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DOUBLE HOPF BIFURCATION AND CHAOS IN LIU SYSTEM WITH DELAYED FEEDBACK

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Abstract In this paper, we consider the stability of equilibria, Hopf and double Hopf bifurcation in Liu system with delay feedback. Firstly, we identify the critical values for stability switches and Hopf bifurcation using the method of bifurcation analysis. When we choose appropriate feedback strength and delay, two symmetrical nontrivial equilibria of Liu system can be controlled to be stable at the same time, and the stable bifurcating periodic solutions occur in the neighborhood of the two equilibria at the same time. Secondly, by applying the normal form method and center manifold theory, the normal form near the double Hopf bifurcation, as well as classifications of local dynamics are analyzed. Furthermore, we give the bifurcation diagram to illustrate numerically that a family of stable periodic solutions bifurcated from Hopf bifurcation occur in a large region of delay and the Liu system with delay can appear the phenomenon of "chaos switchover".

Keywords Liu system, Chaos, Hopf bifurcation, Double Hopf bifurcation, Delayed feedback control.

MSC(2000) 93C10, 37G10, 34C28, 37G05.

1. Introduction

As one of the most fascinating nonlinear phenomena, in the last four decades chaos has been extensively studied in the field of mathematics, physics, astronomy, etc. The Lorenz chaotic attractor was discovered in a three-dimensional autonomous system in 1963 [16]. Another famous three-dimensional chaotic system, Chen system, which is not topologically equivalent to the Lorenz system, was constructed in 1999 [3]. In 2002, the Lü system [17] which represents the transition between the Lorenz system and the Chen system was reported. Afterwards, the so-called unified system [18] and the generalized Lorenz canonical form [2] were presented.

In 2004, Liu *et al.* [15] proposed the following three-dimensional quadratic chaotic system called Liu system:

$$\begin{cases} \dot{x} = a(y-x), \\ \dot{y} = bx - kxz, \\ \dot{z} = -cz + hx^2, \end{cases} \tag{1.1}$$

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where a, b, c, h and k are parameters. They discussed some basic dynamical properties of Liu system and numerically studied the continuous spectrum and chaotic behaviors of this new butterfly attractor. From then on, a lot of researches have been done on the Liu system. For example, Matouk [20] used linear feedback control technique to stabilize and synchronize the chaotic Liu system. Zhu and Chen [34] proposed three feedback control strategies of Liu system to its unstable equilibria and compared the effectiveness of these strategies. Xu *et al.* [32] analyzed the stability of impulsive control Liu system and gave some sufficient conditions which guaranteed the global asymptotical stability for the controlled system.

Delayed feedback control (DFC) method has been receiving considerable attention recently since it was proposed by Pyragas [21, 22]. It provides an alternative effective method for feedback control of chaos. The basic idea of DFC is to realize a continuous control for a dynamical system by applying a feedback signal which is proportional to the difference between the dynamical variable $X(t)$ and its delayed value. In other words, we use a perturbation of the form $F(t) = K[X(t)-X(t-\tau)]$, where K is feedback strength and τ is time delay. DFC method has been successfully applied to many practical chaotic systems [12–17].

Recently, much attention has been focused on high-codimensional bifurcations, since they may reveal some complex dynamical behaviors, such as quasi-periodic solutions and chaos [18–20]. The normal form method can be used to analyze the dynamical behaviors of systems, therefore, it has been applied effectively in the study of singularities of vector fields [21–29].

In this paper, we consider Liu system with delay as follows using the DFC method:

$$\begin{cases} \dot{x} = a(y-x) + K[x(t-\tau) - x(t)], \\ \dot{y} = bx - kxz, \\ \dot{z} = -cz + hx^2. \end{cases} \tag{1.2}$$

There exist Hopf bifurcations and double Hopf bifurcations in system (1.2). Using the normal form theory and center manifold theorem, we obtain the stability of bifurcating periodic solutions and the direction of Hopf bifurcation, and derive the normal forms of double Hopf bifurcation and their unfolding with perturbation parameters. There are the same characteristic equation of linearized system for two nontrivial symmetrical equilibria, therefore, for fixed feedback strength, we obtain the region of delay in which the two unstable equilibria can be controlled to be stable at the same time, and we also obtain the critical values of delay near which stable bifurcation periodic solutions occur at the two equilibria at the same time. Furthermore, we analyze the dynamical behaviors in Liu system (1.2) near the double Hopf bifurcation point. We show the coexistence of a pair of stable periodic solutions, a pair of unstable periodic solutions, a pair of stable quasi-periodic solutions, or a pair of unstable quasi-periodic solutions.

Moreover, our concern is that if the family of stable bifurcating periodic solutions bifurcated from Hopf bifurcation occur in a large region of delay and the controlled system can appear chaos again. By numerical simulation, we show the specific regions in which a family of stable bifurcating periodic solutions occur, and the phenomenon of "chaos switchover" existed in Liu system (1.2).

The rest of the paper is organized as follows. In section 2, we consider the stability of equilibria and bifurcating periodic solutions, and the direction of Hopf bifurcation in Liu system (1.2). In section 3, we derive the normal forms of double Hopf bifurcation and their unfolding with perturbation parameters near the double Hopf bifurcation point.

2. Hopf bifurcation

2.1. Stability of equilibria

In this section, system (1.2) is considered. First of all, we determine the equilibria of this system. System (1.2) has three equilibria:

$$E_1 = (x_1, y_1, z_1) = (0,0,0), \quad E_{2,3} = (x_{2,3}, y_{2,3}, z_{2,3}) = (\pm\sqrt{\frac{bc}{hk}}, \pm\sqrt{\frac{bc}{hk}}, \frac{b}{k}).$$

We obtain the characteristic equation of linearized system for E_1 as follows:

$$\lambda^3 + (a+K+c)\lambda^2 + (ac+Kc-ab)\lambda - abc - K(\lambda^2 + c\lambda)e^{-\lambda\tau} = 0. \tag{2.1}$$

When $\tau = 0$, (2.1) becomes

$$\lambda^3 + (a+c)\lambda^2 + (ac-ab)\lambda - abc = 0. \tag{2.2}$$

When $a,\ b,\ c > 0$, $\lambda = 0$ is not the root of (2.1), and (2.2) have two roots with negative real parts and one root with positive real part, and the equilibrium E_1 is unstable for all $\tau \geq 0$.

We obtain the same characteristic equation of linearized system for E_2 and E_3 as follows:

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 - K(\lambda^2 + c\lambda)e^{-\lambda\tau} = 0, \tag{2.3}$$

where $a_2 = c + a + K$, $a_1 = ac + cK$, $a_0 = 2abc$. When $\tau = 0$, (2.3) becomes

$$\lambda^3 + (a+c)\lambda^2 + ac\lambda + 2abc = 0. \tag{2.4}$$

By Routh–Hurwitz criterion, if

$$\begin{cases} a + c > 0, \\ (a+c)ac - 2abc > 0, \end{cases}$$

all the roots of equation (2.4) have negative real parts, and the equilibria E_2 and E_3 are stable.

Let $\lambda = \mathrm{i}\omega$ ($\omega > 0$) be a root of (2.3). Applying the analysis results of [4], we give the following assumptions:

(H1) $a + c \leq 0$ or $a^2c + ac^2 - 2abc \leq 0$; $abc \neq 0$; $a + c - 2b \neq 0$.

(H2) $Z^* > 0$; $h(Z^*) < 0$.

where $c_2 = a_2^2 - 2a_1 - K^2$, $c_1 = a_1^2 - 2a_0a_2 - K^2c^2$, $c_0 = a_0^2$, $h(Z) = Z^3 + c_2Z^2 + c_1Z + c_0$, $Z^* = \dfrac{-c_2 + \sqrt{c_2^2 - 3c_1}}{3}$.

