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NEW ITERATIVE METHODS FOR SOLVING LINEAR SYSTEMS

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Abstract In this paper, we introduce some new iterative methods to solve linear systems $Ax = b$. We show that these methods, comparing to the classical Jacobi or Gauss-Seidel method, can be applied to more systems and have faster convergence.

Keywords Linear systems, Jacobi method, Gauss-Seidel method, convergence.

MSC(2000) 15A06.

1. Introduction

Let R^n be n-dimensional vector space with real vectors and $M_n(R)$ be the linear space of all real $n \times n$ matrices. Consider the following linear system

$$Ax = b, \tag{1.1}$$

where $A \in M_n(R)$ is invertible, and $x, b \in R^n$. Let $A = M - E$, where M is invertible, be a splitting of A. Then the iterative method based on this splitting is given by

$$x^{(k+1)} = Tx^{(k)} + M^{-1}b,\ k = 0, 1, 2, \cdots, \tag{1.2}$$

where $T = M^{-1}E$. It is well known that the iteration (1.2) converges to the solution of (1.1) if and only if the spectral radius $\rho(T)$ of T satisfies $\rho(T) < 1$ (see, e.g. [8, 11]). One sufficient condition for $\rho(T) < 1$ is $||T|| < 1$, where $||T||$ is a norm of T. In general, a smaller norm will results in a smaller spectral radius. If we write $A = D-L-U$, where D =diag(A), $-L$ the strictly lower and $-U$ the strictly upper triangular matrices of A, respectively, then the classical Jacobi iterative method is defined if $M = D$, and the Gauss-Seidel iterative method is defined if $M = D - L$.

Definition 1.1. Let $A = (a_{ij}) \in M_n(R)$, then the matrix A is called diagonally dominant if

$$\sum_{j=1, j\neq i}^{n} |a_{ij}| \leq |a_{ii}|, \quad i = 1, 2, \cdots, n;$$

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strictly diagonally dominant if

$$\sum_{j=1, j\neq i}^{n} |a_{ij}| < |a_{ii}|, \quad i = 1, 2, \cdots, n;$$

and irreducibly diagonally dominant if A is irreducible and diagonally dominant with strict inequality for at least one i.

The following result regarding Jacobi and Gauss-Seidel methods can be found in [8, 11].

Theorem 1.1. *If A is strictly diagonally dominant, or irreducibly diagonally dominant, then both Jacobi and Gauss-Seidel methods converge regardless of the choice of the initial guess $x^{(0)}$.*

Research has been done extensively on Jacobi method to either expand the systems that the Jacobi method can be applied or to improve the rate of convergence. Gauss-Seidel method, Successive Over relaxation method, and many Jacobi or Gauss-Seidel type preconditioned iterative methods have been proposed and studied, see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] and references therein. In this research, we use a different approach than preconditioning to propose a new scheme that not only extend the application but also improve the rate of convergence.

2. Preliminaries

To prove our main results, we need the following lemmas. Note that $||\cdot||$ represents a vector norm in R^n and the induced matrix norm in $M_n(R)$.

Lemma 2.1. *Let $x_1, x_2, \cdots, x_m \in R^n$, $b_1, b_2, \cdots, b_m \in R$ and $b_i > 0$. If*

$$||b_1x_1 + b_2x_2 + \cdots + b_mx_m|| = (b_1 + b_2 + \cdots + b_m) \max_{1\leq i\leq n} ||x_i||, \tag{2.1}$$

then $x_1 = x_2 = \cdots = x_m$. In particular, if $x_1, x_2, \cdots, x_m \in C$, the set of complex numbers, and if

$$|b_1x_1 + b_2x_2 + \cdots + b_mx_m| = (b_1 + b_2 + \cdots + b_m) \max_{1\leq i\leq n} |x_i|,$$

then $x_1 = x_2 = \cdots = x_m$.

