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RANDOM ATTRACTOR OF NONLINEAR STRAIN WAVES WITH WHITE NOISE

Xianyun Du

Abstract In this paper, we consider the long time behaviors of nonlinear strain waves in elastic waveguides with white noise. We show that the initial boundary value problem has a global solution and a compact global attractor.

Keywords Nonlinear Strain Waves, random Attractor, white noise.

MSC(2000) 38B41, 38B40.

1. Introduction

In some problems of nonlinear wave propagation in waveguides, the interaction of waveguides, the external medium and the possibility of energy exchange through lateral surface of waveguide cannot be neglected. When the energy exchange between the rod and the medium is considered, there is a dissipation of deformation wave in the viscous external medium. The general cubic double dispersion equation (CDDE) can be derived from Hamilton principle:

$$w_{tt} - w_{xx} = \frac{1}{4}(cw^3 + 6w^2 + aw_{tt} - bw_{xx} + dw_t)_{xx}, \tag{1.1}$$

where a, b, c, d are some positive constants depending on Young modulus E_0. The equation (1.1) was studied in the literatures [5, 6, 12, 13, 14, 15]. In this paper, we consider the following stochastic nonlinear wave equation perturbed by a random forcing term

$$du_t - (\alpha du_t + \gamma du)_{xx} = (u - \beta u_{xx} + f(u))_{xx}dt + g(x)dt + \sum_{j=1}^{m} h_j dw_j, \tag{1.2}$$

where α, β, γ are positive constants, f is a sufficiently smooth real valued function with $f(0) = 0$, g and h_j ($j \in \{1, 2, \cdots, m\}$) are given functions defined on $\mathbb{R}$ and $\{w_j\}_{j=1}^m$ are independent two side real-valued Wiener processes on a probability space which will be specified later.

Attractor is an important concept in the study of asymptotic behavior of deterministic dynamical system. Crauel, Debussche and Flandoli [4] present a general theroy to study the random attractor by defining an attracting set as a set that attracts any orbit starting from $-\infty$. The random attractors are compact invariant

Email address: duxy2011@foxmail.com
College of mathematics, Chengdu university of information technology, Chengdu, P.R.China
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sets, which depend on chance and move with time. The authors applied the theory to prove the existence of random attractors for the two-dimensional stochastic Navier-Stokes equation. In this paper, we apply another method prove the existence of random attractors for nonlinear strain waves in elastic waveguides with white noise. We establish the asymptotic compactness of solutions for system (1.2) by applying the method of operator decomposition (see [11]), which is a crucial step to get the global attractor.

This paper is arranged as follows. In section 2, some relevant concepts and theories are given. In section 3, we introduce the Ornstein-Ohlenbeck process and some properties and provide some basic settings about (1.2). In section 4, We prove results on the existence of a unique random attractor of the random dynamical system generated by (1.2).

Throughout this paper, we denote by $||\cdot||$ the norm of $H = L^2(0, l)$, with the inner product $(\cdot, \cdot)$, $||\cdot||_p$ denotes the norm of $L^p(0, l)$ for all $1 \leq p \leq \infty$, and $||\cdot||_{k,p}$ the norm of any Banach space $W^{k,p}(0, l)$.

2. Preliminaries

In this section, we recall some basic notions of the theory of random dynamical system (RDS) (see [3, 4, 7, 16, 17]) and the Kuratowski measure of non-compactness (see [8]), which is a useful tool to study the attractor (see [11], [18]).

Let $(X, ||.||_X)$ be a separable Banach space with Borel $\sigma-$algebra $\mathcal{B}(X)$ and $(\Omega, \mathcal{F}, P, (\vartheta_t)_{t\in\mathbb{R}})$ be the ergodic metric dynamical system.

Definition 2.1 A continuous random dynamical system over $(\Omega, \mathcal{F}, P, (\vartheta_t)_{t\in R})$ is a $(\mathcal{B}(R^+) \times \mathcal{F} \times \mathcal{B}(X),\ \mathcal{B}(X))$-measurable mapping

$$S : \mathbb{R}^+ \times \Omega \times X \to X\ (t, \omega, x) \to S(t, \omega, x)$$

satisfying the following properties:

(1) $S(0, \omega, x) = x$ for $\omega \in \Omega$ and $x \in X$;

(2) $S(t + \tau, \omega, \cdot) = S(t, \vartheta_\tau\omega, \cdot) \circ S(\tau, \omega, \cdot)$ for $\tau, t \geq 0$, and $\omega \in \Omega$;

(3) S is continuous with respect to x for $t \geq 0$ and $\omega \in \Omega$.

A set-valued map $B : \Omega \to 2^X$ is called a random closed set if $B(\omega)$ is a nonempty closed set and $\omega \to d(x, B(\omega))$ is measurable for $x \in X$. A random set $B(\omega)$ is called tempered if for P-a.s. $\omega \in \Omega$ and all $\beta > 0$

$$\lim_{t\to\infty} e^{-\beta t} \sup\{||b||_X : b \in B(\vartheta_{-t}\omega)\} = 0.$$

Let $\mathcal{D}$ be the collection of all tempered random subsets in X and $\{K(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$. Then $\{K(\omega)\}_{\omega\in\Omega}$ is called a random absorbing set for S in $\mathcal{D}$ if for $B(\omega) \in \mathcal{D}$ and P -a.e. $\omega \in \Omega$, there exists $t_B(\omega) > 0$ such that

$$S(t, \vartheta_{-t}\omega, B(\vartheta_{-t}\omega)) \subset K(\omega) \ \text{ for all } t \geq t_B(\omega).$$

Definition 2.2. A random set $\{\mathcal{A}(\omega)\} \in \mathcal{D}$ is random attractor (or pullback attractor) for a RDS S if the following conditions are satisfied, for P -a.e. $\omega \in \Omega$,