Under (H1), the equilibria E_2 and E_3 of system (1.2) with $\tau = 0$ are unstable. Under (H2), $h(Z) = 0$ has two positive roots Z_1 and Z_2. Suppose $Z_1 < Z_2$, then $h'(Z_1) < 0$, $h'(Z_2) > 0$. Therefore, similar to [4], note that $\omega_l = \sqrt{Z_l}$, $(l = 1,2)$, we get

$$\tau_l^{(j)} = \begin{cases} \frac{1}{\omega_l}[\arccos(P) + 2j\pi], & Q \geq 0, \\ \frac{1}{\omega_l}[2\pi - \arccos(P) + 2j\pi], & Q < 0, \end{cases} \tag{2.5}$$

where

$$Q = \sin(\omega_l \tau_l) = \frac{-\omega_l^4 + (a_1 - a_2 c)\omega_l^2 + a_0 c}{K\omega_l^3 + c^2 K\omega_l}, \ P = \cos(\omega_l \tau_l) = \frac{(a_2 - c)\omega_l^2 + a_1 c - a_0}{K\omega_l^2 + c^2 K}.$$

Lemma 2.1. *If (H1) and (H2) hold, when $\tau = \tau_l^{(j)}$ $(l = 1, 2;\ j = 0, 1, 2\cdots)$, then (2.3) has a pair of pure imaginary roots $\pm \mathrm{i}\omega_l$, and all the other roots of (2.3) have nonzero real parts.*

Furthermore, let $\lambda(\tau) = \alpha(\tau) + \mathrm{i}\omega(\tau)$ be the root of (2.3) satisfying $\alpha(\tau_l^{(j)}) = 0$, $\omega(\tau_l^{(j)}) = \omega_l$ $(l = 1, 2;\ j = 0, 1, 2\cdots)$.

Lemma 2.2. *If (H2) holds, then we have the following transversality conditions: $\alpha'(\tau_1^{(j)}) < 0$, $\alpha'(\tau_2^{(j)}) > 0$, where $j = 0, 1, 2\cdots$.*

Proof. Substituting $\lambda(\tau)$ into (2.3) and taking the derivative with respect to τ, we get

$$(\alpha'(\tau))^{-1}_{\tau=\tau_l^{(j)}} = \frac{Z_l}{\Lambda}(3Z_l^2 + 2c_2 Z_l + c_1) = \frac{Z_l}{\Lambda} h'(Z_l),$$

where $\Lambda = c^2 K^2 \omega_l^4 + K^2 \omega_l^6 > 0$ $(l = 1, 2)$. For $Z_l > 0$, $\alpha'(\tau_l^{(j)})$ and $h'(Z_l)$ have the same signs. Note that $h'(Z_1) < 0$ and $h'(Z_2) > 0$, then the proof is complete. □

Theorem 2.1. *For system (1.2),*

(1) Equilibrium E_1 is unstable for all $\tau \geq 0$ when $a,\ b,\ c > 0$.

(2) If (H1) and (H2) hold, then system (1.2) undergoes Hopf bifurcations at the equilibria E_2 and E_3 at the same time when $\tau = \tau_l^{(j)}(l = 1, 2;\ j = 0, 1, 2\cdots)$.

(a) If $\tau_2^{(0)} < \tau_1^{(0)}$, then equilibria E_2 and E_3 of system (2) are unstable for $\tau \geq 0$.

(b) If $\tau_1^{(0)} < \tau_2^{(0)}$, then there exists $m \in N$ such that $\tau_1^{(0)} < \tau_2^{(0)} < \tau_1^{(1)} < \tau_2^{(1)} < \cdots < \tau_1^{(m)} < \tau_2^{(m)} < \tau_2^{(m+1)} < \tau_1^{(m+1)}$, and equilibria E_2 and E_3 of system (2) are unstable for $\tau \in [0, \tau_1^{(0)}) \cup \bigcup_{l=1}^m (\tau_2^{(l-1)}, \tau_1^{(l)}) \cup (\tau_2^{(m)}, +\infty)$ and asymptotically stable for $\tau \in \bigcup_{l=0}^m (\tau_1^{(l)}, \tau_2^{(l)})$.

Let $a = 10$, $b = 40$, $c = 2.5$, $h = 4$, $k = 1$, which satisfy the assumptions (H1) and (H2), and system (1.1) has a butterfly-shaped attractor [15] (see Figure 1). Under these parameters, $E_2 = (5, 5, 40)$ and $E_3 = (-5, -5, 40)$.

When $K \in (-\infty, -38.9549) \cup (5.9825, +\infty)$, (H2) holds, and $h(Z) = 0$ has two positive real roots. We plot the bifurcation diagram for the feedback strength and the time delay (see Figure 2).

Let $K = 15 \in (-\infty, -38.9549) \cup (5.9825, +\infty)$. By (2.5) and Lemma 2.2, we get

$w_1 = 6.7962, \quad \tau_1^{(j)} = 0.3882 + 0.9245j,\ j = 0, 1, 2\cdots, \quad \alpha'(\tau_1^{(j)}) < 0,$

$w_2 = 12.0402, \quad \tau_2^{(j)} = 0.4664 + 0.5219j,\ j = 0, 1, 2\cdots, \quad \alpha'(\tau_2^{(j)}) > 0.$

Especially, $\tau_1^{(0)} = 0.3882 < \tau_2^{(0)} = 0.4664 < \tau_2^{(1)} = 0.9883 < \tau_1^{(1)} = 1.3127$. By Theorem 2.3, we know that the equilibria E_2 and E_3 are unstable for $\tau \in [0, \tau_1^{(0)}) \cup (\tau_2^{(0)}, +\infty)$ and stable for $\tau \in (\tau_1^{(0)}, \tau_2^{(0)})$. We give the numerical simulation for E_2 and E_3 when $\tau = 0.42 \in (\tau_1^{(0)} = 0.3882, \tau_2^{(0)} = 0.4664)$, and the equilibria E_2 and E_3 are both asymptotically stable (see Figure 3), and the waveform plot for E_2 and E_3 overlap in $z-$axis.

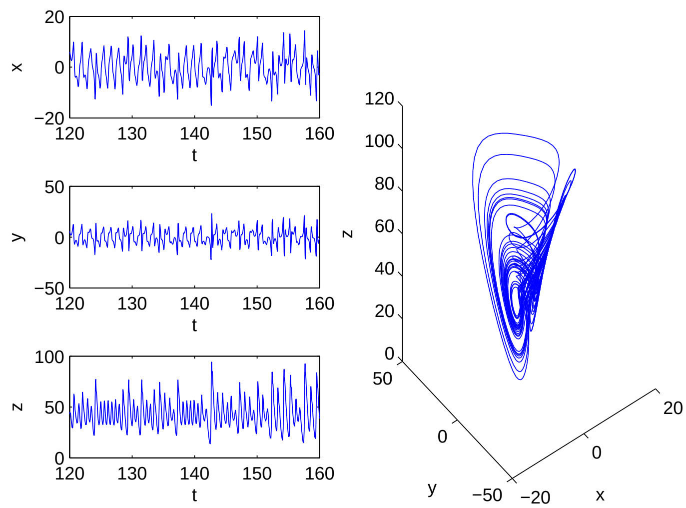

Figure 1. Waveform plot and phase for Liu system (1.2) with $\tau = 0$: there is a chaotic attractor.

