \gamma_2(a)^q da < +\infty, \text{ if } q \in (1, +\infty),$$

and

$$\limsup_{\varepsilon \to 0} \frac{1}{\varepsilon} \sup_{a \in [0,\varepsilon]} \gamma_2(a) < +\infty, \text{ if } q = +\infty.$$

Therefore the condition (1.6) combined with (1.5) implies

$$\gamma_2(0) = 0, \tag{1.7}$$

and the condition (1.6) is satisfied if (for example) there exists a constant $\delta > 0$, such that

$$\gamma_2(a) \leq \delta a,$$

for all $a > 0$ small enough.

Under the above conditions we obtain the following convergence result.

Theorem 1.4. *Let Assumptions 1.1 and 1.3 be satisfied. Let* $\tau > 0$. *Then for each* $\varphi \in L^p_+ ((0, +\infty); \mathbb{R})$

$$\sup_{t \in [0,\tau]} \left\| \widehat{U}_\varepsilon(t)\varphi - \widehat{U}(t)\varphi \right\|_{L^p} \to 0 \text{ as } \varepsilon(> 0) \to 0.$$

Remark 1.1. Beyond this convergence result, this class of examples provides a class of non-densely defined Cauchy problem that can be approximated by a singular limit of densely defined semi-linear Cauchy problem. Indeed, it is well know that problem (1.4) has a densely defined Cauchy problem formulation, and system (1.1) has "only" a non-densely defined Cauchy problem formulation (see Proposition 2.2 for more presicions).

Theorem 1.4 shows that Ricker's type boundary conditions can be interpreted as a fast predation (or limitation) process. But it seems that some extra-conditions on γ_2 (see Assumption 1.3) are needed to derive such a result.

We now turn to convergence results uniformly with respect to the initial distribution φ in a bounded set. In order consider this problem, we will need to introduce the following definition.

Definition 1.1. Let $p \in [1, \infty)$ be given. Let $\mathcal{B}$ be a subset of $L^p_+ ((0, +\infty); \mathbb{R})$. We define the quantity $\kappa_p(\mathcal{B}) \in [0, \infty)$ as

$$\kappa_p(\mathcal{B}) := \lim_{\delta \searrow 0^+} \sup_{\varphi \in \mathcal{B}} \left(\int_0^{\delta} |\varphi(a)|^p \, da \right)^{\frac{1}{p}}.$$

We will say that $\mathcal{B}$ is a $(p-)$**non-atomic set at** 0 if

$$\kappa_p(\mathcal{B}) = 0.$$

The first main result of this article is the following theorem.

Theorem 1.5. *Let Assumptions 1.1 and 1.3 be satisfied. Let* $p \in [1,\infty)$ *and* $\tau > 0$. *Then for each* $\mathcal{B}$ *bounded subset of* $L^p_+(0,\infty)$ *there exists a constant* $\tilde{C} = \tilde{C}\left(\gamma_1, \gamma_2, \beta, \tau, \mathcal{B}\right) > 0$ *such that*

$$\limsup_{\varepsilon(>0)\to 0} \sup_{t\in[0,\tau]} \sup_{\varphi\in\mathcal{B}} \left\| \widehat{U}_\varepsilon(t)\varphi - \widehat{U}(t)\varphi \right\|_{L^p} \leq \tilde{C}\kappa_p(\mathcal{B}).$$

Set

$$\mathcal{B}_{|(0,\delta)} := \left\{ \varphi_{|(0,\delta)} : \varphi \in \mathcal{B} \right\} \subset L^p\left(0,\delta\right),$$

for each constant $\delta > 0$.

In section 2.1, we will see that the functional $\kappa_p(\mathcal{B})$ satisfies most of the properties of a measure of non-compactness. In particular, if $\mathcal{B}_{|(0,\delta)}$ is compact in $L^p\left(0,\delta\right)$ for some $\delta > 0$, then

$$\kappa_p(\mathcal{B}) = 0.$$

Of course the converse implication is false. This question will be studied in section 2.1.

As a direct consequence of Theorem 1.5 we obtain the following corollary.

Corollary 1.1. ***(Strong uniform convergence)*** *Let Assumptions 1.1 and 1.3 be satisfied. Let* $\tau > 0$. *Assume that*

$$p \in [1,\infty),$$

and $\mathcal{B}$ *is bounded subset of* $L^p_+(0,\infty)$ *satifying*

$$\kappa_p(\mathcal{B}) = 0.$$

Then

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,\tau]} \sup_{\varphi\in\mathcal{B}} \left\| \widehat{U}_\varepsilon(t)\varphi - \widehat{U}(t)\varphi \right\|_{L^p} = 0.$$

Remark 1.2. From this corollary we deduce that if $\mathcal{B}$ is a compact subset of $L^p_+(0,\infty)$, then

$$\widehat{U}_\varepsilon(t)\varphi \to \widehat{U}(t)\varphi \text{ as } \varepsilon \to 0 \text{ in } L^p(0,\infty),$$

and the convergence is uniform with respect to t in bounded intervals and uniform with respect to φ in $\mathcal{B}$.

Note that when $\mathcal{B}$ consists in a single point $\mathcal{B} = \{\varphi\}$, it is compact and Corollary 1.1 implies Theorem 1.4.

The second main result of this article is the following theorem.

Theorem 1.6. ***(Weak uniform convergence)*** *Let Assumptions 1.1 and 1.3 be satisfied. Let* $\tau > 0$ *and* $M > 0$. *Assume that*

$$p \in (1,\infty).$$

Then

$$\widehat{U}_\varepsilon(t)\varphi \rightharpoonup \widehat{U}(t)\varphi \textit{ as } \varepsilon \to 0 \textit{ for the weak topology in } L^p$$

uniformly with respect to $t \in [0,\tau]$, *and* $\varphi \in L^p_+(0,\infty)$ *with* $\|\varphi\|_{L^p} \leq M$. *That is to say that, for each* $\psi \in L^q\left(0,+\infty\right)$,

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,\tau]} \sup_{\varphi\in L^p_+(0,\infty):\|\varphi\|_{L^p}\leq M} \int_0^{+\infty} \psi\left(a\right) \left(\widehat{U}_\varepsilon(t)\varphi\left(a\right) - \widehat{U}(t)\varphi\left(a\right) \right) da = 0.$$

The plan of the paper is the following. The section 2 is spitted into two parts. The section 2.1 summarizes some properties of κ_p. The section 2.2 is devoted to the existence of a continuous semiflow by applying integrated semigroups theory. Section 2.3 is focusing on Volterra's integral equations formulations. In section 3, we will prove Theorem 1.5 and Theorem 1.6.

2. Preliminary

2.1. Pseudo measure of non compactness

Let $\mathcal{B}$ be a bounded subset of $L^p\left((0,+\infty),\mathbb{R}\right)$ for some $p\in[1,\infty)$. For each $\delta_0>0$ we set

$$\mathcal{B}^p_{|(0,\delta_0)} := \left\{ |\varphi|^p_{|(0,\delta_0)} : \varphi\in\mathcal{B}\right\} \subset L^1_+\left(0,\delta_0\right).$$

We summarize some properties of κ_p in the following proposition.

Proposition 2.1. *Let $p\in[1,\infty)$ and $\delta_0>0$. Let $\mathcal{B}$ and $\mathcal{B}'$ be two bounded subsets of $L^p\left((0,+\infty);\mathbb{R}\right)$. Then the following properties are satisfied:*

(i) *If $\mathcal{B}^p_{|(0,\delta_0)}$ is relatively compact for the weak topology $\sigma(L^1,L^\infty)$, then $\kappa_p(\mathcal{B})=0$;*

(ii) *If there exists $p'\in(p,+\infty)$, such that $\mathcal{B}_{|(0,\delta_0)}$ is a bounded in $L^{p'}_+\left(0,\delta_0\right)$ then $\kappa_p(\mathcal{B})=0$;*

(iii) *If $\mathcal{B}_{|(0,\delta_0)}$ is compact for the strong topology of $L^p\left(0,\delta_0\right)$ then $\kappa_p(\mathcal{B})=0$;*

(iv) $\kappa_p(\mathcal{B}+\mathcal{B}')\leq\kappa_p(\mathcal{B})+\kappa_p(\mathcal{B}')$;

(v) $\mathcal{B}\subset\mathcal{B}'\Longrightarrow\kappa_p(\mathcal{B})\leq\kappa_p(\mathcal{B}')$;

(vi) $\kappa_p\left(\overline{\mathcal{B}}\right)=\kappa_p\left(\mathcal{B}\right)$;

(vii) $\kappa_p\left(conv\left(\mathcal{B}\right)\right)=\kappa_p\left(\mathcal{B}\right)$;

(viii) $\kappa_p(\mathcal{B}\cup\mathcal{B}')=\max\left\{\kappa_p(\mathcal{B}),\kappa_p(\mathcal{B}')\right\}$;

(ix) $\kappa_p(c\mathcal{B})=|c|\,\kappa_p(\mathcal{B}),\ \forall c\in\mathbb{R}$.