(i) $\mathcal{A}(\omega)$ is a random compact set. i.e. $\omega \to d(x, \mathcal{A}(\omega))$ is measurable for every $x \in X$ and $\mathcal{A}(\omega)$ is compact;

(ii) $\{\mathcal{A}(\omega)\}$ is strictly invariant, i.e.

$$S(t,\omega,\mathcal{A}(\omega)) = \mathcal{A}(\vartheta_t\omega) \ \text{ for all } t \geq 0;$$

(iii) $\{\mathcal{A}(\omega)\}$ attracts every set in $\mathcal{D}$, i.e., for all $B = \{B(\omega)\} \in \mathcal{D}$,

$$\lim_{t\to\infty} d_H(\varphi(t,\vartheta_{-t}\omega, B(\vartheta_{-t}\omega)), \mathcal{A}(\omega)) = 0,$$

where d_H is the Hausdorff semi-distance.

Let B be a bounded set in a Banach space X. The Kuratowski measure of non-compactness $\alpha(B)$ of B is defined by

$$\alpha(B) = \inf\{d > 0 : B \ admits\ a\ finite\ cover\ by\ sets\ of\ diameter \leq d\}.$$

We define $\alpha(B) = \infty$, if B is unbounded, see [8].

Definition 2.3. [11] A random dynamical system S on a Polish space (X, d) is almost surely $\mathcal{D}-\alpha-$contracting if

$$\lim_{t\to\infty} \alpha(S(t,\vartheta_{-t}\omega, A(\vartheta_{-t}\omega))) = 0 \text{ for } A \in \mathcal{D}.$$

Lemma 2.4. *For a random dynamical system $S(t,\omega)$ on a separabal Banach space $(X, ||.||_X)$, if almost surely the following hold:*

(1) $S(t,\omega) = S_1(t,\omega) + S_2(t,\omega)$;

(2) For any tempered random variable $a \geq 0$, there exist $r(a)$ $(0 \leq r < \infty)$, a.s. such that for the closed ball B_a with radius a in X, $S_1(t,\vartheta_{-t}\omega, B_a(\vartheta_{-t}\omega))$ is precompact in X for all $t > r(a)$.

(3) $||S_2(t,\vartheta_{-t}\omega, u)||_X \leq K(t,\vartheta_{-t}\omega, a)$, $t > 0$, $u \in B_a(\omega)$ and $K(t,\omega,a)$ is a measurable function with respect to (t,ω,x) which satisfies

$$\lim_{t\to\infty} K(t,\vartheta_{-t}\omega, a) = 0.$$

Then $S(t,\omega)$ is almost surely $\mathcal{D}-\alpha-$contracting (see [11]).

Lemma 2.5. *Let $S(t,\omega)$ be a random dynamical system on a Polish space $(X, ||.||_X)$. Assume that*

(1) $S(t,\omega)$ has an absorbing set $B(\omega) \in \mathcal{D}$;

(2) $S(t,\omega)$ is almost surely $\mathcal{D}-\alpha-$contracting.

Then $S(t,\omega)$ possesses a global random attractor in X.

3. The basic setting and O-U processes

In this section, we present the existence of continuous random dynamical system for the stochastic nonlinear strain wave equation in elastic waveguides:

$$du_t - \alpha d(\Delta u_t) - \gamma d(\Delta u) - \Delta u + \beta^2\Delta^2 u - \Delta f(u) = g + \sum_{j=1}^{m} h_j dw_j \tag{3.1}$$

subject to the initial conditions

$$u(x,0) = u_0,\ \ u_t(x,0) = u_1 \text{ for } x \in (0,l) \tag{3.2}$$

and boundary condition

$$u(0,t) = u(l,t) = 0, \tag{3.3}$$

where $\Delta = \partial_{xx}$, α, β, γ are positive constants, g is a given function in $L^2(0.l)$, for $j \in \{1, 2, \cdots, m\}$, $h_j \in H_0^1 \cap W^{2,q}(0,l)$ for some $q \geq 2$ and $\{w_j\}_{j=1}^m$ are independent two-sided real valued Wiener processes on a probability space, which will be specified below and f is a nonlinear function satisfying the following conditions: for all $s \in \mathbb{R}$

$$f(s)s \geq c_1 F(s) \geq c_2 |s|^{2p+2} \geq 0, \tag{3.4}$$

$$|f(s)| \leq c_3(|s|^{2p+1} + |s|), \tag{3.5}$$

where $F(s) = \int_0^s f(\tau)d\tau$ and $c_i (i = 1, 2, 3)$ are positive constants.

In the sequel, we consider the probability space $(\Omega, \mathcal{F}, P)$, where

$$\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}^m) : \omega(0) = 0\},$$

the Borel $\sigma-$algebra $\mathcal{F}$ on Ω is generated by the compact open topology, and P is the corresponding Wiener measure on $\mathcal{F}$. Then we identify $\omega(t)$ with $(w_1, w_2, \cdots, w_m)$, i.e.,

$$(w_1, w_2, \cdots, w_m) = \omega(t) \text{ for } t \in \mathbb{R}.$$

The time shift is defined by

$$\vartheta_s \omega(t) = \omega(t+s) - \omega(s) \text{ for } t, s \in \mathbb{R}.$$

It is a family of ergodic terms formations. Now we consider the one-dimensional Ornstein-Uhlenbeck equation

$$dz_j + \lambda z_j dt = dw_j(t). \tag{3.6}$$

It is easy to check that for each $j = 1, 2, \cdots, m$

$$z_j(t) = z_j(\vartheta_t \omega_j) \equiv -\lambda \int_{-\infty}^0 e^{\lambda\tau} (\vartheta_t \omega_j)(\tau) d\tau, \text{ for} t \in \mathbb{R}.$$

is a solution of (3.6). Putting $z(\vartheta_t \omega) = \sum_{j=1}^m (I - \alpha\Delta)^{-1} h_j z_j(\vartheta_t \omega_j)$, where Δ is the Laplacian with domain $H_0^1 \cap H^2(0,l)$, By (3.6) we find that

$$dz - \alpha d(\Delta z) + \alpha(z - \alpha\Delta z)dt = \sum_{j=1}^m h_j dw_j. \tag{3.7}$$