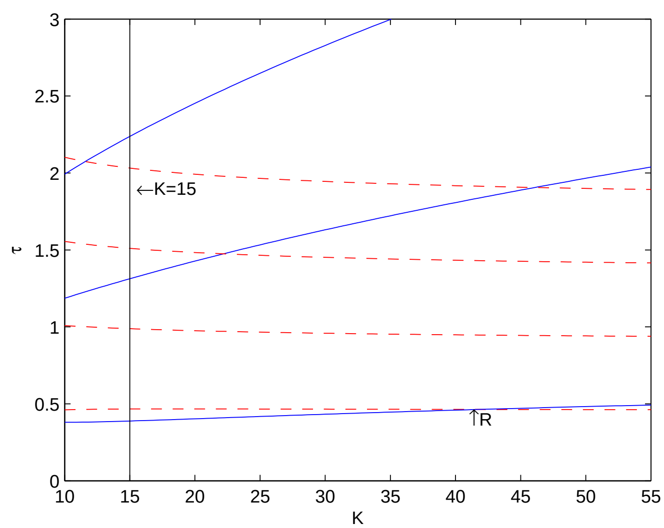

Figure 2. (Color online) Bifurcation diagram for the feedback strength K and the time delay τ, where $\tau_1^{(j)}$ $(j = 0, 1, 2)$ (blue lines) and $\tau_2^{(j)}$ $(j = 0, 1, 2, 3)$ (red dashed lines) are Hopf bifurcation critical curves in respect to K.

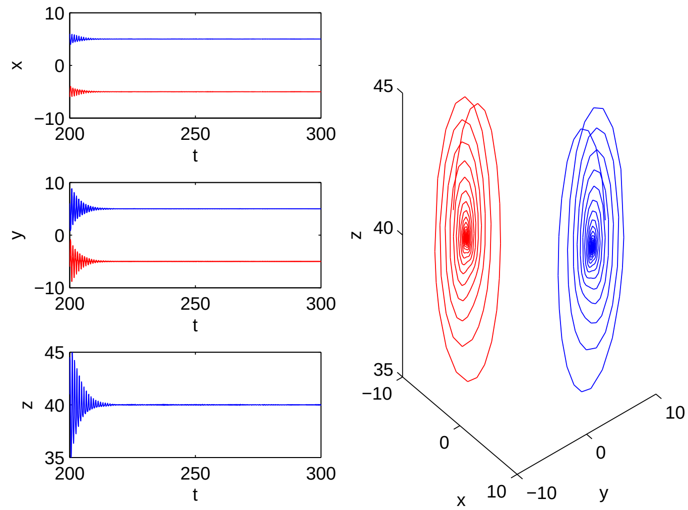

Figure 3. Waveform plots and phases for Liu system (1.2) with $\tau = 0.42$ for E_2 and E_3 respectively: two asymptotically stable equilibria are coexisted.

2.2. Stability and direction of Hopf bifurcation

In this section, according to the theories of [11, 10], we obtain the direction of Hopf bifurcation and the stability of bifurcating periodic solutions on the center manifold. Without loss of generality, we denote the critical value $\tau = \tau^*$, at which system (1.2) undergoes a Hopf bifurcation at equilibrium (x^*, y^*, z^*).

We first let $\mu = \tau - \tau^*$, then rescale the time by $t \mapsto (t/\tau)$ to normalize the delay so that system (1.2) can be written as

$$\dot{X}(t) = L_\mu(X_t) + f(\mu, X_t), \tag{2.6}$$

where $X(t) = (x(t), y(t), z(t))^{\mathrm{T}} \in \mathrm{R}^3$, $L_\mu : \mathrm{C} \mapsto \mathrm{R}$, $f : \mathrm{R} \times \mathrm{C} \mapsto \mathrm{R}$, for $\varphi = (\varphi_1, \varphi_2, \varphi_3)^{\mathrm{T}} \in \mathrm{C}([-1, 0], \mathrm{R}^3)$,

$$L_\mu\varphi = (\tau^*+\mu)N_1\varphi(0)+(\tau^*+\mu)N_2\varphi(-1), \quad f(\mu,\varphi) = (\tau^*+\mu)\begin{pmatrix} 0 \\ -k\varphi_1(0)\varphi_3(0) \\ h\varphi_1^2(0) \end{pmatrix},$$

where

$$N_1 = \begin{pmatrix} -K-a & a & 0 \\ b-kz^* & 0 & -kx^* \\ 2hx^* & 0 & -c \end{pmatrix}, \quad N_2 = \begin{pmatrix} K & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By the Riesz representation theorem, there exists a function $\eta(\theta, \mu)$ of bounded variation for $(-1 \le \theta \le 0)$ such that

$$L_\mu(\varphi) = \int_{-1}^{0} d\eta(\theta,\mu)\varphi(\theta) \qquad \text{for} \;\; \varphi \in \mathrm{C}([-1,0],\ \mathrm{R}^3).$$

In fact, we can choose

$$\eta(\theta,\mu) = (\tau^* + \mu)N_1\delta(\theta) - (\tau^* + \mu)N_2\delta(\theta + 1),$$

where $\delta(\theta) = \begin{cases} 0, & \theta \neq 0, \\ 1, & \theta = 0. \end{cases}$

For $\varphi \in \mathrm{C}([-1,0],\mathrm{R}^3)$, define

$$A(\mu)\varphi = \begin{cases} d\varphi(\theta)/d\theta, & \theta \in [-1,0), \\ \int_{-1}^{0} d\eta(t,\mu)\varphi(t), & \theta = 0. \end{cases} \quad R(\mu)\varphi = \begin{cases} 0, & \theta \in [-1,0), \\ f(\mu,\varphi), & \theta = 0. \end{cases}$$

Then (2.6) can be rewritten as

$$\dot{u}_t = A(\mu)u_t + R(\mu)u_t, \tag{2.7}$$

where $u = (x, y, z)^{\mathrm{T}}$ and $u_t = u(t + \theta)$ for $\theta \in [-1, 0]$.

For $\psi \in \mathrm{C}([0,1],(\mathrm{R}^3)^*)$, define

$$A^*\psi(s) = \begin{cases} -d\psi(s)/ds, & s \in (0,1], \\ \int_{-1}^{0} \psi(-s)d\eta(s,0), & s = 0, \end{cases}$$

and in a bilinear form

$$\langle \psi(s)\ ,\ \varphi(\theta)\rangle = \bar{\psi}(0)\varphi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{\psi}(\xi - \theta)d\eta(\theta)\varphi(\xi)d\xi,$$

where $\eta(\theta) = \eta(\theta, 0)$. Then $A(0)$ and A^* are adjoint operators. We know that $\pm \mathrm{i}\omega^*\tau^*$ are eigenvalues of $A(0)$, and they are also eigenvalues of A^*.

We can obtain that $q(\theta) = (1,\alpha,\beta)^{\mathrm{T}} e^{\mathrm{i}\omega^*\tau^*\theta}$ ($\theta \in [-1,0]$) and $q^*(s) = D(1,\alpha^*,\beta^*)e^{\mathrm{i}\omega^*\tau^* s}$ ($s \in [0,1]$) are the eigenvectors of $A(0)$ and A^* corresponding to the eigenvalues $\mathrm{i}\omega^*\tau^*$ and $-\mathrm{i}\omega^*\tau^*$, where

$$\begin{aligned} &\alpha = \frac{\mathrm{i}\omega^* + a + K - Ke^{-\mathrm{i}\tau^*\omega^*}}{a}, \beta = \frac{2hx^*}{\mathrm{i}\omega^* + c}, \alpha^* = -\frac{a}{\mathrm{i}\omega^*}, \beta^* = \frac{akx^*}{c\omega^*\mathrm{i} + \omega^{*2}}, \\ &D = (1 + \bar{\alpha}\alpha^* + \bar{\beta}\beta^* + K\tau^* e^{\mathrm{i}\omega^*\tau^*})^{-1}, \end{aligned} \tag{2.8}$$

such that $\langle q^*(s), q(\theta)\rangle = 1$, $\langle q^*(s), \bar{q}(\theta)\rangle = 0$.