Proof. The proof for (i) and (ii) are respectively direct consequences of Dunford-Pettis's Theorem (see Brezis [7, Theorem 4.30 p.115]) and De la Vallée-Poussin's theorem (see Brezis [7, Problem 23-D p.468]). To prove (iii), let $\eta>0$ be given. Then since $\mathcal{B}_{|(0,\delta_0)}$ is compact without loss of generality we may assume that

$$\mathcal{B}_{|(0,\delta_0)} \subset \bigcup_{k=1}^{m} B\left(\varphi_k,\eta\right),$$

where $B\left(\varphi_k,\eta\right)$ is a open ball in $L^p_+\left(0,\delta_0\right)$. Hence for $\psi\in\mathcal{B}_{|(0,\delta_0)}$, there exists

$k_0 \in \{1, .., m\}$ such that $\psi \in B(\varphi_{k_0}, \eta)$ and we have for each $\delta \in (0, \delta_0)$

$$\begin{aligned} \left(\int_0^\delta |\psi(a)|^p \, da\right)^{\frac{1}{p}} &\leq \left(\int_0^\delta |\varphi_{k_0}(a) - \psi(a)|^p \, da\right)^{\frac{1}{p}} + \left(\int_0^\delta |\varphi_{k_0}(a)|^p \, da\right)^{\frac{1}{p}}, \\ &\leq \eta + \left(\int_0^\delta |\varphi_{k_0}(a)|^p \, da\right)^{\frac{1}{p}} \\ &\leq \eta + \max_{k=1,\dots,m} \left(\int_0^\delta |\varphi_k(a)|^p \, da\right)^{\frac{1}{p}}, \quad \forall \delta \in (0, \delta_0). \end{aligned}$$

Thus letting $\delta \searrow 0$ leads us to

$$\kappa_p(\mathcal{B}) \leq \eta, \ \forall \eta > 0$$

and the results follows. The proof of properties (iv)-(ix) are similar to the proofs for measure of non-compactness (see Deimling [10], Martin [19], Sell and You [27]). □

To conclude this subsection, we will characterize the fact that $\mathcal{B}$ is atomic set at 0, that is to say that

$$\kappa_1(\mathcal{B}) > 0.$$

To do so, let us first recall some definitions and basic fact on bounded variation functions.

Definition 2.1. A map $\eta : [0, 1] \to \mathbb{R}$ has a **bounded variation on** $[0, 1]$ if

$$V(\eta, [0, 1]) = \sup \sum_{i=1}^{n} |\eta(x_{i+1}) - \eta(x_i)| < +\infty$$

where the supremum is taken over all the partitions $x_1 = 0 < x_2 < ... < x_{n+1} = 1$.

Recall that by the Riesz's representation theorem, for each $\varphi^* \in C([0, 1], \mathbb{R})^*$ the dual space of $C([0, 1], \mathbb{R})$, we can find $\eta : [0, 1] \to \mathbb{R}$ a function with bounded variation on $[0, 1]$ such that for each $\chi \in C([0, 1], \mathbb{R})$,

$$\langle \varphi^*, \chi \rangle_{C^*, C} = \int_0^1 d\eta(x)\chi(x),$$

where the last integral is a Stieltjes integral. The Stieltjes integral is defined as follows

$$\int_0^1 d\eta(x)\chi(x) = \lim_{\Delta(\Gamma)\to 0} \sum_{i=1}^{n} [\eta(x_{i+1}) - \eta(x_i)] \chi(y_i)$$

where the limit is taken over all the partition $\Gamma = (x_1, x_2, ..., x_{n+1})$ satisfying

$$x_1 = 0 < x_2 < ... < x_{n+1} = 1,$$

with

$$y_i \in [x_i, x_{i+1}],$$

and

$$\Delta(\Gamma) = \max_{i=1,\ldots,n} x_{i+1} - x_i.$$

As a direct consequence of the definition of the Stieltjes integral, one deduces that

$$\langle \varphi^*, \chi \rangle_{C^*,C} \leq V(\eta, [0,1]) \|\chi\|_\infty .$$

Definition 2.2. An element of φ^* of $C([0,1],\mathbb{R})^*$ is said to have a **non null mass at** 0, if and only if

$$\lim_{\varepsilon \to 0} \sup_{\chi \in C \text{ with Support}(\chi) \subset [0,\varepsilon)} \langle \varphi^*, \chi \rangle_{C^*,C} > 0.$$

We also recall that

$$\varphi^* \geq 0 \Leftrightarrow \langle \varphi^*, \chi \rangle_{C^*,C} \geq 0, \forall \chi \in C_+([0,1],\mathbb{R}).$$

We remark that if $\chi \in C([0,1],\mathbb{R})$ with Support$(\chi) \subset [0,\varepsilon)$ then

$$\langle \varphi^*, \chi \rangle_{C^*,C} = \int_0^\varepsilon d\eta(x)\, \chi(x).$$

Furthermore, if $\varphi^* \in C_+([0,1],\mathbb{R})^*$, then the map $x \to \eta(x)$ is a non-negative and increasing function from $[0,1]$ into $\mathbb{R}$. Hence one deduces that

$$V(\eta, [0,\varepsilon]) \geq \sup_{\chi \in C \text{ with Support}(\chi) \subset [0,\varepsilon)} \langle \varphi^*, \chi \rangle_{C^*,C} \geq \lim_{\widehat{\varepsilon} \nearrow \varepsilon} V(\varphi^*, [0,\widehat{\varepsilon}]) .$$

As a consequence, φ^* has a non null mass at 0 if and only if

$$c := \lim_{\varepsilon(>0) \to 0} V(\eta, [0,\varepsilon]) > 0.$$

This implies that

$$\eta(x) - \eta(0) > c, \forall x > 0.$$

therefore

$$\langle \varphi^*, \chi \rangle_{C^*,C} \geq c\delta_0(\chi), \forall \chi \in C_+([0,1],\mathbb{R}),$$

where δ_0 is the Dirac mass a 0.

The main theorem of this section is the following.

Theorem 2.1. *Let $\mathcal{B}$ be a bounded subset of $L^1_+((0,1),\mathbb{R})$. Then*

$$\kappa_1(\mathcal{B}) > 0$$

if and only if the weak closure of $\mathcal{B}$ considered as a subset of the dual space of $C([0,1],\mathbb{R})$ contains at least one element with non zero mass at 0. More precisely $\kappa_1(\mathcal{B}) > 0$ if and only if $\mathcal{B}$ contains a sequence φ_n such that*

$$\varphi_n \overset{weak*}{\rightharpoonup} \varphi^* \text{ in } C([0,1],\mathbb{R})^*$$

and there exists a positive constant $c > 0$ such that

$$\varphi^* \geq c\delta_0 \Leftrightarrow \langle \varphi^*, \chi \rangle_{C^*,C} \geq c\delta_0(\chi), \forall \chi \in C_+([0,1],\mathbb{R}).$$

Proof. Let $\mathcal{B} \subset L^1_+ \left((0,+\infty), \mathbb{R}\right)$ be a given bounded. Assume that it satisfies

$$\kappa_1(\mathcal{B}) = \lim_{\delta \searrow 0^+} \sup_{\varphi \in \mathcal{B}} \int_0^\delta \varphi(a)da > 0.$$

Set

$$\rho := \kappa_1(\mathcal{B}).$$

Then we can find a decreasing sequence of positive numbers $\{\delta_n\}_{n\in\mathbb{N}}$ such that

$$\delta_n \searrow 0, \text{ as } n \to +\infty,$$

and a sequence

$$\varphi_n \in \mathcal{B}$$

such that

$$\int_0^{\delta_n} \varphi_n(a)da > \rho/2.$$

Let $\varepsilon > 0$ be given. Let $\chi \in C([0,1], \mathbb{R})$ be given such that

$$\begin{cases} \chi \geq 0, \\ \chi(x) = 1 \text{ if } x \in [0, \varepsilon/2] \\ Support(\chi) \subset [0, \varepsilon). \end{cases}$$

Then recall that

$$\lim_{n\to\infty} \int_0^1 \varphi_n(a)\chi(a)da = \int_0^\varepsilon \chi(a)d\varphi^*(a).$$