Lemma 3.1. *For $\epsilon > 0$, there exists a tempered random variable $\rho_1 : \Omega \to \mathbb{R}$ such that*

$$||z(\vartheta_t \omega)||_{2p+2} \leq e^{\epsilon|t|} \rho_1(\omega) \text{ for } t \in \mathbb{R} \text{ and } \omega \in \Omega, \tag{3.8}$$

where $p \geq 0$ and $\rho_1(\omega),\ \omega \in \Omega$ satisfies

$$\rho_1(\vartheta_t \omega) \leq e^{\epsilon|t|} \rho_1(\omega) \text{ for } t \in \mathbb{R}. \tag{3.9}$$

Proof. Let $j = 1, 2, \cdots, m$. Since $|z_j(\omega_j)|$ is a tempered random variable and the mapping $t \to \ln |z_j(\vartheta_t \omega_j)|$ is P-a.s.continuous, it follows from Proposition 4.3.3 in

[13] that for any $\epsilon_j > 0$, there is a tempered random variable $r_j(\omega_j) > 0$ such that $|z_j(\omega_j)| \le r_j(\omega_j)$ and

$$r_j(\vartheta_t\omega_j) \le e^{\epsilon_j|t|} r_j(\omega_j),\ t \in R,\ \text{for P-a.s. } \omega \in \Omega. \tag{3.10}$$

Since $(I - \alpha\Delta)z(\vartheta_t\omega) = \sum_{j=1}^m h_j z_j(\vartheta_t\omega_j)$ we find

$$\begin{aligned}
&||z(\vartheta_t\omega)||_{2p+2}^{2p+2} + \alpha(2p+1)\int_0^l z(\vartheta_t\omega)^{2p}|\nabla z(\vartheta_t\omega)|^2 dx \\
\le &\int_0^l z(\vartheta_t\omega)^{2p+1}(\sum_{j=1}^m h_j z_j(\vartheta_t\omega_j))dx \\
\le &\frac{2p+1}{2p+2}||z(\vartheta_t\omega)||_{2p+2}^{2p+2} + \frac{1}{2p+2}\int_0^l (\sum_{j=1}^m h_j z_j(\vartheta_t\omega_j))^{2p+2} dx.
\end{aligned}$$

Let $\epsilon > 0$ and $\epsilon_1 = \epsilon_2 = \cdots = \epsilon_m = \epsilon$, then we have

$$\begin{aligned}
||z(\vartheta_t\omega)||_{2p+2} \le ||\sum_{j=1}^m h_j z_j(\vartheta_t\omega_j)||_{2p+2} &\le \sum_{j=1}^m ||h_j||_{2p+2}|z_j(\vartheta_t\omega_j)| \\
&\le \sum_{j=1}^m ||h_j||_{2p+2} r_j(\vartheta_t\omega_j) \le e^{\epsilon|t|}\sum_{j=1}^m r_j(\omega_j)||h_j||_{2p+2}.
\end{aligned}$$

Let $\rho_1(\omega) = \sum_{j=1}^m r_j(\omega_j)||h_j||_{2p+2}$ then (3.8) holds and (3.9) follows from (3.10). □

Corollary 3.2. *For $\epsilon > 0$, there exists a tempered random variable $\rho_2 : \Omega \to \mathbb{R}$ such that for $\sigma = 0$ or 1,*

$$||A^{\frac{\sigma}{2}} z(\vartheta_t\omega)|| + \alpha\sqrt{\lambda_1}||A^{\frac{1+\sigma}{2}} z(\vartheta_t\omega)|| \le e^{\epsilon|t|}\rho_2(\omega), \tag{3.11}$$

for $t \in \mathbb{R}$ and $\omega \in \Omega$, where λ_1 is the first eigenvalue of $-\Delta$.

Proof. Let $\xi = ||A^{\frac{\sigma}{2}} z(\vartheta_t\omega)||^2 + \alpha||A^{\frac{1+\sigma}{2}} z(\vartheta_t\omega)||^2$, $\rho_2(\omega) = \sum_{j=1}^m r_j(\omega_j)||A^{\frac{\sigma}{2}} h_j||$. Since $(I - \alpha\Delta)z(\vartheta_t\omega) = \sum_{j=1}^m h_j z_j(\vartheta_t\omega_j)$, we get

$$\begin{aligned}
\xi \le &||A^{\frac{\sigma}{2}} z(\vartheta_t\omega)||(\sum_{j=1}^m ||A^{\frac{\sigma}{2}} h_j||\ |z_j(\vartheta_t\omega_j)|) \le ||A^{\frac{\sigma}{2}} z(\vartheta_t\omega)||(\sum_{j=1}^m ||A^{\frac{\sigma}{2}} h_j||\ r_j(\vartheta_t\omega_j)) \\
\le &||A^{\frac{\sigma}{2}} z(\vartheta_t\omega)||e^{\epsilon|t|}(\sum_{j=1}^m r_j(\omega_j)||A^{\frac{\sigma}{2}} h_j||\ ).
\end{aligned}$$

By the Poincare inequality

$$||A^{\frac{1+\sigma}{2}} z(\vartheta_t\omega)|| \ge \sqrt{\lambda_1}||A^{\frac{\sigma}{2}} z(\vartheta_t\omega)||.$$

Hence, we have

$$\xi \le e^{\epsilon|t|}(\sum_{j=1}^m r_j(\omega_j)||A^{\frac{\sigma}{2}} h_j||\ ) \le e^{\epsilon|t|}\rho_2(\omega),$$

the corollary holds. □

Now transform the problem (3.1)-(3.3) to a deterministic system with a random parameter and show that it generates a random dynamical system.