Proof. If (2.1) holds, then we have

$$||c_1x_1 + c_2x_2 + \cdots + c_mx_m|| = \max_{1\leq i\leq n} ||x_i||,$$

where $c_i = b_i/(b_1 + b_2 + \cdots + b_m) > 0$ and $\sum_{i=1}^{m} c_i = 1$. The lemma follows immediately from the convex set theory. □

Lemma 2.2. *Let matrix $B \in M_n(R)$ be in the form*

$$B = \begin{bmatrix} -b_{11} & -b_{12} & -b_{13} & \cdots & -b_{1n} \\ b_{21} & 0 & b_{23} & \cdots & b_{2n} \\ b_{31} & b_{32} & 0 & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & 0 \end{bmatrix}$$

where $b_{ij} > 0$, $\sum_{j=1}^{n} b_{1j} \leq 1$, *and* $\sum_{j=1, j\neq i}^{n} b_{ij} \leq 1, i = 2, \cdots, n$. *Then* $\rho(B) < 1$.

Proof. Obviously $\rho(B) \leq ||B||_\infty \leq 1$. If $\rho(B) = 1$, then B has an eigenvalue e^{iw} for some real number w. Let $x = (x_1, x_2, \cdots, x_n)^T$ be an eigenvector corresponding to e^{iw}. We will look at the following two different cases.

(a) $x_1 = x_2 = \cdots = x_n$. In this case, $\bar{x} = (1, 1, \cdots, 1)^T$ is also an eigenvector, thus from $B\bar{x} = e^{iw}\bar{x}$, we have

$$\begin{bmatrix} -\sum_{j=1}^{n} b_{1j} \\ \sum_{j=1, j\neq 2}^{n} b_{2j} \\ \vdots \\ \sum_{j=1, j\neq n}^{n} b_{nj} \end{bmatrix} = e^{iw} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix},$$

which yields a contradiction.

(b) $x_1, x_2, \cdots, x_n$ are not all equal. Assume $|x_m| = \max_{1\leq i\leq n} |x_i|$. If $m = 1$, the first component of $Bx = e^{iw}x$ gives

$$|b_{11}x_1 + \cdots + b_{1n}x_n| = |x_1|. \tag{2.2}$$

But

$$|b_{11}x_1 + \cdots + b_{1n}x_n| \leq (b_{11} + b_{12} + \cdots + b_{1n})|x_1| \leq |x_1|. \tag{2.3}$$

It follows from (2.2) and (2.3) that

$$|b_{11}x_1 + \cdots + b_{1n}x_n| = (b_{11} + b_{12} + \cdots + b_{1n})|x_1|$$

Lemma 2.1 implies $x_1 = x_2 = \cdots = x_n$, a contradiction. If $m > 1$, let $|x_k| = \max_{i\neq m, 1\leq i\leq n} |x_i|$. Obviously, $|x_k| \leq |x_m|$. From the mth component of the equation $Bx = e^{iw}x$, we get

$$|b_{m1}x_1 + \cdots + b_{m\ m-1}x_{m-1} + b_{m\ m+1}x_{m+1} + \cdots + b_{mn}x_n| = |x_m| \geq |x_k|.$$

But

$$|b_{m1}x_1 + \cdots + b_{m\ m-1}x_{m-1} + b_{m\ m+1}x_{m+1} + \cdots + b_{mn}x_n| \leq \left(\sum_{j=1, j\neq m}^{n} b_{mj}\right) |x_k| \leq |x_k|.$$

Therefore, $|x_k| = |x_m|$ and

$$|b_{m1}x_1 + \cdots + b_{m\ m-1}x_{m-1} + b_{m\ m+1}x_{m+1} + \cdots + b_{mn}x_n| = \left(\sum_{j=1, j\neq m}^{n} b_{mj}\right) |x_k|.$$

Lemma 2.1 means $x_1 = \cdots = x_{m-1} = x_{m+1} = \cdots = x_n$. Now (2.2) becomes

$$\left| b_{1m}x_m + \left(\sum_{j=1, j\neq m}^{n} b_{1j}\right) x_k \right| = |x_1| = |x_k| = |x_m|.$$