Since $\chi \geq 0$ then one obtains that

$$\int_0^1 \varphi_n(a)\chi(a)da \geq \int_0^{\varepsilon/2} \varphi_n(a)da, \quad \forall n \geq 0.$$

Now since $\delta_n \to 0$, there exists $N = N_\varepsilon$ such that for all $n \geq N_\varepsilon$ one has $\delta_n \leq \varepsilon/2$ and therefore

$$\frac{\rho}{2} < \int_0^{\delta_n} \varphi_n(a)da \leq \int_0^{\frac{\varepsilon}{2}} \varphi_n(a)da$$

This implies that

$$\frac{\rho}{2} \leq \int_0^1 \chi(a)d\varphi^*(a),$$

and the result follows. The converse easily holds true. □

2.2. Integrated semigroup formulation

In order to explain the meaning of mild solutions for systems (1.1) and (1.4) we first introduce a suitable functional framework. Let $p \in [1, \infty)$. Consider the Banach space X as well as its positive cone X_+ respectively defined by

$$X = \mathbb{R} \times L^p(0,\infty), \quad X_+ = \mathbb{R}_+ \times L^p_+(0,\infty),$$

and endowed with the usual product norm

$$\left\| \begin{pmatrix} \alpha \\ \psi \end{pmatrix} \right\| = |\alpha| + \|\psi\|_{L^p}, \ \forall \begin{pmatrix} \alpha \\ \psi \end{pmatrix} \in X_p.$$

Next consider the linear operator $A : D(A) \subset X \to X$ defined by

$$A \begin{pmatrix} 0 \\ \psi \end{pmatrix} = \begin{pmatrix} -\psi(0) \\ -\psi' - \mu(.)\psi \end{pmatrix},$$

with

$$D(A) = \{0_{\mathbb{R}}\} \times W^{1,p}(0,\infty).$$

Set

$$X_0 := \overline{D(A)} = \{0_{\mathbb{R}}\} \times L^p(0,\infty).$$

Consider the bounded linear operator $B : X_0 \to X$ defined by

$$B \begin{pmatrix} 0 \\ \psi \end{pmatrix} = \begin{pmatrix} \int_0^{+\infty} \beta(a)\psi(a)\, da \\ 0_{L^p} \end{pmatrix}.$$

Set

$$X_{0+} := X_0 \cap X_+ = \{0_{\mathbb{R}}\} \times L^p_+(0,\infty).$$

Recall that, $(A+B)_0 : D((A+B)_0) \subset X_0 \to X_0$ the part of $A+B : D(A) \subset X \to X$ in X_0 is defined by

$$(A+B)_0 \begin{pmatrix} 0 \\ \psi \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}} \\ -\psi' - \mu(.)\psi \end{pmatrix}$$

and

$$D((A+B)_0) = \left\{ \begin{pmatrix} 0 \\ \psi \end{pmatrix} \in D(A) : \psi(0) = \int_0^{+\infty} \beta(a)\psi(a)\, da \right\}.$$

By combining the bounded perturbation result of section 3 in [16], and by applying the results of section 6 in [16] we obtain the following result.

Lemma 2.1. *Let Assumption 1.1 be satisfied. Then the linear* $(A+B)_0 : D((A+B)_0) \subset X_0 \to X_0$ *is the infinitesimal generator a strongly continuous semigroup* $\left\{T_{(A+B)_0}(t)\right\}_{t\geq 0}$ *of bounded linear operators on* X_0.

Next consider the map $F : X_0 \to X$ by

$$F \begin{pmatrix} 0 \\ \psi \end{pmatrix} = \begin{pmatrix} \exp\left(-\int_0^{+\infty} \gamma_2(a)\psi(a)\, da\right) \int_0^{+\infty} \beta(a)\psi(a)\, da \\ -m\left(\int_0^{+\infty} \gamma_1(a)\psi(a)\, da\right) h(a)\psi(a) \end{pmatrix}.$$

For each $\varepsilon > 0$, we consider the map $F_\varepsilon : X_0 \to X$

$$F_\varepsilon \begin{pmatrix} 0 \\ \psi \end{pmatrix} = \begin{pmatrix} \int_0^{+\infty} \beta(a)\psi(a)\, da \\ -\left[m\left(\int_0^{+\infty} \gamma_1(a)\psi(a)\, da\right) h(a) - \frac{1}{\varepsilon}\mathbf{1}_{[0,\varepsilon]}(a) \int_0^{+\infty} \gamma_2(a)\psi(a)da\right] \psi(a) \end{pmatrix}$$

and $G_\varepsilon : X_0 \to X_0$ defined by

$$G_\varepsilon \begin{pmatrix} 0 \\ \psi \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}} \\ -\left[m\left(\int_0^{+\infty} \gamma_1(a)\psi(a)\, da\right) h(a) - \frac{1}{\varepsilon}\mathbf{1}_{[0,\varepsilon]}(a) \int_0^{+\infty} \gamma_2(a)\psi(a)da\right] \psi(a) \end{pmatrix}.$$

By identifying $u(t,.)$ to $v(t) = \binom{0}{u(t,.)}$, the problem (1.1) can be reformulated as the following abstract non-densely Cauchy problem

$$\frac{dv(t)}{dt} = Av(t) + F(v(t)), \;\; v(0) = \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \in X_{0+}, \tag{2.1}$$

and $u_\varepsilon(t,.)$ to $v_\varepsilon(t) = \binom{0}{u_\varepsilon(t,.)}$, the problem (1.4) can be reformulated as the following abstract non-densely Cauchy problem

$$\frac{dv_\varepsilon(t)}{dt} = Av_\varepsilon(t) + F_\varepsilon(v_\varepsilon(t)), \;\; v_\varepsilon(0) = \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \in X_{0+}. \tag{2.2}$$

This former problem is also equivalent to the following densely defined Cauchy problem

$$\frac{dv_\varepsilon(t)}{dt} = (A+B)_0 \, v_\varepsilon(t) + G_\varepsilon(v_\varepsilon(t)), \;\; v_\varepsilon(0) = \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \in X_{0+}. \tag{2.3}$$

Now in order to derive a global existence result for the semiflow generated by (2.1) and (2.2) it is sufficient to use the following arguments combined together with the results in [17]. First since F and F_ε are Lipschitz continuous on bounded sets, the existence and uniqueness of a maximal semiflow follows. Secondly, the positivity of solutions is obtained by observing

$$(\lambda I - A)^{-1} X_+ \subseteq X_+ \text{ for each } \lambda > 0 \text{ large enough,}$$

and that for each constant $M > 0$ there exists $\lambda = \lambda(M) > 0$ such that

$$(F + \lambda I) \begin{pmatrix} 0 \\ \psi \end{pmatrix} \in X_{0+}, \tag{2.4}$$

whenever $\begin{pmatrix} 0 \\ \psi \end{pmatrix} \in X_{0+}$ and $\left\| \begin{pmatrix} 0 \\ \psi \end{pmatrix} \right\| \leq M$. Finally, the global existence of positive solutions follows from the sub-linearity argument

$$F \begin{pmatrix} 0 \\ \psi \end{pmatrix} \leq B \begin{pmatrix} 0 \\ \psi \end{pmatrix} \text{ and } F_\varepsilon \begin{pmatrix} 0 \\ \psi \end{pmatrix} \leq B \begin{pmatrix} 0 \\ \psi \end{pmatrix}$$

whenever $\begin{pmatrix} 0 \\ \psi \end{pmatrix} \in X_{0+}$.

Now by applying the results in [17] we obtain the following theorem.