Let $v(t,\omega) = u_t(t,\omega) + \varepsilon u(t,\omega) - z(\vartheta_t\omega)$. Then (3.1)-(3.3) is equivalent to the following random partial differential system

$$u_t = v - \varepsilon u + z(\vartheta_t\omega), \tag{3.12}$$

$$\begin{aligned} &v_t - \alpha\Delta v_t - \varepsilon v - (\gamma - \alpha\varepsilon)\Delta v + \varepsilon^2 u - (1 + \alpha\varepsilon^2 - \gamma\varepsilon)\Delta u \\ &+\beta\Delta^2 u - \Delta f(u) = g + (\alpha + \varepsilon)z(\vartheta_t\omega) + (\gamma - \alpha^2 - \alpha\varepsilon)\Delta z(\vartheta_t\omega), \end{aligned} \tag{3.13}$$

$$(u, v)|_{t=0} = (u_0, v_0), \tag{3.14}$$

$$u(0, t) = u(l, t) = v(0, t) = v(l, t) = 0, \tag{3.15}$$

where $v_0 = u_1 + \varepsilon u_0 - z(\omega)$, ε is a positive constant. We set $E_0 = H_0^1 \times L^2(0, l)$, $E_1 = H^2 \cap H_0^1 \times H_0^1 \cap L^2(0, l)$. Then $E_1 \hookrightarrow E_0$ with compact imbedding.

By a Galerkin method as in [6], it can be proved that under assumptions (3.4) and (3.5), for P-a.e.$\omega \in \Omega$ and for every $(u_0, v_0) \in E_0$, problem (3.12)-(3.15) have a unique solution $(u, v) \in C(R^+, E_0)$ and the solution (u, v) is continuous with respect to x in E_0 for all $t \geq 0$. Hence, the solution mapping generates a RDS. It is called stochastic flow associated with the nonlinear strain wave equation with additive noise.

4. Uniform time a priori estimates and random attractors

In this section, we derive uniform estimates on the solutions of (3.12)-(3.15) when $t \to \infty$ and prove the existence of a bounded random absorbing set and the asymptotic compactness of the random dynamical system associated with the equation. From now on, we always assume that $\mathcal{D}$ is the collection of all tempered subsets of E_0 with respect to $(\Omega, \mathcal{F}, P, (\vartheta_t)_{t\in R})$. Let $E_0 = H_0^1 \times L^2(0, l)$ endowed with the inner product and norm $(Y_1, Y_2)_{E_0} = (u_1, u_2)_{H_0^1} + (v_1, v_2)_{L^2}$, $||Y||_{E_0} = ||u||_{H_0^1} + ||v||_{L^2}$, $Y_j = (u_j, v_j)$, $Y_0 = (u_0, v_0)$. We first derive the following uniform estimates in E_0.

Lemma 4.1. *Suppose that f satisfies (3.4) and (3.5), and $g \in H^{-1}(0.l)$. Then for $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$, $Y_0 = (u_0, v_0) \in B(\omega)$ and for P-a.e.$\omega \in \Omega$, there exists $T = T(B, \omega) > 0$, such that*

$$||Y(t, \vartheta_{-t}\omega, Y_0(\vartheta_{-t}\omega))||_{E_0} \leq R(\omega) \text{ for } t \geq T,$$

where $R(\omega) = c(1 + \rho_1^{2p+2}(\omega) + \rho_2^2(\omega))$ is a positive random function.

Proof. Taking the inner product of (3.14) with $(-\Delta)^{-1}v$ and using $v = u_t + \varepsilon u - z(\vartheta_t\omega)$, we have

$$\frac{d}{dt}\phi_0(t, \omega) + H_0(t, \omega) = 0, \tag{4.1}$$

where

$$\begin{aligned} \phi_0(t, \omega) =& \frac{1}{2}(||v||_{-1,2}^2 + \alpha||v||^2 + \varepsilon^2||u||_{-1,2}^2 + (1 + \alpha\varepsilon^2 - \gamma\varepsilon)||u||^2 \\ &+ \beta||\nabla u||^2 + 2\int_0^l F(u)dx), \end{aligned} \tag{4.2}$$

$$
\begin{aligned}
H_0(t,\omega) =& -\varepsilon||v||^2_{-1,2} + (\gamma-\alpha\varepsilon)||v||^2 + \varepsilon^3||u||^2_{-1,2} + (1+\alpha\varepsilon^2-\gamma\varepsilon)\varepsilon||u||^2 \\
&+ \beta\varepsilon||\nabla u||^2 + \varepsilon\int_0^l f(u)udx - \varepsilon^2(u,(-\Delta)^{-1}z(\vartheta_t\omega)) \\
&+ (1+\alpha\varepsilon^2-\gamma\varepsilon)(u,-z(\vartheta_t\omega)) - \beta(\nabla u,\nabla z(\vartheta_t\omega)) - (f(u),z(\vartheta_t\omega)) \\
&+ (-g-(\alpha+\varepsilon)z(\vartheta_t\omega) + (\gamma-\alpha^2-\alpha\varepsilon)z(\vartheta_t\omega),(-\Delta)^{-1}v).
\end{aligned}
\tag{4.3}
$$