Let u_t be the solution of (2.6) when $\mu = 0$. Define

$$\begin{aligned} z(t) &= \langle q^*, u_t\rangle, \\ w(z(t), \bar{z}(t), \theta) &= u_t(\theta) - 2\mathrm{Re}\{z(t)q(\theta)\}. \end{aligned}$$

For solution $u_t \in C_0$ (C_0 denotes the center manifold), we have

$$\begin{aligned} \dot{z}(t) &= \mathrm{i}\tau^*\omega^* z + \bar{q}^*(0)f(0, w(z,\bar{z},0) + 2\mathrm{Re}\{z(t)q(0)\}) \\ &\triangleq \mathrm{i}\tau^*\omega^* z + g_{20}\tfrac{z^2}{2} + g_{11}z\bar{z} + g_{02}\tfrac{\bar{z}^2}{2} + g_{21}\tfrac{z^2\bar{z}}{2} + \cdots, \end{aligned}$$

then we can get

$$
\begin{aligned}
g_{20} &= 2\bar{D}\tau^*(h\bar{\beta^*} - k\beta\bar{\alpha^*}),\\
g_{11} &= \bar{D}\tau^*(2h\bar{\beta^*} - k\bar{\alpha^*}\beta - k\bar{\alpha^*}\bar{\beta}),\\
g_{02} &= 2\bar{D}\tau^*(h\bar{\beta^*} - k\bar{\beta}\bar{\alpha^*}),\\
g_{21} =& \frac{4\bar{D}\tau^*\bar{\beta^*}hak[c(\beta+\bar{\beta}) + 2hx^*]}{abc - ackz^* - 2ahkx^{*2}z^*}\\
&- \frac{2\bar{D}\tau^*\bar{\alpha^*}ak[2hkx^*(2\beta+\bar{\beta}) - 2h(kz^*-b) + ck\beta(\beta+\bar{\beta})]}{abc - ackz^* - 2ahkx^{*2}z^*}\\
&- \frac{4\bar{D}\tau^*akh\bar{\beta^*}(2\beta\omega^*\mathrm{i} + \beta c + hx^*) - 2\bar{D}\tau^*\bar{\alpha^*}\bar{\beta}ak^2(2\beta\omega^*\mathrm{i} + \beta c - hx^*)}{(2\omega^*\mathrm{i}+c)[2\omega^*\mathrm{i}(2\omega^*\mathrm{i} + K + a - Ke^{-2\omega^*\tau^*\mathrm{i}}) + a(kz^*-b)] + 2akhx^{*2}z^*}\\
&- \frac{2\bar{D}\tau^*k\bar{\alpha^*}[2\mathrm{i}\omega^*h(2\mathrm{i}\omega^* + K + a - Ke^{-2\omega^*\tau^*\mathrm{i}}) + ah(kz^*-b)]}{(2\omega^*\mathrm{i}+c)[2\omega^*\mathrm{i}(2\omega^*\mathrm{i} + K + a - Ke^{-2\omega^*\tau^*\mathrm{i}}) + a(kz^*-b)] + 2akhx^{*2}z^*}\\
&+ \frac{6\bar{D}^2\tau^{*2}(h\bar{\beta^*} - k\beta\bar{\alpha^*})[2h\bar{\beta^*} - k\bar{\alpha^*}(\beta+\bar{\beta})]\mathrm{i}}{3\omega^*\tau^*}\\
&+ \frac{2D\bar{D}\tau^{*2}h\bar{\beta^*}(h\beta^* - k\beta\alpha^*)(3+\bar{\beta})\mathrm{i}}{3\omega^*\tau^*}.
\end{aligned}
\tag{2.9}
$$

By the normal form method and the center manifold theory introduced by Hassard et al. [11], define

$$
\begin{aligned}
C_1(0) &= \frac{\mathrm{i}}{2\omega^*\tau^*}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2},\\
\mu_2 &= -\frac{\mathrm{Re}(C_1(0))}{\mathrm{Re}(\lambda'(\tau^*))},
\end{aligned}
\tag{2.10}
$$

then they can determine the properties of bifurcating periodic solutions at the critical value τ^*. In fact, μ_2 determines the direction of the Hopf bifurcation: if $\mu_2 > 0$ ($\mu_2 < 0$), the bifurcating periodic solutions are forward (backward) when $\tau = \tau^*$. $\mathrm{Re}(C_1(0))$ determines the stability of bifurcating periodic solutions: if $\mathrm{Re}(C_1(0)) < 0$ ($\mathrm{Re}(C_1(0)) > 0$), the bifurcating periodic solutions on the center manifold are stable (unstable). Therefore, we have the following theorem.

Theorem 2.2. *If (H2) holds, then system (1.2) undergoes Hopf bifurcations at the equilibria E_2 and E_3 at the same time when $\tau = \tau_l^{(j)}$, where $l = 1, 2;\ j = 0, 1, 2\cdots$.*

(1) If $\mathrm{Re}(C_1(0)) > 0$ (< 0) when $\tau = \tau_1^{(j)}$, the bifurcating periodic solutions are forward (backward), and they are unstable (stable) on the center manifold.

(2) If $\mathrm{Re}(C_1(0)) > 0$ (< 0) when $\tau = \tau_2^{(j)}$, the bifurcating periodic solutions are backward (forward), and they are unstable (stable) on the center manifold.

Remark: For specific parameters values satisfying (H1) and (H2), we can obtain ω_l ($\omega_l = \sqrt{z_l}$) and $\tau_l^{(j)}(l = 1, 2; j = 0, 1, 2\cdots)$ respectively, and get $\mathrm{Re}(C_1(0))$ from (10–12), therefore, by Theorems 2.3 and 2.4, we can obtain the region of delay in which the two equilibria E_2 and E_3 are stable, the stability and the direction of bifurcating periodic solutions.

For the typical parameter set: $a = 10$, $b = 40$, $c = 2.5$, $h = 4$, $k = 1$. When $K = 15$, we can work out $\mathrm{Re}(C_1(0)) < 0$ at $\tau = \tau_l^{(0)}$, where $l = 1, 2$. Therefore, by

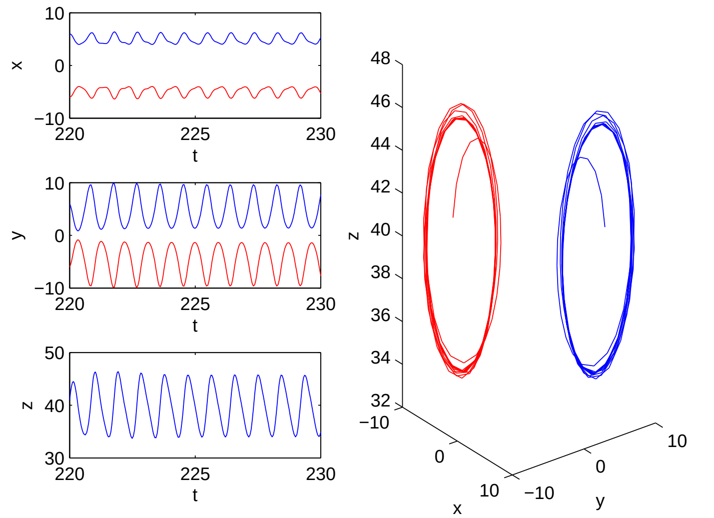

Figure 4. Waveform plots and phases for Liu system (1.2) with $\tau = 0.38$ for E_2 and E_3 respectively: two stable periodic solutions are coexisted.

Theorems 2.3 and 2.4, we have that Hopf bifurcations occur at equilibria E_2 and E_3 when $\tau = \tau_l^{(j)}$ $(l = 1, 2;\ j = 0, 1, 2 \cdots)$. When $\tau = \tau_1^{(0)}$ $(\tau = \tau_2^{(0)})$ the bifurcating periodic solutions are backward (forward), and they are stable. Therefore, when we choose appropriate regions of delay, a pair of asymptotically stable equilibria or a pair of stable periodic solutions are coexisted in system (1.2). We give the numerical simulation when $\tau = 0.38 < \tau_1^{(0)} = 0.3882$ and $\tau = 0.32 < \tau_1^{(0)} = 0.3882$ for E_2 and E_3 (see Figures. 4 and 5), and the waveform plot for E_2 and E_3 overlap in $z-$axis.