Proposition 2.2. *Let Assumption 1.1 be satisfied. Then there exists a unique continuous semiflow $\{U(t)\}_{t\geq 0}$ on X_{0+} such that for each $x \in X_{0+}$ the map $t \to U(t)x$ is the unique integrated solution (or mild solution) of (2.1) that is to say that $U(.)x \in C([0,+\infty), X_{0+})$ and satisfies the following properties*

$$\int_0^t U(s)x ds \in D(A), \forall t \geq 0,$$

and

$$U(t)x = x + A\int_0^t U(s)x ds + \int_0^t F(U(s)x)ds, \;\; \forall t \geq 0.$$

Similarly, for each $\varepsilon \in (0,1]$, there exists a unique continuous semiflow $\{U_\varepsilon(t)\}_{t\geq 0}$ on X_{0+} such that for each $x \in X_{0+}$ the map $t \to U_\varepsilon(t)x$ is a mild solution of (2.2), that is to say that $U_\varepsilon(.)x \in C([0,+\infty), X_{0+})$ and satisfies the following properties

$$\int_0^t U_\varepsilon(s)xds \in D(A),\ \forall t \geq 0,$$

and

$$U_\varepsilon(t)x = x + A\int_0^t U_\varepsilon(s)xds + \int_0^t F_\varepsilon(U_\varepsilon(s)x)ds,\ \forall t \geq 0.$$

Moreover $U_\varepsilon(.)x \in C([0,+\infty), X_{0+})$ is a mild solution of the densely defined Cauchy problem of (2.3), that is to say that

$$U_\varepsilon(t)x = T_{(A+B)_0}(t)x + \int_0^t T_{(A+B)_0}(t-s)G_\varepsilon(U_\varepsilon(s)x)ds,\ \forall t \geq 0.$$

2.3. Volterra's integral equation formulation

In this subsection, we present a Volterra's integral equation formulation of the age-structured systems (1.1) and (1.4). We refer to the book of Webb [32] and Iannelli [13] for more results and informations on this subject. Set

$$\Gamma_\gamma^\varphi(t) := \int_0^{+\infty} \gamma(a)u^\varphi(t,a)da,\ \forall t \geq 0, \text{ for } \gamma = \gamma_1, \gamma_2, \beta. \tag{2.5}$$

for $\gamma = \gamma_1, \gamma_2$ or β and each initial distribution $\varphi \in L_+^p((0,+\infty), \mathbb{R})$.
The solution of system (1.1) integrated along the characteristics is given by

$$u^\varphi(a,t) = \begin{cases} I^\varphi(t,a,a-t)\varphi(a-t), \text{ if } a \geq t, \\ I^\varphi(t,a,0)\exp\left(-\Gamma_{\gamma_2}^\varphi(t-a)\right)\Gamma_\beta^\varphi(t-a), \text{ if } a \leq t. \end{cases} \tag{2.6}$$

where

$$I^\varphi(t,a,s) = \exp\left(-\int_s^a \left[\mu(r) + m\left(\Gamma_{\gamma_1}^\varphi(r+t-a)\right)h(r)\right]dr\right), \tag{2.7}$$

for each $a \geq s \geq 0$. Combining (2.5) and (2.6) we deduce that $t \to \Gamma_{\gamma_1}^\varphi(t),\ t \to \Gamma_{\gamma_2}^\varphi(t)$ and $t \to \Gamma_\beta^\varphi(t)$ are the unique continuous solutions of the system of Volterra's integral equations:

$$\begin{array}{rcl} \Gamma_{\gamma_1}^\varphi(t) & = & \int_t^{+\infty} \gamma_1(a)I^\varphi(t,a,a-t)\varphi(a-t)da \\ & & + \int_0^t \gamma_1(a)I^\varphi(t,a,0)\Gamma_\beta^\varphi(t-a)e^{-\Gamma_{\gamma_2}^\varphi(t-a)}da, \\ \Gamma_{\gamma_2}^\varphi(t) & = & \int_t^{+\infty} \gamma_2(a)I^\varphi(t,a,a-t)\varphi(a-t)da \\ & & + \int_0^t \gamma_2(a)I^\varphi(t,a,0)\Gamma_\beta^\varphi(t-a)e^{-\Gamma_{\gamma_2}^\varphi(t-a)}da, \\ \Gamma_\beta^\varphi(t) & = & \int_t^{+\infty} \beta(a)I^\varphi(t,a,a-t)\varphi(a-t)da \\ & & + \int_0^t \beta(a)I^\varphi(t,a,0)\Gamma_\beta^\varphi(t-a)e^{-\Gamma_{\gamma_2}^\varphi(t-a)}da. \end{array} \tag{2.8}$$

Similarly, set

$$\Gamma_{\gamma,\varepsilon}^\varphi(t) := \int_0^{+\infty} \gamma(a)u_\varepsilon^\varphi(t,a)da, \tag{2.9}$$

for $\gamma = \gamma_1, \gamma_2$ or β and each initial distribution $\varphi \in L_+^p((0,+\infty), \mathbb{R})$. As before we have

$$u_\varepsilon^\varphi(t,a) = \begin{cases} I_\varepsilon^\varphi(t,a,a-t)\varphi(a-t), \text{ if } a \geq t, \\ I_\varepsilon^\varphi(t,a,0)\Gamma_{\beta,\varepsilon}^\varphi(t-a), \text{ if } a \leq t, \end{cases} \tag{2.10}$$

wherein

$$I_{\varepsilon}^{\varphi}(t,a,s) := \exp\left(-\int_{s}^{a}\left[\mu(r)+m\left(\Gamma_{\gamma_1,\varepsilon}^{\varphi}(r+t-a)\right)h(r)\right]dr - \int_{\min(\varepsilon,s)}^{\min(\varepsilon,a)}\frac{1}{\varepsilon}\Gamma_{\gamma_2,\varepsilon}^{\varphi}(r+t-a)\,dr\right), \tag{2.11}$$

for each $a \geq s \geq 0$, and $t \to \Gamma_{\gamma_1,\varepsilon}^{\varphi}(t)$, $t \to \Gamma_{\gamma_2,\varepsilon}^{\varphi}(t)$ and $t \to \Gamma_{\beta,\varepsilon}^{\varphi}(t)$ are the unique continuous solutions of the system of Volterra's integral equations:

$$\begin{aligned}
\Gamma_{\gamma_1,\varepsilon}^{\varphi}(t) = {} & \int_t^{+\infty}\gamma_1(a)\,I_{\varepsilon}^{\varphi}(t,a,a-t)\,\varphi(a-t)\,da \\
& + \int_0^t \gamma_1(a)\,I_{\varepsilon}^{\varphi}(t,a,0)\,\Gamma_{\beta,\varepsilon}^{\varphi}(t-a)\,da \\
\Gamma_{\gamma_2,\varepsilon}^{\varphi}(t) = {} & \int_t^{+\infty}\gamma_2(a)\,I_{\varepsilon}^{\varphi}(t,a,a-t)\,\varphi(a-t)\,da \\
& + \int_0^t \gamma_2(a)\,I_{\varepsilon}^{\varphi}(t,a,0)\,\Gamma_{\beta,\varepsilon}^{\varphi}(t-a)\,da \\
\Gamma_{\beta,\varepsilon}^{\varphi}(t) = {} & \int_t^{+\infty}\beta(a)\,I_{\varepsilon}^{\varphi}(t,a,a-t)\,\varphi(a-t)\,da \\
& + \int_0^t \beta(a)\,I_{\varepsilon}^{\varphi}(t,a,0)\,\Gamma_{\beta,\varepsilon}^{\varphi}(t-a)\,da.
\end{aligned} \tag{2.12}$$

3. Proofs of the main results

We start this section with a preliminary estimate.

Lemma 3.1. *Let Assumption 1.1 be satisfied. Let $p \in [1,+\infty)$. Then for each $\varphi \in L_+^p(0,\infty)$ we have the following upper bound*

$$\left\|\widehat{U}(t)\varphi\right\|_{L^p} \leq \|\varphi\|_{L^p(0,\infty)}\, e^{\frac{\|\beta\|_{L^q}^p}{p}t},\ \forall t \geq 0, \tag{3.1}$$

and

$$\left\|\widehat{U}_{\varepsilon}(t)\varphi\right\|_{L^p} \leq \|\varphi\|_{L^p(0,\infty)}\, e^{\frac{\|\beta\|_{L^q}^p}{p}t},\ \forall t \geq 0. \tag{3.2}$$

As a consequence we deduce that for each $\gamma \in L_+^q((0,\infty);\mathbb{R})$ we have for each $\varepsilon > 0$, $\varphi \in L_+^p((0,\infty);\mathbb{R})$

$$0 \leq \Gamma_{\gamma,\varepsilon}^{\varphi}(t) \leq \|\gamma\|_{L^q}\|\varphi\|_{L^p}\, e^{\frac{\|\beta\|_{L^q}^p}{p}t},\ \forall t \geq 0. \tag{3.3}$$