Choose δ and ε such that

$$
0<\delta\le\min\{\frac{c_1}{2},1\},\ 0<\varepsilon\le\min\{\frac{1}{2\gamma},\frac{\gamma\lambda_1}{(1+\delta)(1+\alpha\lambda_1)}\},
\tag{4.4}
$$

where c_1 is defined in (3.4) and λ_1 is the first eigenvalue of $-\Delta$. It follows from (3.4) and (3.5) that

$$
\varepsilon\int_0^l f(u)udx - \delta\varepsilon\int_0^l F(u)dx \ge \varepsilon(c_1-\delta)c_2||u||^{2p+2}_{2p+2}
$$

and

$$
\begin{aligned}
|-\int_0^l f(u)z(\vartheta_t\omega)dx| \le& \frac{1}{2}\varepsilon c_2\delta||u||^{2p+2}_{2p+2} + \frac{1}{2}\delta\varepsilon(1+\alpha\varepsilon^2-\gamma\varepsilon)||u||^2 \\
&+ c(||z(\vartheta_t\omega)||^{2p+2}_{2p+2} + ||z(\vartheta_t\omega)||^2).
\end{aligned}
$$

Using (4.4)and computing, we get

$$
\begin{aligned}
&H_0(t,\omega) - \delta\varepsilon\phi_0(t,\omega) \\
\ge&(\gamma\lambda_1 - \varepsilon(1+\delta)(1+\alpha\lambda_1))||v||^2_{-1,2} + \varepsilon^2(1-\delta)||u||^2_{-1,2} \\
&+ \varepsilon(1+\alpha\varepsilon^2-\gamma\varepsilon)(1-\delta)||u||^2 + \beta\varepsilon(1-\delta)||\nabla u||^2 \\
&+ \varepsilon c_2(c_1-2\delta)||u||^{2p+2}_{2p+2} - c||g||^2_{-1,2} \\
&- c(||z(\vartheta_t\omega)||^2 + ||\nabla z(\vartheta_t\omega)||^2 + ||z(\vartheta_t\omega)||^{2p+2}_{2p+2}) \\
\ge& - c(||z(\vartheta_t\omega)||^2 + ||\nabla z(\vartheta_t\omega)||^2 + ||z(\vartheta_t\omega)||^{2p+2}_{2p+2} + ||g||^2_{-1,2})
\end{aligned}
\tag{4.5}
$$

and

$$
a(||Y(t,\omega)||^2_{E_0} + ||u(t,\omega)||^{2p+2}_{2p+2}) \le \phi_0(t,\omega) \le b(||Y(t,\omega)||^2_{E_0} + ||u(t,\omega)||^{2p+2}_{2p+2}),
\tag{4.6}
$$

where

$$
a = \frac{1}{2}\min\{\alpha,(1+\alpha\varepsilon^2-\gamma\varepsilon),\beta,2c_2/c_1\}
$$

and

$$
b = \frac{1}{2}\max\{\alpha+1/\lambda_1,\varepsilon^2/\lambda_1+1+\alpha\varepsilon^2-\gamma\varepsilon+2c_3/c_1,\beta\}.
$$

By (4.1) and (4.5), we have

$$
\frac{d}{dt}\phi_0(t,\omega) + \lambda\phi_0(t,\omega) \le c_4p_0(\vartheta_t\omega) + c_5,
\tag{4.7}
$$

where $\lambda = \delta\varepsilon > 0$ and $p_0(\vartheta_t\omega) = ||z(\vartheta_t\omega)||^2 + ||\nabla z(\vartheta_t\omega)||^2 + ||z(\vartheta_t\omega)||^{2p+2}_{2p+2}$. By Lemma 3.1 with $\epsilon = \frac{\lambda}{2(2p+2)}$ and Corollary 3.2 with $\epsilon = \frac{\lambda}{4}$, for P-a.c. $\omega\in\Omega$ and $t\in\mathbb{R}$ we obtain

$$
p_0(\vartheta_t\omega) \le e^{\frac{1}{2}\lambda|t|}(\rho_1^{2p+2}(\omega) + \rho_2^2(\omega)).
\tag{4.8}
$$

It follows from (4.7) that for all $t \geq 0$,

$$\phi_0(t,\omega) \leq e^{-\lambda t}\phi_0(0,\omega) + c_4 \int_0^t e^{\lambda(\tau - t)} p_0(\vartheta_\tau \omega) d\tau + \frac{c_5}{\lambda}. \tag{4.9}$$

Replacing ω by $\vartheta_{-t}\omega$ with $t \geq 0$ in (4.9) and using (4.8), we obtain

$$\begin{aligned} \phi_0(t,\vartheta_{-t}\omega) \leq & e^{-\lambda t}\phi_0(0,\vartheta_{-t}\omega) + c_4 \int_0^t e^{\lambda(\tau-t)} p_0(\vartheta_{\tau - t}\omega) d\tau + \frac{c_5}{\lambda} \\ \leq & e^{-\lambda t}\phi_0(0,\vartheta_{-t}\omega) + c_4 \int_{-t}^0 e^{\lambda\tau} p_0(\vartheta_{\tau}\omega) d\tau + \frac{c_5}{\lambda} \\ \leq & e^{-\lambda t}\phi_0(0,\vartheta_{-t}\omega) + c_4 \int_{-t}^0 e^{\frac{1}{2}\lambda\tau} (\rho_1^{2p+2}(\omega) + \rho_2^2(\omega)) d\tau + \frac{c_5}{\lambda} \\ \leq & e^{-\lambda t}\phi_0(0,\vartheta_{-t}\omega) + c_*(1 + \rho_1^{2p+2}(\omega) + \rho_2^2(\omega)), \end{aligned} \tag{4.10}$$

where $c_* = \frac{1}{\lambda}(2c_4 + c_5)$ is a deterministic positive constant. This together with (4.6) shows that

$$\begin{aligned} a||Y(t,\vartheta_{-t}\omega, Y_0(\vartheta_{-t}\omega))||_{E_0}^2 \leq & b e^{-\lambda t}(||Y_0(\vartheta_{-t}\omega)||_{E_0}^2 + ||u_0(\vartheta_{-t}\omega)||_{2p+2}^{2p+2}) \\ & + c_*(1 + \rho_1^{2p+2}(\omega) + \rho_2^2(\omega)). \end{aligned} \tag{4.11}$$