But

$$\left| b_{1m}x_m + \left(\sum_{j=1, j\neq m}^{n} b_{1j}\right) x_k \right| \leq \left(\sum_{j=1}^{n} b_{1j}\right) |x_m| \leq |x_m|,$$

therefore, we have

$$\left| b_{1m}x_m + \left(\sum_{j=1, j\neq m}^{n} b_{1j} \right) x_k \right| = \left(\sum_{j=1}^{n} b_{1j} \right) |x_m|.$$

Again Lemma 2.1 means $x_k = x_m$, i.e. $x_1 = x_2 = \cdots = x_n$, a contradiction. That completes the proof of the Lemma. □

3. New Iterative Methods

To solve system (1.1), we propose a new iterative method. First, we split the matrix A in the following way.

$$A = \bar{D} - \bar{E},$$

where

$$\bar{D} = \begin{bmatrix} a_{11} & & & & & \\ & a_{22} & & & & \\ & & \ddots & & & \\ a_{k1} & a_{k2} & \cdots & a_{kk} & \cdots & a_{kn} \\ & & & & \ddots & \\ & & & & & a_{nn} \end{bmatrix},$$

where $1 \leq k \leq n$, all other entries are zero, and $\bar{E} = -(A - \bar{D})$. Then the iterative method is given by

$$\text{(I)} \qquad x^{(k+1)} = \bar{T}x^{(k)} + \bar{D}^{-1}b, \;\; k = 0, 1, 2, \cdots, \tag{3.1}$$

where $\bar{T} = \bar{D}^{-1}\bar{E}$. The following theorem shows that in terms of the infinity norm, the norm of the iterative matrix of method (I) is less than or equal to the norm of the iterative matrix of the Jacobi method.

Theorem 3.1. *Assume that* $A = (a_{ij})_{n\times n}$, $\sum_{j=1,j\neq i}^{n} |a_{ij}| \leq |a_{ii}|$, $|a_{ii}| > 0$, $i = 1, 2, \cdots, n, n \geq 2$. *Let* $T = D^{-1}E$ *and* $\bar{T} = \bar{D}^{-1}\bar{E}$ *be the iteration matrices for the Jacobi method and the iterative method* (I). *Then* $||\bar{T}||_\infty \leq ||T||_\infty$.

Proof. Without loss of generality, we assume $k = 1$. Set

$$D = \begin{bmatrix} a_{11} & 0 \\ 0 & D_1 \end{bmatrix}, \qquad E = -(A - D) = -\begin{bmatrix} 0 & \alpha \\ \beta & E_1 \end{bmatrix}$$

and

$$\bar{D} = \begin{bmatrix} a_{11} & \alpha \\ 0 & D_1 \end{bmatrix}, \qquad \bar{E} = -(A - \bar{D}) = -\begin{bmatrix} 0 & 0 \\ \beta & E_1 \end{bmatrix}.$$

Then calculation shows

$$D^{-1} = \begin{bmatrix} a_{11}^{-1} & 0 \\ 0 & D_1^{-1} \end{bmatrix}, \qquad T = D^{-1}E = -\begin{bmatrix} 0 & a_{11}^{-1}\alpha \\ D_1^{-1}\beta & D_1^{-1}E_1 \end{bmatrix}$$

and

$$\bar{D}^{-1} = \begin{bmatrix} a_{11}^{-1} & -a_{11}^{-1}\alpha D_1^{-1} \\ 0 & D_1^{-1} \end{bmatrix}, \; \bar{T} = \bar{D}^{-1}\bar{E} = -\begin{bmatrix} -a_{11}^{-1}\alpha D_1^{-1}\beta & -a_{11}^{-1}\alpha D_1^{-1}E_1 \\ D_1^{-1}\beta & D_1^{-1}E_1 \end{bmatrix}.$$

Let T be partitioned into row vectors, i.e.

$$T = -\begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_n \end{bmatrix}^T.$$