Proof. The proof of the two inequalities are similar. Let us for example consider inequality (3.2). By using (2.10) we obtain

$$\begin{aligned}
\int_0^{+\infty}|u_{\varepsilon}^{\varphi}(t,a)|^p\,da &\leq \int_0^{+\infty}|\varphi(l-t)|^p\,dl + \int_0^t\left|\Gamma_{\beta,\varepsilon}^{\varphi}(t-r)\right|^p dr \\
&\leq \|\varphi\|_{L^p}^p + \int_0^t\left|\int_0^{+\infty}\beta(\sigma)u^{\varphi}(t-r,\sigma)d\sigma\right|^p dr \\
&\leq \|\varphi\|_{L^p}^p + \|\beta\|_{L^q}^p\int_0^t\|u^{\varphi}(r,.)\|_{L^p}^p\,dr,
\end{aligned}$$

and the result follows from Gronwall's lemma. □

3.1. Preliminary estimates

To derive the main results of this paper it will be important to give some estimates of the quantity

$$\int_0^{+\infty} \psi(a)\left(u_\varepsilon^\varphi(t,a) - u^\varphi(t,a)\right) da,$$

for some given test function $\psi \in L^q(0,\infty)$. To do so we shall re-write the above quantity as follows

$$\begin{aligned}\int_0^{+\infty} \psi(a)\left(u_\varepsilon^\varphi(t,a) - u^\varphi(t,a)\right) da = \int_0^t \psi(a)\,\mathcal{H}_\varepsilon^\varphi(a,t-a)\,da \\ + \int_t^{+\infty} \psi(a)\,\mathcal{J}_\varepsilon^\varphi(a,a-t)\,\varphi(a-t)\,da,\end{aligned} \tag{3.4}$$

where we have set for each $t \in [0,\tau]$,

$$\mathcal{H}_\varepsilon^\varphi(a,t-a) := I^\varphi(t,a,0)\,e^{-\Gamma_{\gamma_2}^\varphi(t-a)}\Gamma_\beta^\varphi(t-a) - I_\varepsilon^\varphi(t,a,0)\,\Gamma_{\beta,\varepsilon}^\varphi(t-a),\ \forall a \le t, \tag{3.5}$$

and

$$\mathcal{J}_\varepsilon^\varphi(a,a-t) := I^\varphi(t,a,a-t) - I_\varepsilon^\varphi(t,a,a-t),\ \forall a \ge t. \tag{3.6}$$

Let us first derive some Lipschitz estimates independent of ε for some useful linear form, namely of the form $\Gamma_{\gamma_2,\varepsilon}^\varphi$.

Lemma 3.2. *Let Assumptions 1.1 and 1.3 be satisfied. Let $\tau > 0$ and $M > 0$. Then for each $\varepsilon \in (0,1]$ and each $\varphi \in L_+^p((0,\infty);\mathbb{R})$ with $\|\varphi\|_{L^p} \le M$, the map $t \mapsto \Gamma_{\gamma_2,\varepsilon}^\varphi(t)$ is Lipschitz continuous on $[0,\tau]$, and we have the following estimate*

$$\left\|\Gamma_{\gamma_2,\varepsilon}^\varphi\right\|_{Lip,[0,\tau]} \le \kappa_{\gamma_2}(\tau,M)\,Me^{\frac{\|\beta\|_{L^q}^p}{p}\tau} \tag{3.7}$$

where

$$\left\|\Gamma_{\gamma_2,\varepsilon}^\varphi\right\|_{Lip,[0,\tau]} := \sup_{t,s\in[0,\tau]:t\neq s} \frac{\left|\Gamma_{\gamma_2,\varepsilon}^\varphi(t) - \Gamma_{\gamma_2,\varepsilon}^\varphi(s)\right|}{|t-s|},$$

and the constant

$$\begin{aligned}\kappa_{\gamma_2}(\tau,M) := \left\|\gamma_2'\right\|_{L^q} + \|\mu\|_\infty \|\gamma_2\|_{L^q} \\ + m_\infty \|h\|_\infty \|\gamma_2\|_{L^q} + \|\gamma_2\|_{L^q}\,\widehat{\kappa} M e^{\frac{\|\beta\|_{L^q}^p}{p}\tau}\end{aligned}$$

where $\widehat{\kappa}$ is defined in (1.6) and

$$m_\infty := \sup_{x\in\left[0,\|\gamma_1\|_{L^q} Me^{\frac{\|\beta\|_{L^q}^p}{p}\tau}\right]} m(x).$$

Proof. By choosing the initial distribution φ smooth enough that is to say that

$$D = \left\{\varphi \in W^{1,p}(0,+\infty) : \varphi(0) = \int_0^{+\infty} \beta(a)\varphi(a)da\right\}$$

then it is well known that $t \to u_\varepsilon^\varphi(t,.)$ is from $[0,+\infty)$ into $L^p(0,+\infty)$, and for $t \ge 0$,

$$u_\varepsilon^\varphi(t,.) \in W^{1,p}(0,+\infty) \text{ with } u_\varepsilon^\varphi(t,0) = \int_0^{+\infty} \beta(a)u_\varepsilon^\varphi(t,a)\,da.$$

Hence for such an initial distribution $t \mapsto \Gamma^{\varphi}_{\gamma_2,\varepsilon}(t)$ is continuously differentiable and we have

$$\begin{aligned}
\frac{d\Gamma^{\varphi}_{\gamma_2,\varepsilon}(t)}{dt} &= \int_0^{+\infty} \gamma_2(a) \frac{\partial u^{\varphi}_{\varepsilon}(t,a)}{\partial t} da \\
&= -\int_0^{+\infty} \gamma_2(a) \frac{\partial u^{\varphi}_{\varepsilon}(t,a)}{\partial a} da - \int_0^{+\infty} \gamma_2(a) \mu(a) u^{\varphi}_{\varepsilon}(t,a)\, da \\
&\quad -m\left(\Gamma^{\varphi}_{\gamma_1,\varepsilon}(t)\right) \int_0^{+\infty} \gamma_2(a) h(a) u^{\varphi}_{\varepsilon}(t,a)\, da \\
&\quad -\Gamma^{\varphi}_{\gamma_2,\varepsilon}(t) \int_0^{+\infty} \gamma_2(a) \frac{1}{\varepsilon} \mathbf{1}_{[0,\varepsilon]}(a) u^{\varphi}_{\varepsilon}(t,a)\, da,
\end{aligned}$$

since $\gamma_2(0) = 0$, by integrating by parts we obtain

$$\begin{aligned}
\frac{d\Gamma^{\varphi}_{\gamma_2,\varepsilon}(t)}{dt} &= \gamma_2(0) \int_0^{+\infty} \beta(a) u^{\varphi}_{\varepsilon}(t,a)\, da + \int_0^{+\infty} \gamma_2'(a) u^{\varphi}_{\varepsilon}(t,a)\, da \\
&\quad -\int_0^{+\infty} \gamma_2(a) \mu(a) u^{\varphi}_{\varepsilon}(t,a)\, da \\
&\quad -m\left(\Gamma^{\varphi}_{1,\varepsilon}(t)\right) \int_0^{+\infty} \gamma_2(a) h(a) u^{\varphi}_{\varepsilon}(t,a)\, da \\
&\quad -\Gamma^{\varphi}_{\gamma_2,\varepsilon}(t) \int_0^{+\infty} \gamma_2(a) \frac{1}{\varepsilon} \mathbf{1}_{[0,\varepsilon]}(a) u^{\varphi}_{\varepsilon}(t,a)\, da, \\
&= \int_0^{+\infty} \gamma_2'(a) u^{\varphi}_{\varepsilon}(t,a)\, da \\
&\quad -\int_0^{+\infty} \gamma_2(a) \mu(a) u^{\varphi}_{\varepsilon}(t,a)\, da \\
&\quad -m\left(\int_0^{+\infty} \gamma_1(a) u^{\varphi}_{\varepsilon}(t,a)\, da\right) \int_0^{+\infty} \gamma_2(a) h(a) u^{\varphi}_{\varepsilon}(t,a)\, da \\
&\quad -\left(\int_0^{+\infty} \gamma_2(a) u^{\varphi}_{\varepsilon}(t,a)\, da\right) \left(\int_0^{+\infty} \gamma_2(a) \frac{1}{\varepsilon} \mathbf{1}_{[0,\varepsilon]}(a) u^{\varphi}_{\varepsilon}(t,a)\, da\right).
\end{aligned}$$

Using Holder's inequality and Lemma 3.1, we obtain (3.7).