Since $B(\omega)_{\omega\in\Omega} \in \mathcal{D}$ is tempered, it follows that if

$$Y_0(\vartheta_{-t}\omega) = (u_0(\vartheta_{-t}\omega), v_0(\vartheta_{-t}\omega)) \in B(\vartheta_{-t}\omega),$$

then there is $T_{B(\omega)} > 0$ such that for all $t \geq T_{B(\omega)}$,

$$b e^{-\lambda t}(||Y_0(\vartheta_{-t}\omega)||_{E_0}^2 + ||u_0(\vartheta_{-t}\omega)||_{2p+2}^{2p+2}) \leq c_*(1 + \rho_1^{2p+2}(\omega) + \rho_2^2(\omega)). \tag{4.12}$$

The result follows from (4.11) and (4.12). □

Denote

$$K(\omega) = \{Y \in E_0 : ||Y||_{E_0} \leq R(\omega)\}.$$

Then $\{K(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ is an absorbing set in E_0.

In order to prove that RDS $S(t,\omega)$ is almost surely $\mathcal{D}-\alpha-$contracting on E_0 by Lemma 2.6, we decompose the solution $Y = (u,v)$ of (3.12)-(3.15) with the initial value $Y_0 = (u_0, v_0)$ into two parts. Define by $Y^a = (u^a, v^a) = S_1(t)(u_0, v_0)$ is the solution of the equations

$$v^a = u_t^a + \varepsilon u^a, \tag{4.13}$$

$$v_t^a - \alpha\Delta v_t^a - \varepsilon v^a - (\gamma - \alpha\varepsilon)\Delta v^a + \varepsilon^2 u^a - (1 + \alpha\varepsilon^2 - \gamma\varepsilon)\Delta u^a + \beta\Delta^2 u^a = 0 \tag{4.14}$$

with the initial data $(u^a, v^a)|_{t=0} = (u_0, v_0) = (u_0, u_1 + \varepsilon u_0 - z(\omega))$ and homogeneous boundary condition. Then

$$Y^b = (u^b, v^b) = S_2(t)(u_0, v_0) = S(t)(u_0, v_0) - S_1(t)(u_0, v_0)$$

is the solution of the problems

$$v^b = u_t^b + \varepsilon u^b - z(\vartheta_t\omega), \tag{4.15}$$

$$\begin{aligned} &v_t^b - \alpha\Delta v_t^b - \varepsilon v^b - (\gamma - \alpha\varepsilon)\Delta v^b + \varepsilon^2 u^b - (1 + \alpha\varepsilon^2 - \gamma\varepsilon)\Delta u^b \\ &+\beta\Delta^2 u^b - \Delta f(u) = g + (\alpha + \varepsilon)z(\vartheta_t\omega) + (\gamma - \alpha^2 - \alpha\varepsilon)\Delta z(\vartheta_t\omega), \end{aligned} \tag{4.16}$$

with the initial data $(u^b, v^b)|_{t=0} = (0, 0)$ and homogeneous boundary conditions.

Lemma 4.2. *Assume $g \in H^{-1}(0, l)$, $Y_0 = (u_0, v_0) \in B(\omega) \in \mathcal{D}$ and* (3.4) *and* (3.5) *hold. Then*

$$||Y^a(t, \vartheta_{-t}\omega, Y_0(\vartheta_{-t}\omega))||^2_{E_0} \leq C||Y_0(\vartheta_{-t}\omega)||^2_{E_0} e^{-\lambda t}.$$

Proof. Taking the inner product of (4.18) with $(-\Delta)^{-1}v^a$ and using $v^a = u_t^a + \varepsilon u^a$, we have

$$\frac{d}{dt}\phi_1(t, \omega) + H_1(t, \omega) = 0, \tag{4.17}$$

where

$$\phi_1(t, \omega) = \frac{1}{2}(||v^a||^2_{-1,2} + \alpha||v^a||^2 + \varepsilon^2||u^a||^2_{-1,2} + (1 + \alpha\varepsilon^2 - \gamma\varepsilon)||u^a||^2 + \beta||\nabla u^a||^2)$$

and

$$\begin{aligned} H_1(t, \omega) = &- \varepsilon||v^a||^2_{-1,2} + (\gamma - \alpha\varepsilon)||v^a||^2 + \varepsilon^3||u^a||^2_{-1,2} + (1 + \alpha\varepsilon^2 - \gamma\varepsilon)||u^a||^2 \\ &+ \beta\varepsilon||\nabla u^a||^2. \end{aligned}$$

By (4.4), we have

$$\begin{aligned} &H_1(t, \omega) - \delta\varepsilon\phi_1(t, \omega) \\ = &- \varepsilon(1 + \frac{1}{2}\delta)||v^a||^2_{-1,2} + (\gamma - \alpha\varepsilon - \frac{1}{2}\delta\alpha\varepsilon)||v^a||^2 \\ &+ \varepsilon^3(1 - \frac{1}{2}\delta)||u^a||^2_{-1,2} + \varepsilon(1 + \alpha\varepsilon^2 - \gamma\varepsilon)(1 - \frac{1}{2}\delta)||u^a||^2 \\ &+ \beta\varepsilon(1 - \frac{1}{2}\delta)||\nabla u^a||^2 \geq 0. \end{aligned} \tag{4.18}$$

By (4.17) and (4.18), we have

$$\frac{d}{dt}\phi_1(t, \omega) + \lambda\phi_1(t, \omega) \leq 0,$$

where $\lambda = \delta\varepsilon > 0$. Applying Gronwall's lemma, we obtain for all $t \geq 0$

$$\phi_1(t, \omega) \leq \phi_1(0, \omega)e^{-\lambda t}.$$

By arguments similar to (4.6), we can derive that

$$a||Y^a(t, \omega, Y_0(\omega))||^2_{E_0} \leq b||Y_0(\omega)||^2_{E_0} e^{-\lambda t}, \tag{4.19}$$

where a, b are same as in Lemma 4.1. Replacing ω by $\vartheta_{-t}\omega$ with $t \geq 0$ in (4.19), implies that the result holds. □

Lemma 4.3. *Assume $g \in H^{-1}(0, l)$, $Y_0 = (u_0, v_0) \in B(\omega) \in \mathcal{D}$ and* (3.4) *and* (3.5) *hold. Then*

$$a||Y^b(t, \vartheta_{-t}\omega, Y_0(\vartheta_{-t}\omega))||^2_{E_1} \leq 2R^2_*(\omega),$$

where $R^2_*(\omega) = c_{**}(1 + \rho_1^{2p+2}(\omega) + \rho_2^2(\omega))$ and c_{**} is a deterministic positive constant.