When $\tau = 0.38$, a periodic solution with small amplitude is bifurcated from E_2 and E_3 respectively. The amplitudes of periodic solutions increase with the delay far away from the critical point $\tau_1^{(0)}$ in the region which the equilibria E_2 and E_3 are unstable. A pair of stable periodic solutions with large amplitudes are coexisted when $\tau = 0.32$ (see Figure 5), and the pair of periodic solutions is reversed phase. We concern the large region of delay in which the family of stable periodic solutions occur. Therefore, we give the bifurcation diagram for equilibrium E_2 when $\tau \in [0, 0.55]$ and $K = 15$ (see Figure 6). It is plotted by Matlab software, which depicts the maximum value of periodic solutions in z-axis following the change of time delay. From this figure, we have that stable bifurcating periodic solutions occur at E_2 when $\tau \in (0.22, 0.3882)$ and $\tau \in (0.4664, 0.47)$, and the equilibrium E_2 is stable when $\tau \in (0.3882, 0.4664)$. These numerical results are in accordance with the above theoretical analysis given by Theorem 2.3 and 2.4, and there exists the phenomenon of "chaos switchover" in system (1.2) when $\tau \in [0, 0.55]$. Because there are the same characteristic equation of linearized system for E_2 and E_3, equilibrium E_3 has the same behaviors as E_2.

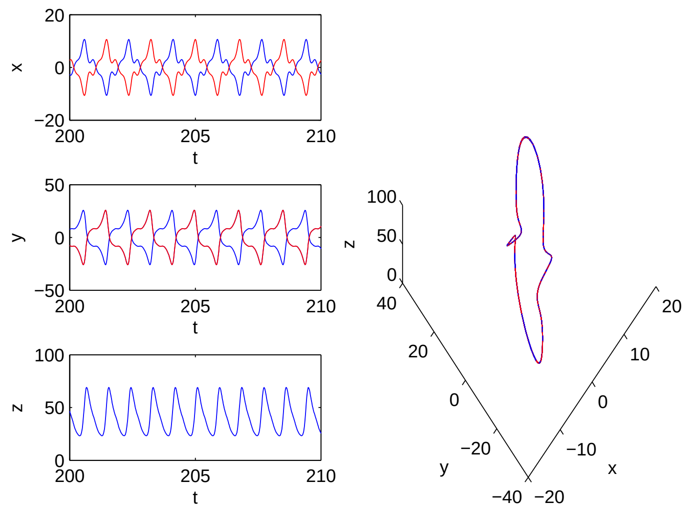

Figure 5. (Color online) Waveform plots and phases for Liu system (1.2) with $\tau = 0.32$ for E_2 (blue lines) and E_3 (red lines) respectively: there is a pair of reversed phase stable periodic solutions.

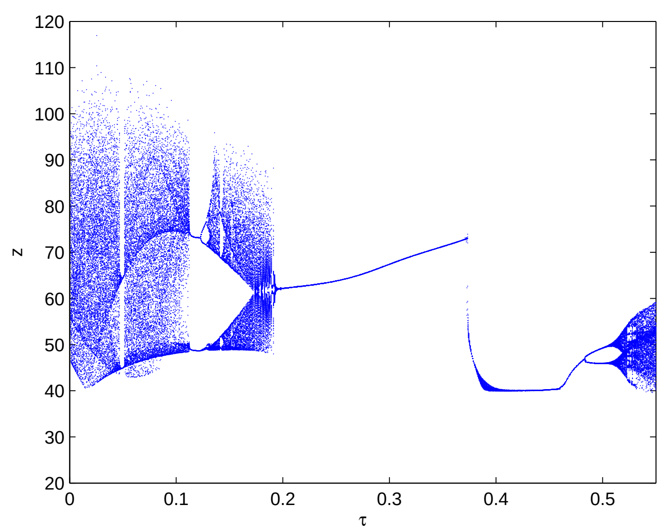

Figure 6. Bifurcation diagram for equilibrium E_2 when $K = 15$ and $\tau \in [0, 0.55]$, where the ordinate depicts the maximum value of periodic solutions in z-axis.

3. Double Hopf bifurcation

From the bifurcation diagram for the feedback strength K and the time delay τ (see Figure 2), we know that there are some points at which two Hopf bifurcation critical curves intersect. For example, $(K_c, \tau_c) = (41.7843, 0.46337)$ (see point R in Figure 2). From the discussion given in Section 2, when $K = K_c$, $\tau = \tau_c$, the characteristic equation (2.3) has two pairs of pure imaginary eigenvalues $\Lambda = \{\pm \mathrm{i}\omega_1, \pm \mathrm{i}\omega_2\}$, and all the other eigenvalues have negative real parts. We denote that system (1.2) undergoes a double Hopf bifurcation at equilibrium (x^*, y^*, z^*) when $K = K_c$, $\tau = \tau_c$. There is only a implicit expression of frequencies $h(\omega^2) = 0$ for the Liu system with parameters. Aiming to a group of determinate parameters, we can solve the values of frequencies, here, we only consider the normal form of non-resonant case.

In this section, in order to determine the bifurcation direction and the stability of bifurcating periodic solutions near the double Hopf bifurcation critical point (K_c, τ_c), we have to compute the normal forms on the center manifold. The method we use is based on the center manifold reduction and normal form theory due to Faria and Magalhaes [5, 6](see Appendix).

In polar coordinates $\eta_1 = r_1 e^{\mathrm{i}\theta_1}$, $\eta_2 = r_2 e^{\mathrm{i}\theta_2}$, the amplitude and phase equations on the center manifold can be derived from (A.8) as

$$\begin{cases} \dot{r_1} = r_1(\mu_1 + \mathrm{Re}(P_{11})r_1^2 + \mathrm{Re}(P_{12})r_2^2), \\ \dot{r_2} = r_2(\mu_2 + \mathrm{Re}(P_{21})r_1^2 + \mathrm{Re}(P_{22})r_2^2), \\ \dot{\theta_1} = \omega_1 + \nu_1 + \mathrm{Im}(P_{11})r_1^2 + \mathrm{Im}(P_{12})r_2^2, \\ \dot{\theta_2} = \omega_2 + \nu_2 + \mathrm{Im}(P_{21})r_1^2 + \mathrm{Im}(P_{22})r_2^2, \end{cases} \tag{3.1}$$

where P_{ij} are given by (A.8); $\mu_j = \mathrm{Re}(d_1^j)K_\varepsilon + \mathrm{Re}(d_2^j)\tau_\varepsilon$, $\nu_j = \mathrm{Im}(d_1^j)K_\varepsilon + \mathrm{Im}(d_2^j)\tau_\varepsilon$, where K_ε, τ_ε are perturbation parameters and d_i^j $(i = 1, 2; j = 1, 2)$ are given by (A.6).

For (3.1), let $\xi_j = \sqrt{|\mathrm{Re}(P_{jj})|}r_j, (j = 1, 2)$, we have the following planar system:

$$\begin{cases} \dot{\xi_1} = \xi_1(\mu_1 + \xi_1 + \vartheta\xi_2), \\ \dot{\xi_2} = \xi_2(\mu_2 + \delta\xi_1 + \xi_2), \end{cases} \tag{3.2}$$

where $\vartheta = \frac{\mathrm{Re}(P_{12})}{\mathrm{Re}(P_{22})}$, $\delta = \frac{\mathrm{Re}(P_{21})}{\mathrm{Re}(P_{11})}$.