Now let $\varphi \in L^p_+(0,+\infty)$. Since $D \cap L^p_+(0,+\infty)$ and dense in $L^p_+(0,+\infty)$, we can find a sequence $\{\varphi_n\}_{n\geq 0} \subset D \cap L^p_+(0,+\infty)$ such that

$$\varphi_n \underset{n\to+\infty}{\to} \varphi \text{ in } L^p(0,+\infty).$$

Therefore from the fact that

$$\left|\Gamma^{\varphi_n}_{\gamma,\varepsilon}(s) - \Gamma^{\varphi_n}_{\gamma,\varepsilon}(l)\right| \leq \kappa_\gamma(t,M)\,|s-l|\,,\ \forall s,l \in [0,t]\,,\ \forall n \geq 0.$$

Passing to the limit $n \to \infty$ and using the continuity of the semiflow with respect to the initial value, we obtain

$$\left|\Gamma^{\varphi}_{\gamma,\varepsilon}(s) - \Gamma^{\varphi}_{\gamma,\varepsilon}(l)\right| \leq \kappa_\gamma(t,M)\,|s-l|\,,\ \forall s,l \in [0,t]\,,\ \forall \varphi \in X_{0+},$$

this completes the proof of Lemma 3.2. □

Now we are ready to give the estimates which are the key point to prove our main results.

Lemma 3.3. *Let Assumptions 1.1 and 1.3 and be satisfied. Let $\tau > 0$ and $M > 0$. Then there exists a constant $C_0 = C_0(\tau, M) > 0$ such that for each $\varepsilon \in (0,1]$,*

$$\begin{aligned}&\int_0^t |\psi(a)| \left|\mathcal{H}_\varepsilon^\varphi(a, t-a)\right| da \\ &\leq C_0 \left[\|\psi\|_{L^q} \varepsilon^{\frac{1}{p}} + \int_0^t \left(|\psi(t-r)| + \|\psi\|_{L^q}\right) F_\varepsilon(\varphi)(r) dr\right]\end{aligned} \tag{3.8}$$

whenever $\varphi \in L_+^p((0,+\infty),\mathbb{R})$ with $\|\varphi\|_{L^p} \leq M$, $t \in [0,\tau]$, $\psi \in L^q((0,+\infty),\mathbb{R})$, and

$$F_\varepsilon(\varphi)(t) = \sum_{\gamma=\gamma_1,\gamma_2,\beta} \left|\Gamma_{\gamma,\varepsilon}^\varphi(t) - \Gamma_\gamma^\varphi(t)\right|, \ \forall r \in [0,\tau]. \tag{3.9}$$

Proof. Let $\varphi \in L_+^p((0,\infty),\mathbb{R})$ be such that $\|\varphi\|_{L^p} \leq M$ and $\psi \in L^q((0,+\infty),\mathbb{R})$. Let $\varepsilon \in (0,1]$. Then for each $t \in [0,\tau]$ and each $a \in [0,t]$,

$$\begin{aligned}\mathcal{H}_\varepsilon^\varphi(a,t-a) &= \left[I^\varphi(t,a,0)\, e^{-\Gamma_{\gamma_2}^\varphi(t-a)} - I_\varepsilon^\varphi(a,0)\right] \Gamma_\beta^\varphi(t-a) \\ &\quad + I_\varepsilon^\varphi(t,a,0) \left[\Gamma_\beta^\varphi(t-a) - \Gamma_{\beta,\varepsilon}^\varphi(t-a)\right].\end{aligned}$$

Recall that

$$\left|I_\varepsilon^\varphi(t,a,0)\right| \leq 1, \ \forall a \in [0,t], \ \forall t \leq \tau,$$

hence

$$\begin{aligned}&\int_0^t |\psi(a)| \left|\mathcal{H}_\varepsilon^\varphi(a,t-a)\right| da \\ \leq\ & \left\|\Gamma_\beta^\varphi\right\|_{\infty,[0,\tau]} \int_0^t |\psi(a)| \left|I^\varphi(t,a,0)\exp\left(-\Gamma_{\gamma_2}^\varphi(t-a)\right) - I_\varepsilon^\varphi(t,a,0)\right| da \\ & + \int_0^t |\psi(a)| \left|\Gamma_\beta^\varphi(t-a) - \Gamma_{\beta,\varepsilon}^\varphi(t-a)\right| da.\end{aligned}$$

It remains to evaluate the integral

$$I_1 = \int_0^t |\psi(a)| \left|I^\varphi(t,a,0)\exp\left(-\Gamma_{\gamma_2}^\varphi(t-a)\right) - I_\varepsilon^\varphi(t,a,0)\right| da.$$

Since

$$\begin{aligned}& I^\varphi(t,a,0)\exp\left(-\Gamma_{\gamma_2}^\varphi(t-a)\right) \\ =\ & \exp\left(-\int_0^a \left[\mu(r) + m\left(\Gamma_{\gamma_1}^\varphi(r+t-a)\right) h(r)\right] dr - \Gamma_{\gamma_2}^\varphi(t-a)\right),\end{aligned}$$

and

$$\begin{aligned}I_\varepsilon^\varphi(t,a,0) &= \exp\left(-\int_0^a \left[\mu(r) + m\left(\Gamma_{\gamma_1,\varepsilon}^\varphi(r+t-a)\right) h(r)\right] dr \right. \\ &\qquad \left. - \int_0^{\min(\varepsilon,a)} \frac{1}{\varepsilon} \Gamma_{\gamma_2,\varepsilon}^\varphi(r+t-a)\, dr\right)\end{aligned}$$

we deduce that

$$\begin{aligned}
& \left|I^{\varphi}\left(t,a,0\right)\exp\left(-\Gamma^{\varphi}_{\gamma_2}(t-a)\right)-I^{\varphi}_{\varepsilon}\left(t,a,0\right)\right| \\
\leq\ & \|m\|_{Lip,[0,C_1]}\|h\|_{L^\infty}\int_0^a\left|\Gamma^{\varphi}_{\gamma_1}\left(r+t-a\right)-\Gamma^{\varphi}_{\gamma_1,\varepsilon}\left(r+t-a\right)\right|dr \\
& +\left|\Gamma^{\varphi}_{\gamma_2}(t-a)-\int_0^{\min(\varepsilon,a)}\frac{1}{\varepsilon}\Gamma^{\varphi}_{\gamma_2,\varepsilon}\left(r+t-a\right)dr\right| \\
\leq\ & \|m\|_{Lip,[0,C_1]}\|h\|_{L^\infty}\int_0^a\left|\Gamma^{\varphi}_{\gamma_1}\left(t-l\right)-\Gamma^{\varphi}_{\gamma_1,\varepsilon}\left(t-l\right)\right|dl \\
& +\left|\Gamma^{\varphi}_{\gamma_2}(t-a)-\Gamma^{\varphi}_{\gamma_2,\varepsilon}\left(t-a\right)\right| \\
& +\left|\Gamma^{\varphi}_{\gamma_2,\varepsilon}\left(t-a\right)-\int_0^{\min(\varepsilon,a)}\frac{1}{\varepsilon}\Gamma^{\varphi}_{\gamma_2,\varepsilon}\left(r+t-a\right)dr\right|
\end{aligned}$$

where $C_1=\|\gamma_1\|_{L^q}Me^{\frac{\|\beta\|^p_{L^q}}{p}\tau}$. Next let us set

$$C_2:=\|m\|_{Lip,[0,C_1]}\|h\|_{L^\infty}\tau^{1/p}.$$

Then by using Holder's inequality, we obtain

$$\begin{aligned}
I_1\ \leq\ & C_2\int_0^t\|\psi\|_{L^q}\left|\Gamma^{\varphi}_{\gamma_1}\left(t-l\right)-\Gamma^{\varphi}_{\gamma_1,\varepsilon}\left(t-l\right)\right|dlda \\
& +\int_0^t\left|\psi\left(a\right)\right|\left|\Gamma^{\varphi}_{\gamma_2}(t-a)-\Gamma^{\varphi}_{\gamma_2,\varepsilon}\left(t-a\right)\right|da \\
& +\int_0^t\left|\psi\left(a\right)\right|\left|\Gamma^{\varphi}_{\gamma_2,\varepsilon}\left(t-a\right)-\int_0^{\min(\varepsilon,a)}\frac{1}{\varepsilon}\Gamma^{\varphi}_{\gamma_2,\varepsilon}\left(r+t-a\right)dr\right|da.
\end{aligned}$$

Now it remains to evaluate

$$I_2:=\int_0^t\left|\psi\left(a\right)\right|\left|\Gamma^{\varphi}_{\gamma_2,\varepsilon}\left(t-a\right)-\int_0^{\min(\varepsilon,a)}\frac{1}{\varepsilon}\Gamma^{\varphi}_{\gamma_2,\varepsilon}\left(r+t-a\right)dr\right|da.$$