Proof. Taking the inner product of (4.16) with v^b and using (4.15), we obtain

$$\frac{d}{dt}\phi_2(t,\omega)+H_2(t,\omega)=0, \tag{4.20}$$

where

$$\begin{aligned}\phi_2(t,\omega)=&\frac{1}{2}[||v^b||^2+\alpha||\nabla v^b||^2+\varepsilon^2||u^b||^2\\&+(1+\alpha\varepsilon^2-\gamma\varepsilon)||u||^2+\beta||\Delta u^b||^2]\end{aligned} \tag{4.21}$$

and

$$\begin{aligned}H_2(t,\omega)=&-\varepsilon||v^b||^2+(\gamma-\alpha\varepsilon)||\nabla v^b||^2+\varepsilon^3||u^b||^2+\beta\varepsilon||\Delta u^b||^2\\&+\varepsilon(1+\alpha\varepsilon^2-\gamma\varepsilon)||u^b||^2+(1+\alpha\varepsilon^2-\gamma\varepsilon)(\Delta u^b,z(\vartheta_t\omega))\\&-\beta(\Delta u^b,\Delta z(\vartheta_t\omega))-\varepsilon^2(u^b,z(\vartheta_t\omega))-(\Delta f(u),v^b)\\&-(g+(\alpha+\varepsilon)z(\vartheta_t\omega)+(\gamma-\alpha^2-\alpha\varepsilon)\Delta z(\vartheta_t\omega),v^b).\end{aligned} \tag{4.22}$$

Note that

$$|-(\Delta f(u),v^b)|=|(f'(u)\nabla u,\nabla v^b)|\leq\frac{\alpha\varepsilon}{2}||\nabla v^b||^2+c||\nabla u||^2$$

and

$$\begin{aligned}&|-((\alpha+\varepsilon)z(\vartheta_t\omega)+(\gamma-\alpha^2-\alpha\varepsilon)\Delta z(\vartheta_t\omega),v^b)|\\\leq&\varepsilon||v^b||^2+c(||z(\vartheta_t\omega)||^2+||\Delta z(\vartheta_t\omega)||^2).\end{aligned}$$

Similar to the arguments used in lemma 4.1, we can get

$$H_2(t,\omega)-\lambda\phi_2(t,\omega)\geq-c(||z(\vartheta_t\omega)||^2+||\Delta z(\vartheta_t\omega)||^2+||g||^2_{-1,2}+||\nabla u||^2). \tag{4.23}$$

It follows from (4.21) and (4.23) that

$$\frac{d}{dt}\phi_2(t,\omega)+\lambda\phi_2(t,\omega)\leq c_6p_2(\vartheta_t\omega)+c_7(1+||\nabla u||^2), \tag{4.24}$$

where $p_2(\vartheta_t\omega)=||z(\vartheta_t\omega)||^2+||\Delta z(\vartheta_t\omega)||^2$. By Gronwall inequality, we obtain

$$\phi_2(t,\omega)\leq c_6\int_0^t e^{\lambda(s-t)}p_2(\vartheta_s\omega)ds+c_7\int_0^t e^{\lambda(s-t)}||\nabla u(s)||^2ds+\frac{1}{\lambda}c_7. \tag{4.25}$$

Noting $\phi_2(t,\omega)\geq a||Y^b(t,\omega)||^2_{E_1}$ and replacing ω by $\vartheta_{-t}\omega$, we have

$$\begin{aligned}&a||Y^b(t,\vartheta_{-t}\omega)||^2_{E_1}\\\leq&c_6\int_0^t e^{\lambda(s-t)}p_2(\vartheta_{s-t}\omega)ds+c_7\int_0^t e^{\lambda(s-t)}||\nabla u(s,\vartheta_{-t}\omega)||^2ds+\frac{1}{\lambda}c_7\\\leq&c_6\int_{-t}^0 e^{\lambda s}p_2(\vartheta_s\omega)ds+c_7\int_0^t e^{\lambda(s-t)}||\nabla u(s,\vartheta_{-t}\omega)||^2ds+\frac{1}{\lambda}c_7.\end{aligned} \tag{4.26}$$

By Corollary lemma 3.2, the first term on the right-hand side of (4.26) satisfies

$$c_6\int_{-t}^0 e^{\lambda s}p_2(\vartheta_s\omega)ds\leq c_6\int_{-t}^0 e^{\frac{1}{2}\lambda s}\rho_2^2(\omega)ds\leq\frac{2}{\lambda}c_6\rho_2^2(\omega)\text{ for }t\geq0. \tag{4.27}$$

By (4.11), we have

$$\begin{aligned}||\nabla u(s,\vartheta_{-t}\omega)||^2 \leq & \frac{b}{a}e^{-\lambda s}[||Y_0(\vartheta_{-t}\omega)||^2_{E_0}+||u_0(\vartheta_{-t}\omega)||^{2p+2}_{2p+2}] \\ & +\frac{1}{a}c_*[1+\rho_1^{2p+2}(\omega)+\rho_2^2(\omega)].\end{aligned} \tag{4.28}$$