It then follows that

$$\bar{T} = -\begin{bmatrix} -a_{11}^{-1}\alpha \begin{bmatrix} \beta_2 & \beta_3 & \cdots & \beta_n \end{bmatrix}^T \\ \begin{bmatrix} \beta_2 & \beta_3 & \cdots & \beta_n \end{bmatrix}^T \end{bmatrix}.$$

Since

$$\left|\left|-a_{11}^{-1}\alpha \begin{bmatrix} \beta_2 & \beta_3 & \cdots & \beta_n \end{bmatrix}^T\right|\right|_1 = \left|\left|\frac{a_{12}}{a_{11}}\beta_2 + \frac{a_{13}}{a_{11}}\beta_3 + \cdots + \frac{a_{1n}}{a_{11}}\beta_n\right|\right|_1$$
$$\leq \frac{|a_{12}| + |a_{13}| + \cdots + |a_{1n}|}{|a_{11}|} \max_{2\leq i\leq n} ||\beta_i||_1 \leq \max_{2\leq i\leq n} ||\beta_i||_1.$$

Therefore,

$$||\bar{T}||_\infty = \max_{2\leq i\leq n} ||\beta_i||_1 \leq \max_{1\leq i\leq n} ||\beta_i||_1 = ||T||_\infty,$$

completing the proof. □

Theorem 3.2. *Let $A = (a_{ij})_{n\times n},\ a_{ij} > 0,\ \sum_{j=1,j\neq i} a_{ij} \leq a_{ii},\ n \geq 3$. Then the iteration matrix $\bar{T}$ of the method* (I) *satisfies $\rho(\bar{T}) < 1$, i.e. the iterative method* (I) *converges.*

Proof. Without loss of generality, we assume $k = 1$. Then as showed in Theorem 3.1, we have

$$\bar{T} = -\begin{bmatrix} -a_{11}^{-1}\alpha^T D_1^{-1}\beta & -a_{11}^{-1}\alpha^T D_1^{-1}E_1 \\ D_1^{-1}\beta & D_1^{-1}E_1 \end{bmatrix}$$
$$\stackrel{def}{=} -\begin{bmatrix} -b_{11} & -b_{12} & -b_{13} & \cdots & -b_{1n} \\ b_{21} & 0 & b_{23} & \cdots & b_{2n} \\ b_{31} & b_{32} & 0 & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & 0 \end{bmatrix}$$

where $b_{ij} = \frac{a_{ij}}{a_{ii}},\ i = 2, 3, \cdots, n,\ j \neq i, j = 1, 2, \cdots, n$, and

$$\sum_{j=1,j\neq i}^{n} b_{ij} = \sum_{j=1,j\neq i}^{n} \frac{a_{ij}}{a_{ii}} \leq 1,\ i = 2, 3, \cdots, n.$$

Now

$$\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \end{bmatrix} = a_{11}^{-1}\alpha D_1^{-1}[\beta \ \ E_1]$$
$$= \begin{bmatrix} \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \cdots & \frac{a_{1n}}{a_{11}} \end{bmatrix} \begin{bmatrix} b_{21} & 0 & b_{23} & \cdots & b_{2n} \\ b_{31} & b_{32} & 0 & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & 0 \end{bmatrix}.$$

Obviously, $b_{1j} > 0,\ j = 1, 2, \cdots, n$ and

$$\sum_{j=1}^{n} b_{1j} = \begin{bmatrix} \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \cdots & \frac{a_{1n}}{a_{11}} \end{bmatrix} \begin{bmatrix} b_{21} & 0 & b_{23} & \cdots & b_{2n} \\ b_{31} & b_{32} & 0 & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \cdots & \frac{a_{1n}}{a_{11}} \end{bmatrix} \begin{bmatrix} \sum_{j=1, j\neq 2}^{n} b_{2j} \\ \sum_{j=1, j\neq 3}^{n} b_{3j} \\ \vdots \\ \sum_{j=1, j\neq n}^{n} b_{nj} \end{bmatrix}$$

$$\leq \begin{bmatrix} \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \cdots & \frac{a_{1n}}{a_{11}} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \sum_{j=2}^{n} \frac{a_{1j}}{a_{11}} \leq 1.$$

By Lemma 2.2, the theorem is proved. □

Remark 3.1. 1. The size n of the matrix A has to satisfy $n \geq 3$. For instance, let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Then

$$\bar{T} = \bar{D}^{-1}\bar{E} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix},$$

and $\rho(\bar{T}) = 1$.