If $t\leq\varepsilon$, we have

$$\begin{aligned}
I_2\ \leq\ & \int_0^t\left|\psi\left(a\right)\right|\left[\Gamma^{\varphi}_{\gamma_2,\varepsilon}\left(t-a\right)+\int_0^a\frac{1}{\varepsilon}\Gamma^{\varphi}_{\gamma_2,\varepsilon}\left(r+t-a\right)dr\right]da \\
\leq\ & C_3\int_0^{\varepsilon}\left|\psi\left(a\right)\right|da
\end{aligned}$$

with $C_3:=2\|\gamma_2\|_{L^q}Me^{\frac{\|\beta\|^p_{L^q}}{p}\tau}$. Hence by using Holder's inequality, we obtain

$$I_2\leq C_3\varepsilon^{1/p}\|\psi\|_{L^q}.$$

If $t \geq \varepsilon$, we have

$$\begin{aligned}
I_2 \quad : \quad &= \int_0^{\varepsilon} |\psi(a)| \left| \Gamma^{\varphi}_{\gamma_2,\varepsilon}(t-a) - \int_0^{a} \frac{1}{\varepsilon} \Gamma^{\varphi}_{\gamma_2,\varepsilon}(r+t-a)\, dr \right| da \\
&\quad + \int_{\varepsilon}^{t} |\psi(a)| \left| \Gamma^{\varphi}_{\gamma_2,\varepsilon}(t-a) - \int_0^{\varepsilon} \frac{1}{\varepsilon} \Gamma^{\varphi}_{\gamma_2,\varepsilon}(r+t-a)\, dr \right| da \\
&\leq C_3 \varepsilon^{1/p} \|\psi\|_{L^q} \\
&\quad + \int_{\varepsilon}^{t} |\psi(a)| \frac{1}{\varepsilon} \int_0^{\varepsilon} \left| \Gamma^{\varphi}_{\gamma_2,\varepsilon}(t-a) - \Gamma^{\varphi}_{\gamma_2,\varepsilon}(r+t-a) \right| drda \\
&\leq C_3 \varepsilon^{1/p} \|\psi\|_{L^q} \\
&\quad + \left\| \Gamma^{\varphi}_{\gamma_2,\varepsilon} \right\|_{Lip,[0,\tau]} \int_{\varepsilon}^{t} |\psi(a)| \frac{1}{\varepsilon} \int_0^{\varepsilon} rdrda \\
&= C_3 \varepsilon^{1/p} \|\psi\|_{L^q} + \left\| \Gamma^{\varphi}_{\gamma_2,\varepsilon} \right\|_{Lip,[0,\tau]} \frac{\varepsilon}{2} \int_0^{\tau} |\psi(a)|\, da \\
&\leq C_3 \varepsilon^{1/p} \|\psi\|_{L^q} + \left\| \Gamma^{\varphi}_{\gamma_2,\varepsilon} \right\|_{Lip,[0,\tau]} \frac{\varepsilon}{2} \tau^{1/p} \|\psi\|_{L^q}.
\end{aligned}$$

the result follows. □

Lemma 3.4. *Let Assumptions 1.1 and 1.3 be satisfied. Let $\tau > 0$ and $M > 0$. Then there exists a constant $\widehat{C}_0 = \widehat{C}_0(\tau, M) > 0$ such that for each $\varepsilon \in (0,1]$,*

$$\begin{array}{l}
\int_t^{+\infty} |\psi(a)| \left| \mathcal{J}^{\varphi}_{\varepsilon}(a, a-t) \right| |\varphi(a-t)|\, da \\
\leq \widehat{C}_0 \left[\|\psi\|_{L^q} \int_0^t \left| \Gamma^{\varphi}_{\gamma_1}(r) - \Gamma^{\varphi}_{\gamma_1,\varepsilon}(r) \right| dr + \int_0^{\varepsilon} |\psi(a+t)| |\varphi(a)|\, da \right],
\end{array} \tag{3.10}$$

whenever $\varphi \in L^p_+((0,+\infty);\mathbb{R})$ with $\|\varphi\|_{L^p} \leq M$, $t \in [0,\tau]$, and $\psi \in L^q((0,+\infty);\mathbb{R})$.

Proof. Let $M > 0$ and $\tau > 0$ be given. Let $\varphi \in L^p_+((0,+\infty);\mathbb{R})$ be given such that $\|\varphi\|_{L^p} \leq M$. Then we have for each $a \geq t$ and for each $t \in [0,\tau]$,

$$\begin{aligned}
\mathcal{J}^{\varphi}_{\varepsilon}(a, a-t) &= I^{\varphi}_{\varepsilon}(t, a, a-t) - I^{\varphi}(t, a, a-t) \\
&= \exp\left(-\int_{a-t}^{a} \left[\mu(r) + m\left(\Gamma^{\varphi}_{\gamma_1,\varepsilon}(r+t-a) \right) h(r) \right] dr \right. \\
&\qquad \left. - \int_{\min(\varepsilon,a-t)}^{\min(\varepsilon,a)} \frac{1}{\varepsilon} \Gamma^{\varphi}_{\gamma_2,\varepsilon}(r+t-a)\, dr \right) \\
&\qquad - \exp\left(-\int_{a-t}^{a} \left[\mu(r) + m\left(\Gamma^{\varphi}_{\gamma_1}(r+t-a) \right) h(r) \right] dr \right) \\
&\leq \left| \int_{a-t}^{a} \left[m\left(\Gamma^{\varphi}_{\gamma_1,\varepsilon}(r+t-a) \right) - m\left(\Gamma^{\varphi}_{\gamma_1}(r+t-a) \right) \right] h(r) dr \right| \\
&\qquad + \left| \int_{\min(\varepsilon,a-t)}^{\min(\varepsilon,a)} \frac{1}{\varepsilon} \Gamma^{\varphi}_{\gamma_2,\varepsilon}(r+t-a)\, dr \right|
\end{aligned}$$

Let $\psi \in L^q\left((0,+\infty),\mathbb{R}\right).$ Then

$$\begin{aligned}
I_1 &:= \int_t^{+\infty} |\psi(a)|\left|J_\varepsilon^\varphi(a,a-t)\right||\varphi(a-t)|\,da \\
&\leq \|m\|_{Lip,[0,C_1]}\|h\|_{L^\infty}\int_t^{+\infty}|\psi(a)|\left(\int_0^t\left|\Gamma_{\gamma_1}^\varphi(l)-\Gamma_{\gamma_1,\varepsilon}^\varphi(l)\right|dl\right)|\varphi(a-t)|\,da \\
&\quad+\int_t^{+\infty}|\psi(a)|\left|\int_{\min(\varepsilon,a-t)+t-a}^{\min(\varepsilon,a)+t-a}\frac{1}{\varepsilon}\Gamma_{\gamma_2,\varepsilon}^\varphi(l)\,dl\right||\varphi(a-t)|\,da \\
&= \|m\|_{Lip,[0,C_1]}\|h\|_{L^\infty}\int_t^{+\infty}|\psi(a)|\left(\int_0^t\left|\Gamma_{\gamma_1}^\varphi(l)-\Gamma_{\gamma_1,\varepsilon}^\varphi(l)\right|dl\right)|\varphi(a-t)|\,da \\
&\quad+\int_t^{t+\varepsilon}|\psi(a)|\left|\int_{\min(\varepsilon,a-t)+t-a}^{\min(\varepsilon,a)+t-a}\frac{1}{\varepsilon}\Gamma_{\gamma_2,\varepsilon}^\varphi(l)\,dl\right||\varphi(a-t)|\,da
\end{aligned}$$

therefore we obtain

$$\begin{aligned}
I_1 &\leq \|m\|_{Lip,[0,C_1]}\|h\|_{L^\infty}\int_t^{+\infty}|\psi(a)|\left(\int_0^t\left|\Gamma_{\gamma_1}^\varphi(l)-\Gamma_{\gamma_1,\varepsilon}^\varphi(l)\right|dl\right)|\varphi(a-t)|\,da \\
&\quad+\left\|\Gamma_{\gamma_2,\varepsilon}^\varphi\right\|_{\infty,[0,\tau]}\int_t^{t+\varepsilon}|\psi(a)||\varphi(a-t)|\,da
\end{aligned}$$

and the result follows. □

3.2. Proof of Theorem 1.5

Lemma 3.5. ***(Gronwall's like inequality)*** *Let $\tau>0$ be fixed and $f\in L_+^p\left((0,\tau),\mathbb{R}\right).$ Assume in addition that there exist two constants $\alpha\geq 0$ and $\beta\geq 0$ such that*

$$0\leq f(t)\leq \alpha+\beta\left(\int_0^t f(s)^p\,ds\right)^{\frac{1}{p}},\ \text{for almost every } t\in(0,\tau). \tag{3.11}$$