Thus, the second term on the right-hand side of (4.26) satisfies for $t \geq 0$

$$\begin{aligned}c_7\int_0^t e^{\lambda(s-t)}||\nabla u(s,\vartheta_{-t}\omega)||^2 ds \leq & \frac{b}{a}c_7 te^{-\lambda t}[||Y_0(\vartheta_{-t}\omega)||^2_{E_0}+||u_0(\vartheta_{-t}\omega)||^{2p+2}_{2p+2}] \\ & +\frac{1}{a\lambda}c_7c_*[1+\rho_1^{2p+2}(\omega)+\rho_2^2(\omega)].\end{aligned} \tag{4.29}$$

Let $R_*^2(\omega)=\frac{1}{\lambda}(2c_6+c_7+\frac{1}{a}c_7c_*)(1+\rho_1^{2p+2}(\omega)+\rho_2^2(\omega))$. It follows from (4.26)-(4.29) that

$$a||Y(t,\vartheta_{-t}\omega)||^2_{E_1} \leq \frac{b}{a}c_7te^{-\lambda t}[||Y_0(\vartheta_{-t}\omega)||^2_{E_0}+||u_0(\vartheta_{-t}\omega)||^{2p+2}_{2p+2}]+R_*^2(\omega).$$

Since $\{B(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$ is tempered and $Y_0(\vartheta_{-t}\omega)\in B(\vartheta_{-t}\omega)$, there exists $T_B^*(\omega)>0$ such that for $t\geq T_B^*(\omega)$,

$$\frac{b}{a}c_7te^{-\lambda t}(||Y_0(\vartheta_{-t}\omega)||^2_{E_0}+||u_0(\vartheta_{-t}\omega)||^{2p+2}_{2p+2})\leq R_*^2(\omega). \tag{4.30}$$

Thus,

$$a||Y(t,\vartheta_{-t}\omega)||^2_{E_1}\leq 2R_*^2(\omega) \quad \text{for } t\geq T_B^*(\omega)$$

and the result holds. □

We are now in a position to present our main result:

Theorem 4.4. *Assume that $g\in H^{-1}(0,l)$ and (3.4) and (3.5) hold. Then the random dynamical system $S(t,\omega)$ has a unique random attractor in E_0.*

Proof. By Lemma 2.4, Lemma 4.2 and Lemma 4.3, the stochastic dynamical system $S(t,\omega)$ of nonlinear strain waves is almost surely $\mathcal{D}-\alpha-$contracting. This together with Lemma 2.5 implies that the existence of a unique $\mathcal{D}-$random attractor for $S(t,\omega)$. □

References

[1] L. Arnold, *Random Dynamical Systems*, Springer-Verlag, 1998.

[2] Peter W. Bates, K. Lu and B. Wang, *Random attractors for stochastic reaction-diffusion equations on unbounded domains*, J. Differential Equations, 246(2009), 845-869.

[3] T. Caraballo, J.A. Langa and J.C. Robinson, *A stochastic pitchfork bifurcation in a reaction-diffusion equation*, Proc. R. Soc. Lond. Ser. A, 457(2001), 2041-2061.

[4] H. Crauel, A. Debussche and F. Flandoli, *Random attractors*, J. Dynam. Differential Equations, 9(1997), 307-341.

[5] Z. Dai and B. Guo, *Long Time Bahavior of Nonlinear Strain Waves in Elastic Waveguides*, J.Partial Diff. Eqs., 12:4(1999), 301-312.

[6] Z. Dai and X. Du, *Global attractor for the nonlinear strain waves in elastic waveguides*, Acta mathematicae applicatae sinica, 17:2(2001), 260-270.

[7] A. Debussche, *Hausdorff dimension of a random invariant set*, J. Math. Pures Appl., 77(1998), 967-988.

[8] K. Deimling, *Nonlinear functional analyis*, Springer, Berlin, 1985.

[9] F. Flandoli and B. Schmalfuss, *Random attractors for the 3D stochastic Navier-Stokes equation with multiplicative noise*, Stoch. Stoch. Rep., 59(1996), 21-45.

[10] Y. Li and B. Guo, *Random attractor for quasi-continous random dynamical systems and applications to stochastic reaction-diffusion equations*, J. Differential Equations, 245(2008), 1175-1800.

[11] Y. Lv and W. Wang, *Limiting dynamics for stochastic wave equations*, J. Differential Equations, 244(2008), 1-23.

[12] G A. Mangin, *Physical and mathematical models of nonlinear waves in solids*, Jeffrey, Engelbrecht eds, Amsterdam et al., Elsevier, 1994.

[13] A. M. Samsonov and E V. Sokurinskaya,*Energy exchange between nonlinear wave in elastic waveguides and extern media*, Nonlinear wave in Active amedia, Springer-Verlag, Berlin, 1989.

[14] A. M. Samsonov, *Nonlinear strain wave in elastic waveguides*, Nonlinear Waves in Solids A, Jeffrey, J Engelbrecht, 1994.

[15] A. M. Samsonov and E V. Sokurinkaya, *On the excitation of a longitudinal deformation soliton in a nonlinear elastic rod*, Sov Phys-TechPhys, 33:8(1988), 989-991.

[16] M. Scheutzow,*Comparison of various concepts of a random attractor: A case study*, Arch. Math. (Basel), 78(2002), 233-240.

[17] S. Zhou, *On dimension of the global attractor for damped nonlinear wave equations*, J. Math. Phys., 40(1999), 1432-1438.

[18] R. Temam, *Infinite-Dimensional Systems in Mechanics and Physics*, New York, Springer-Verlag, 1988.

[19] J.M. Ghidaglia, *A note on the strong convergence towards attractors for damped forced KdV equations*, J. Differential Equations, 110(1994), 356-359.