2. The condition $a_{ij} > 0$ in the theorem is necessary. For instance, let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Then

$$\bar{D} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ and } \bar{E} = -\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$\bar{T} = \bar{D}^{-1}\bar{E} = -\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

whose eigenvalues are $0, 0, 1$, and -1. So $\rho(\bar{T}) = 1$.

3. Notice that the iterative method (I) converges even if the matrix A is just diagonally dominant.

Corollary 3.1. *Let* $A = (a_{ij})_{n\times n},\ a_{ij} > 0,\ \sum_{j=1,j\neq i} a_{ij} \le a_{ii},\ n \ge 3$. *Then* A *is invertible.*

Proof. Suppose that A is not invertible, then there exists a vector $\bar{x} \neq 0 \in R^n$ such that $A\bar{x} = (\bar{D} - \bar{E})\bar{x} = 0$. Thus, $\bar{D}^{-1}\bar{E}\bar{x} = \bar{x}$. This implies that 1 is an eigenvalue of $\bar{T}$, and $\rho(\bar{T}) \ge 1$, a contradiction. □

The following is a simplified version of the method (I) which we will call it method (II). To show that the method converges, we use a result about an a-transformation, φ_a, on vector space $M_n(R)$, introduced by Zheng and Wang [14]. In this method, we split $A = \bar{D}_1 - \bar{E}_1$ as

$$\bar{D}_1 = \begin{bmatrix} a_{11} & & & & & & \\ & \ddots & & & & & \\ & & a_{ii} & & a_{ij} & & \\ & & & \ddots & & & \\ & & & & a_{jj} & & \\ & & & & & \ddots & \\ & & & & & & a_{nn} \end{bmatrix}$$

where $1 \le i, j \le n$ and all other entries are zero, and $\bar{E}_1 = -(A - \bar{D}_1)$. Then the iterative scheme is given by

$$\text{(II)} \qquad x^{(k+1)} = \bar{T}_1 x^{(k)} + \bar{D}_1^{-1} b,\ k = 0, 1, 2, \cdots, \tag{3.2}$$

where $\bar{T}_1 = \bar{D}_1^{-1}\bar{E}_1$. We have the following result.

Theorem 3.3. *Let* $A = (a_{ij})_{n\times n},\ a_{ij} > 0,\ \sum_{j=1,j\neq i} a_{ij} \le a_{ii},\ n \ge 3$. *Then the iteration matrix* $\bar{T}_1$ *of the method* (II) *satisfies* $\rho(\bar{T}_1) < 1$, *i.e. the iterative method* (II) *converges.*

Proof. Without loss of generality, we assume $i = 1, j = 2$. Then we have

$$\bar{T}_1 = \begin{bmatrix} -\frac{a_{12}a_{21}}{a_{11}a_{22}} & 0 & \frac{a_{13}}{a_{11}} - \frac{a_{12}a_{23}}{a_{11}a_{22}} & \cdots & \frac{a_{1n}}{a_{11}} - \frac{a_{12}a_{2n}}{a_{11}a_{22}} \\ \frac{a_{21}}{a_{22}} & 0 & \frac{a_{23}}{a_{22}} & \cdots & \frac{a_{2n}}{a_{22}} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ \frac{a_{n1}}{a_{nn}} & \frac{a_{n2}}{a_{nn}} & \frac{a_{n3}}{a_{nn}} & \cdots & 0 \end{bmatrix}.$$