Then

$$f(t)\leq 2\alpha\exp\left(\frac{2^p\beta^p}{p}t\right),\ \text{for almost every } t\in(0,\tau).$$

Proof. We first observe that

$$(a+b)^p\leq(2\max(a,b))^p\leq 2^p\left(a^p+b^p\right) \tag{3.12}$$

whenever $a\geq 0$, $b\geq 0$ and $p\in[1,\infty).$ The inequality (3.11) implies

$$f(t)^p\leq\left(\alpha+\beta\left(\int_0^t f(s)^p\,ds\right)^{\frac{1}{p}}\right)^p$$

and by using (3.12) we obtain

$$f(t)^p\leq 2^p\left(\alpha^p+\beta^p\int_0^t f(s)^p\,ds\right)$$

and by using Gronwall inequality in L^1 the result follows. $\square$

Proof. **(of Theorem 1.5)** Let $\mathcal{B} \subset L^p_+ ((0,+\infty);\mathbb{R})$ be a given bounded set. Let $\tau > 0$ be given and fixed. Since $\mathcal{B}$ is bounded, we set

$$M := \sup_{\varphi\in\mathcal{B}} \|\varphi\|_{L^p}.$$

Recalling that

$$F_\varepsilon(\varphi)(t) := \sum_{\gamma=\gamma_1,\gamma_2,\beta} \left|\Gamma^\varphi_{\gamma,\varepsilon}(t) - \Gamma^\varphi_\gamma(t)\right|,$$

we first observe that

$$F_\varepsilon(\varphi)(t) \leq \widetilde{C}_0 \left\|u^\varphi_\varepsilon(t,.) - u^\varphi(t,.)\right\|_{L^p},$$

with

$$\widetilde{C}_0 := \left(\|\gamma_1\|_{L^q} + \|\gamma_2\|_{L^q} + \|\beta\|_{L^q}\right).$$

Moreover it is well known that

$$\left\|u^\varphi_\varepsilon(t,.) - u^\varphi(t,.)\right\|_{L^p} = \sup_{\substack{\psi\in L^q((0,+\infty);\mathbb{R})\\ \|\psi\|_{L^q}\leq 1}} \int_0^{+\infty} \psi(a)\left(u^\varphi_\varepsilon(t,a) - u^\varphi(t,a)\right)da.$$

Let $\psi \in L^q((0,+\infty);\mathbb{R})$ with $\|\psi\|_{L^q} \leq 1$. We have

$$\begin{aligned}\int_0^{+\infty} \psi(a)\left(u^\varphi_\varepsilon(t,a) - u^\varphi(t,a)\right)da &= \int_0^t \psi(a)\,\mathcal{H}^\varphi_\varepsilon(a,t-a)\,da \\ &\quad + \int_t^{+\infty} \psi(a)\,\mathcal{J}^\varphi_\varepsilon(a,a-t)\,\varphi(a-t)\,da.\end{aligned}$$

therefore due to Lemma 3.4 and Lemma 3.3, we obtain for each $\varepsilon \in (0,1]$, each $t \in [0,\tau]$ and each $\varphi \in \mathcal{B}$ that

$$F_\varepsilon(\varphi)(t) \leq C_1 \left[\varepsilon^{\frac{1}{p}} + \int_0^t \left(|\psi(t-r)| + 2\right) F_\varepsilon(\varphi)(r)dr + \int_0^\varepsilon |\psi(a+t)|\,|\varphi(a)|\,da\right],$$

where

$$C_1 := \widetilde{C}_0\left(C_0 + \widehat{C}_0\right).$$

So by using Holder's inequality, we deduce that

$$\begin{aligned}F_\varepsilon(\varphi)(t) \quad &\leq \quad C_1 \left[\left[\varepsilon^{\frac{1}{p}} + \left(\int_0^\varepsilon |\varphi(a)|^p\,da\right)^{1/p}\right]\right. \\ &\quad \left. + \left[\left\||\psi| + 2\right\|_{L^q(0,\tau)}\right]\left(\int_0^t F_\varepsilon(\varphi)(r)^p dr\right)^{1/p}\right],\end{aligned}$$

and this implies the following:

$$F_\varepsilon(\varphi)(t) \leq C_1 \left[\left[\varepsilon^{\frac{1}{p}} + \left(\int_0^\varepsilon |\varphi(a)|^p\,da\right)^{1/p}\right] + \left[1 + 2\tau^{1/q}\right]\left(\int_0^t F_\varepsilon(\varphi)(r)^p dr\right)^{1/p}\right].$$

By applying Gronwall's like inequality given in Lemma 3.5, we obtain

$$F_\varepsilon(\varphi)(t) \leq 2C_1 \left[\varepsilon^{\frac{1}{p}} + \left(\int_0^\varepsilon |\varphi(a)|^p\,da\right)^{1/p}\right] \exp\left(\frac{2^p C_1^p \left[1 + 2\tau^{1/q}\right]^p}{p}\tau\right),$$

the proof of Theorem 1.5 is completed. $\square$

3.3. Proof of Theorem 1.6

By definition we have

$$\begin{aligned} F_\varepsilon(\varphi)(t) &= \sum_{\gamma=\gamma_1,\gamma_2,\beta} \left| \int_0^\infty \gamma(a)\left(u_\varepsilon^\varphi(t,a) - u^\varphi(t,a)\right) da \right| \\ &\le \int_0^\infty \left(\sum_{\gamma=\gamma_1,\gamma_2,\beta} |\gamma(a)| \right) \left| u_\varepsilon^\varphi(t,a) - u^\varphi(t,a) \right| da. \end{aligned}$$

Set

$$\chi := \sum_{\gamma=\gamma_1,\gamma_2,\beta} |\gamma(.)| \in L_+^q(0,+\infty).$$

By using the same argument as in the proof of Theorem 1.5 with $\psi = \chi$, we obtain

$$F_\varepsilon(\varphi)(t) \le \widehat{C}_1 \left[\varepsilon^{\frac{1}{p}} + \int_0^t \left(|\chi(t-r)| + 2\right) F_\varepsilon(\varphi)(r) dr + \int_0^\varepsilon |\chi(a+t)| \, |\varphi(a)| \, da \right],$$

for some constant $\widehat{C}_1 = \widehat{C}_1(\tau, M, \|\chi\|_{L^q}) > 0$.

Now Holder's inequality leads us to

$$\begin{aligned} F_\varepsilon(\varphi)(t) \le \ & \widehat{C}_1 \left[\left(\varepsilon^{\frac{1}{p}} + M \left(\int_0^\varepsilon |\chi(a+t)|^q \, da \right)^{1/q} \right) \right. \\ & \left. + \left[\| |\chi| + 2 \|_{L^q(0,\tau)} \right] \left(\int_0^t F_\varepsilon(\varphi)(r)^p dr \right)^{1/p} \right]. \end{aligned}$$

Hence Gronwall's inequality (see Lemma 3.5) provides

$$F_\varepsilon(\varphi)(t) \le 2\widehat{C}_1 \left[\varepsilon^{\frac{1}{p}} + M \left(\int_t^{t+\varepsilon} |\chi(a)|^q \, da \right)^{1/q} \right] \exp\left(\frac{2^p \widehat{C}_1^p \left[\| |\chi| + 2 \|_{L^q(0,\tau)} \right]^p}{p} \tau \right).$$

Next, we infer from the continuity of the map $t \to \int_0^t |\chi(a)|^q \, da$ that

$$\sup_{t\in[0,\tau] \text{ and } \varphi\in\mathcal{B}} F_\varepsilon(\varphi)(t) \to 0 \text{ as } \varepsilon \to 0.$$

Let $\psi \in L^p(0,+\infty)$ be given and fixed. We have

$$\begin{aligned} I(t) := \textstyle\int_0^{+\infty} \psi(a)\left(u_\varepsilon^\varphi(t,a) - u^\varphi(t,a)\right) da = \int_0^t \psi(a) \mathcal{H}_\varepsilon^\varphi(a, t-a) \, da \\ + \textstyle\int_t^{+\infty} \psi(a) \mathcal{J}_\varepsilon^\varphi(a, a-t) \varphi(a-t) \, da. \end{aligned}$$

And by using the same arguments as in the proof Theorem 1.5 we obtain

$$|I(t)| \le \widehat{C}_1 \left[\varepsilon^{\frac{1}{p}} + \int_0^t \left(|\psi(t-r)| + 2\right) F_\varepsilon(\varphi)(r) dr + \int_0^\varepsilon |\psi(a+t)| \, |\varphi(a)| \, da \right],$$

that completes the proof of the result.

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