When $a = 10$, $b = 40$, $c = 2.5$, $h = 4$, $k = 1$, $K = K_c = 41.7843$, $\tau = \tau_c = 0.46337$, After calculations we have $\mathrm{Re}(d_1^1) = 0.0002$, $\mathrm{Re}(d_1^2) = 0.0037$, $\mathrm{Re}(d_2^1) = -0.1636$, $\mathrm{Re}(d_2^2) = 2.5005$, $\mathrm{Re}(P_{11}) = -0.0329$, $\mathrm{Re}(P_{12}) = -0.0700$, $\mathrm{Re}(P_{21}) = 0.0670$, $\mathrm{Re}(P_{22}) = 0.0588$, $\vartheta\delta = 2.4276$.

We note that $M_0 = (0, 0)$ is always an equilibrium of (3.2). The two semi-trivial equilibria given by perturbation parameters are

$$M_1 = (0, \sqrt{-\mathrm{Re}(d_1^2)K_\varepsilon - \mathrm{Re}(d_2^2)\tau_\varepsilon}), \quad M_2 = (\sqrt{\mathrm{Re}(d_1^1)K_\varepsilon + \mathrm{Re}(d_2^1)\tau_\varepsilon}, 0),$$

which bifurcate from the origin at the bifurcation lines

$$L_1 = \{(K_\varepsilon, \tau_\varepsilon) : \mathrm{Re}(d_1^2)K_\varepsilon + \mathrm{Re}(d_2^2)\tau_\varepsilon = 0\}, L_2 = \{(K_\varepsilon, \tau_\varepsilon) : \mathrm{Re}(d_1^1)K_\varepsilon + \mathrm{Re}(d_2^1)\tau_\varepsilon = 0\},$$

respectively. There may also exist a nontrivial equilibrium

$$M_3 = \Big(\sqrt{\frac{(\mathrm{Re}(d_1^1) - \vartheta\mathrm{Re}(d_1^2))K_\varepsilon + (\mathrm{Re}(d_2^1) - \vartheta\mathrm{Re}(d_2^2))\tau_\varepsilon}{1-\vartheta\delta}},$$
$$\sqrt{\frac{(\delta\mathrm{Re}(d_1^1) - \mathrm{Re}(d_1^2))K_\varepsilon + (\delta\mathrm{Re}(d_2^1) - \mathrm{Re}(d_2^2))\tau_\varepsilon}{1-\vartheta\delta}}\Big).$$

For this expression to be valid, we need to assure that $\vartheta\delta - 1 \neq 0$. The nontrivial equilibrium M_3 collides with a semi-trivial one and from the positive quadrant on the bifurcation curves

$$T_1 = \{(K_\varepsilon, \tau_\varepsilon) : (\mathrm{Re}(d_1^1) - \vartheta\mathrm{Re}(d_1^2))K_\varepsilon + (\mathrm{Re}(d_2^1) - \vartheta\mathrm{Re}(d_2^2))\tau_\varepsilon = 0,$$
$$\mathrm{Re}(d_1^2)K_\varepsilon + \mathrm{Re}(d_2^2)\tau_\varepsilon < 0\},$$

$$T_2 = \{(K_\varepsilon, \tau_\varepsilon) : (\mathrm{Re}(d_1^2) - \delta\mathrm{Re}(d_1^1))K_\varepsilon + (\mathrm{Re}(d_2^2) - \delta\mathrm{Re}(d_2^1))\tau_\varepsilon = 0,$$
$$\mathrm{Re}(d_1^1)K_\varepsilon + \mathrm{Re}(d_2^1)\tau_\varepsilon > 0\}.$$

If $((\vartheta-1)\mathrm{Re}(d_1^2) + (\delta-1)\mathrm{Re}(d_1^1))K_\varepsilon + ((\vartheta-1)\mathrm{Re}(d_2^2) + (\delta-1)\mathrm{Re}(d_2^1))\tau_\varepsilon > 0$, the fixed point M_3 is sink, else M_3 is source. Therefore, we need consider the following bifurcation curve.

$$T_3 = \{(K_\varepsilon, \tau_\varepsilon) : ((\vartheta-1)\mathrm{Re}(d_1^2) + (\delta-1)\mathrm{Re}(d_1^1))K_\varepsilon +$$
$$((\vartheta-1)\mathrm{Re}(d_2^2) + (\delta-1)\mathrm{Re}(d_2^1))\tau_\varepsilon = 0\}.$$

Therefore, $L_1 : \ K_\varepsilon = -672.0840\tau_\varepsilon$, $L_2 : \ K_\varepsilon = 875.9254\tau_\varepsilon$, $T_1 : \ K_\varepsilon = -609.5047\tau_\varepsilon$, $T_2 : \ K_\varepsilon = -528.4533\tau_\varepsilon$, $T_3 : \ K_\varepsilon = -571.3934\tau_\varepsilon$.

According to the conclusion of [9], we give the bifurcation diagram (see Figure 7).

Since there exists no unstable manifold containing equilibrium, according to the center manifold theory, (3.1) on the center manifold determine the asymptotic behavior of solutions of the full equations (1.2). Therefore, if (3.2) has one or two asymptotically stable (unstable) semi-trivial equilibria M_1 and M_2, then (1.2) has one or two asymptotically stable (unstable) periodic solutions in the neighborhood of (x^*, y^*, z^*). If (3.2) has an asymptotically stable (unstable) equilibrium M_3, then (1.2) has an asymptotically stable (unstable) quasi-periodic solution in the neighborhood of (x^*, y^*, z^*). So, we shall call the periodic solution the source (respectively, saddle, sink) periodic solution of (1.2) when the semi-trivial equilibrium of (3.2) is a source (respectively, saddle, sink), and call the quasi-periodic solution the source (respectively, saddle, sink) quasi-periodic solution of (1.2) when the nontrivial equilibrium of (3.2) is a source (respectively, saddle, sink).

For the original system (1.2) in the neighborhood of (x^*, y^*, z^*), the above bifurcation criteria divide the parameters plane $(K_\varepsilon, \tau_\varepsilon)$ into seven regions (see Figure 7). In region D_1, there is only one trivial equilibrium which is saddle; when the parameters vary across the line L_1 from region D_1 to D_2, the trivial equilibrium becomes a sink point, and an unstable periodic solution O_1 (saddle) appears by Hopf bifurcation from the trivial solution; with the variation of the parameters from region D_2 to D_3, another sink periodic solution O_2 appears by Hopf bifurcation from the trivial solution; in region D_4, a sink quasi-periodic solution appears

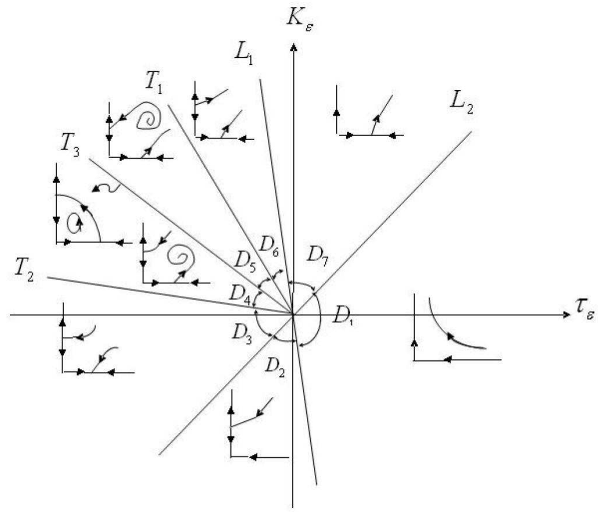

Figure 7. The bifurcation diagram with parameters K_ε and τ_ε near (K_c, τ_c).

by Neimark-Sacker bifurcation from the periodic solution O_2, and O_2 becomes a saddle from a sink; when the parameters vary across line T_3 from region D_4 to D_5, the sink quasi-periodic become a source one; in region D_6; quasi-periodic solution collides with the periodic solution O_1 and then disappears, and O_1 becomes a source solution; when the parameters vary across line L_1 from region D_6 to D_7, the periodic solution O_1 collides with the trivial solution and then disappears, and the trivial solution become a source from a saddle; when the parameters vary across line L_2 from region D_7 to D_1, the saddle periodic solution O_2 collides with the trivial solution and then disappears, and the trivial solution become a saddle from a source.