Obviously, the sum of the absolute values of the entries of the ith row, $i = 2, 3, \cdots, n$, is less than or equal to 1. The entries of the first row satisfy

$$\begin{aligned}
&\left|-\frac{a_{12}a_{21}}{a_{11}a_{22}}\right| + \left|\frac{a_{13}}{a_{11}} - \frac{a_{12}a_{23}}{a_{11}a_{22}}\right| + \cdots + \left|\frac{a_{13}}{a_{11}} - \frac{a_{12}a_{2n}}{a_{11}a_{22}}\right| \\
<&\frac{a_{12}}{a_{11}a_{22}}(a_{21} + a_{23} + \cdots + a_{2n}) + \frac{a_{13}}{a_{11}} + \frac{a_{14}}{a_{11}} + \cdots + \frac{a_{1n}}{a_{11}} \\
\leq&\frac{a_{12}}{a_{11}} + \frac{a_{13}}{a_{11}} + \cdots + \frac{a_{1n}}{a_{11}} \leq 1.
\end{aligned} \tag{3.3}$$

Since $\frac{a_{i1}}{a_{ii}} > 0$, $i = 2, 3, \cdots, n$, it follows that the 1-transformation φ_1 satisfies $\varphi_1^n(\bar{T}_1) = 0$. Therefore, $\rho(\bar{T}_1) < 1$ by Theorem 3, [14]. □

The following theorem shows that the requirement of $a_{ij} > 0$ in Theorem 3.4 is not necessary.

Theorem 3.4. *Let $A = (a_{ij})_{n\times n}, n \geq 3$, satisfy*

(a) $\sum_{j=1, j\neq i} |a_{ij}| \leq |a_{ii}|$, $a_{ii} \neq 0$, $\quad i = 1, 2, \cdots, n.$

(b) *There exist three different indexes p, q, r such that $a_{ip} \neq 0$, $i = 1, 2, \cdots, n$, and $a_{pq}a_{pr}a_{qq}a_{qr} > 0$.*

If $\bar{D}_1$ and $\bar{E}_1$ are chosen such that

$$\bar{D}_1 = \begin{bmatrix}
a_{11} & & & & & & \\
& \ddots & & & & & \\
& & a_{pp} & & a_{pq} & & \\
& & & \ddots & & & \\
& & & & a_{qq} & & \\
& & & & & \ddots & \\
& & & & & & a_{nn}
\end{bmatrix}$$

and the other entries are zero, and $\bar{E}_1 = A - \bar{D}_1$. Then $\rho(\bar{D}_1^{-1}\bar{E}_1) < 1$.

Proof. The proof of the theorem follows immediately by noticing that the strict inequality (3.3) holds if condition (b) holds. □

Remark 3.2. The following matrix A shows that what condition (b) means.

$$A = \begin{bmatrix}
a_{11} & & a_{1p} & & & & & & \\
& \ddots & \vdots & & & & & & \\
& & a_{pp} & & a_{pq} & & a_{pr} & & \\
& & & \ddots & & & & & \\
& & \vdots & & a_{qq} & & a_{qr} & & \\
& & & & & \ddots & & & \\
& & \vdots & & & & a_{rr} & & \\
& & & & & & & \ddots & \\
& & a_{np} & & & & & & a_{nn}
\end{bmatrix},$$

where the entries shown are nonzero entries.

Example: Let A be the following matrix.

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Then A is not strictly diagonally dominant, nor is irreducibly diagonally dominant. Jacobi or Gauss-Seidel method may not converge. But A satisfies all conditions in Theorem 3.5. If the split given in Theorem 3.5 is performed, it gives $\rho(\bar{T}_1) = 1/2$, the method is convergent.

4. Conclusions

In this paper, we propose some new methods for solving linear systems based on the classical Jacobi method. Comparing to Gauss-Seidel method, SOR method and other preconditioning methods, these methods are easy to construct and the conditions for convergence are easy to check. The proposed new methods are convergent as long as the coefficient matrix is diagonally dominant, while the classical methods require that the matrix be either strictly diagonally dominant or irreducibly diagonally dominant. While the norm of the iterative matrix can be used to give a sufficient condition for convergence, we show that the infinity norm of the iterative matrix of the new methods are less than or equal to that of the iterative matrix of the Jacobi method.

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