Because system (1.2) has the same characteristic equation of linearized system for equilibria E_2 and E_3, there are the same dynamical behaviors in the neighborhood of E_2 and E_3. Therefore, for system (1.2), there are the coexistence of a pair of unstable periodic solutions in region D_2 and D_7, the coexistence of a pair of stable periodic solutions and a pair of unstable periodic solutions in region D_3, the coexistence of two pairs of unstable periodic solutions in region D_6, the coexistence of a pair of unstable periodic solutions and a pair of stable quasi-periodic solutions in region D_4, and the coexistence of a pair of unstable periodic solutions and a pair of unstable quasi-periodic solutions in region D_5.

We give the numerical simulation for E_2 and E_3 when $\tau_\varepsilon = -0.0001$, $K_\varepsilon = -0.9$, $\tau_\varepsilon = -0.00337$, $K_\varepsilon = 0$ and $\tau_\varepsilon = -0.001, K_\varepsilon = 0.52$ respectively. The two parameters belong to regions D_2, D_3 and D_4 respectively (see Figures 8-10), and there exists a pair of stable fixed points, a pair of stable periodic solutions and a pair of stable quasi-periodic solutions respectively.

Figure 11 is the bifurcation diagram for equilibrium E_2 when $\tau \in [0, 0.48]$ and $K = K_c$. From this figure, we know that a family of stable bifurcating periodic solutions occur when $\tau \in (0.1, 0.42)$, and the inverse period doubling bifurcation

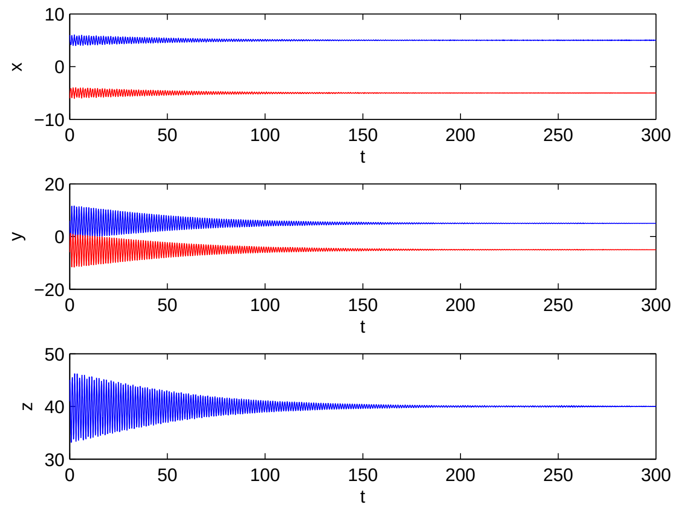

Figure 8. Waveform plots for Liu system (1.2) with $\tau_\varepsilon = -0.0001$ and $K_\varepsilon = -0.9$ for E_2 and E_3 respectively: two stable fixed points are coexisted.

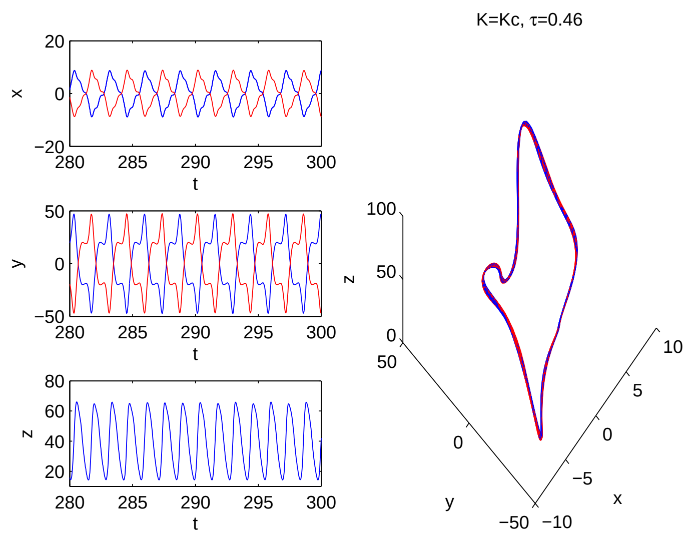

Figure 9. Waveform plots and phases for Liu system (1.2) with $\tau_\varepsilon = -0.00337$ and $K_\varepsilon = 0$ for E_2 and E_3 respectively: a pair of reversed phase stable periodic solutions are coexisted.

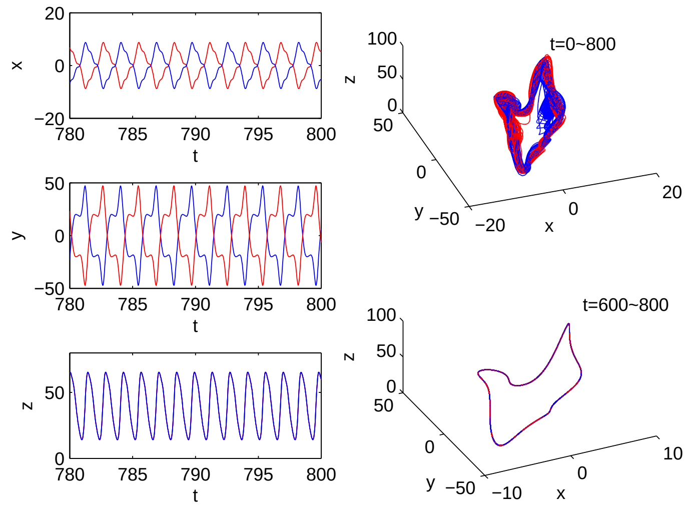

Figure 10. Waveform plots and phases for Liu system (1.2) with $\tau_\varepsilon = -0.001$ and $K_\varepsilon = 0.52$, for E_2 and E_3 respectively: a pair of reversed phase stable quasi-periodic solutions are coexisted.

approaches to chaos is found in (1.2). When τ approaches to $\tau_c = 0.46337$, the stable Hopf bifurcating periodic solutions disappear, and chaos occurs again with the increase of delay. There does not exist the region in which the equilibrium E_2 is stable, which is different from Figure 6 and is in accordance with Figure 2. Similarly, equilibrium E_3 has the same dynamical behaviors as equilibrium E_2.

4. Conclusion

In this paper, we discuss the stability of equilibria and bifurcating periodic solutions, the direction of Hopf bifurcation and dynamical behaviors near the double Hopf bifurcation critical point in Liu system with delay feedback.

We propose a realizable technique to control chaotic Liu system (1.1) to be stable using delayed feedback control method. Namely, if delay τ and feedback strength K satisfy the conditions of Theorems 2.3 and 2.4, then the system (1.1) can be controlled in new states, which have a pair of stable equilibria or a pair of stable bifurcating periodic solutions, and the chaos disappears. Furthermore, we derive the normal forms of double Hopf bifurcation and their unfolding with perturbation parameters, and then give the bifurcation diagram for the perturbation parameters and analyze the dynamical behaviors near the double Hopf bifurcation point. Therefore, with the change of delay and feedback strength, we can obtain the coexistence of a pair of stable periodic solutions, a pair of unstable periodic solutions, two pairs of unstable periodic solutions, a pair of stable quasi-periodic solutions, or a pair of unstable quasi-periodic solutions.

We give the bifurcation diagrams (Figures 6 and 11) to illustrate numerically that a family of stable bifurcating periodic solutions occur in a large region of delay

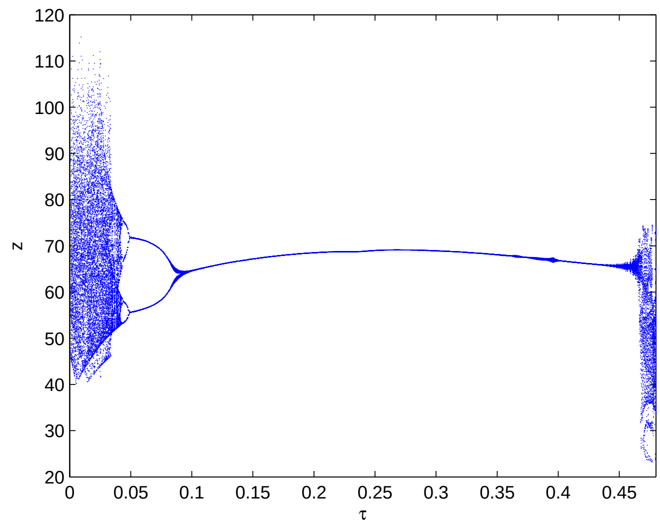

Figure 11. Bifurcation diagram for equilibrium E_2 when $\tau \in [0, 0.48]$ and $K = K_c$.

and the Liu system can appear the phenomenon of "chaos switchover". It is a helpful trying for studying the complex phenomena of delay differential equation.

Since the time delay is easy to be controlled and realized in real applications, our paper proves once again that DFC method is an effective method to control or generate chaos.

Acknowledgments

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Appendix

Rescaling the time by $t \mapsto (t/\tau)$ to normalize the delay so that system (1.2) can be written as

$$\begin{cases} \dot{x} = a\tau(y - x) + K\tau[x(t-1) - x(t)], \\ \dot{y} = b\tau x - k\tau xz, \\ \dot{z} = -c\tau z + h\tau x^2. \end{cases} \tag{A.1}$$

We let $K = K_c$ and $\tau = \tau_c$, and choose

$$\eta(\theta) = \begin{cases} \tau_c A, & \theta = 0, \\ 0, & \theta \in (-1, 0), \\ -\tau_c B, & \theta = -1, \end{cases}$$

with

$$A=\begin{pmatrix} -a-K_c & a & 0 \\ b-kz^* & 0 & -kx^* \\ 2hx^* & 0 & -c \end{pmatrix}, \quad B=\begin{pmatrix} K_c & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then the linearization equation at the equilibrium of (A.1) is

$$\dot{X}(t)=L_0X_t,$$

where $L_0\phi=\int_{-1}^{0}d\eta(\theta)\phi(\theta),\ \varphi\in \mathrm{C}=\mathrm{C}([-1,0],\ \mathrm{R}^3)$, and the bilinear form on $\mathrm{C}^*\times\mathrm{C}$ is

$$\langle\psi(s)\ ,\ \phi(\theta)\rangle=\psi(0)\phi(0)-\int_{-1}^{0}\int_{\xi=0}^{\theta}\psi(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi,$$

where $\phi\in\mathrm{C}$, $\psi\in\mathrm{C}^*$. Then the phase space C is decomposed by $\Lambda=\{\pm\mathrm{i}\omega_1,\pm\mathrm{i}\omega_2\}$ as $\mathrm{C}=P\oplus Q$, where $Q=\{\varphi\in\mathrm{C}:\ (\psi,\varphi)=0,\text{forall }\psi\in P^*$, and the bases for P and its adjoint P^* are

$$\Phi(\theta)=(q_1(\theta),\bar{q}_1(\theta),q_2(\theta),\bar{q}_2(\theta)),\quad \Psi(s)=(q_1^*(s),\bar{q}_1^*(s),q_2(s),\bar{q}_2^*(s))^{\mathrm{T}},$$

with

$$\langle q_1^*,q_1\rangle=1,\ \langle q_1^*,\bar{q}_1\rangle=0,\ \langle q_1^*,q_2\rangle=0,\ \langle q_1^*,\bar{q}_2\rangle=0,$$
$$\langle q_2^*,q_1\rangle=0,\ \langle q_2^*,\bar{q}_1\rangle=0,\ \langle q_2^*,q_2\rangle=1,\ \langle q_2^*,\bar{q}_2\rangle=0.$$

It can be computed directly that

$$\begin{aligned} q_j&=(1,\frac{b-kz^*}{\mathrm{i}\omega_j}-\frac{2hkx^{*2}}{(\mathrm{i}\omega_j+c)\mathrm{i}\omega_j},\frac{2hx^*}{\mathrm{i}\omega_j+c})^{\mathrm{T}}e^{\mathrm{i}\omega_j\tau_c\theta}\\ &\triangleq(p_{j1},p_{j2},p_{j3})^{\mathrm{T}}e^{\mathrm{i}\omega_j\tau_c\theta},\ \ -1\le\theta\le0,\\ q_j^*&=D_j(1,\frac{a}{\mathrm{i}\omega_j},-\frac{akx^*}{\mathrm{i}\omega_j(\mathrm{i}\omega_j+c)})e^{-\mathrm{i}\omega_j\tau_cs}\triangleq(q_{j1},q_{j2},q_{j3})^{\mathrm{T}}e^{-\mathrm{i}\omega_j\tau_cs},\ 0\le s\le1, \end{aligned}\tag{A.2}$$

where

$$D_j=(1-\frac{ab-akz^*}{\omega_j^2}+\frac{2ahkx^{*2}}{(\mathrm{i}\omega_j+c)\omega_j^2}-\frac{2akhx^{*2}}{(\mathrm{i}\omega_j+c)^2\mathrm{i}\omega_j}+K_c\tau_ce^{-\mathrm{i}\omega_j\tau_c})^{-1},\ (j=1,2).$$

We now introduction two bifurcation parameters by $K=K_c+K_\varepsilon$ and $\tau=\tau_c+\tau_\varepsilon$ in (A.1), and denote $\varepsilon=(K_\varepsilon,\tau_\varepsilon)$. Then (A.1) can be written as

$$\dot{X}(t)=L(\varepsilon)X_t+F(X_t,\varepsilon),\tag{A.3}$$

where

$$L(\varepsilon)X_t=\begin{pmatrix} a(\tau_c+\tau_\varepsilon)[y_t(0)-x_t(0)]+(\tau_c+\tau_\varepsilon)(K_c+K_\varepsilon)[x_t(-1)-x_t(0)] \\ b(\tau_c+\tau_\varepsilon)x_t(0) \\ -c(\tau_c+\tau_\varepsilon)z_t(0) \end{pmatrix},$$

and

$$F(X_t,\varepsilon)=\begin{pmatrix} 0 \\ -k(\tau_c+\tau_\varepsilon)x_t(0)z_t(0) \\ h(\tau_c+\tau_\varepsilon)x_t^2(0) \end{pmatrix},$$

As in Faria and Magalhães [5, 6], we consider the enlarged phase space BC of function from $[-1,0]$ to R^3, which are continuous on $[-1,0)$ and with a possible jump discontinuity at zero. This space can be identified with $\mathrm{C}\times\mathrm{R}^3$. Thus its elements can be written in the form $\phi=\varphi+X_0c$, where $\phi\in\mathrm{C}$, $c\in\mathrm{R}^3$ and X_0 is the 3×3 matrix-valued function defined by $X_0(\theta)=0$ for $\theta\in[-1,0)$ and $X_0(0)=\mathrm{I}$. In BC, (A.3) becomes an abstract ODE,

$$\frac{\mathrm{d}}{\mathrm{d}t}u=Au+X_0\tilde{F}(u,\varepsilon), \tag{A.4}$$

where $u\in\mathrm{C}$, and A is defined by

$$A:\ C^1\to BC,\quad Au=\dot{u}+X_0[L_0u-\dot{u}(0)],\ \text{and}\ \tilde{F}(u,\varepsilon)=[L(\varepsilon)-L_0]u+F(u,\varepsilon).$$

By the continuous projection $\pi:\ BC\mapsto P$, $\pi(\varphi+X_0c)=\Phi[(\Psi,\varphi)+\Psi(0)c]$, we can decompose the enlarged phase space by $\Lambda=\{\pm\mathrm{i}\omega_1\tau_c,\pm\mathrm{i}\omega_2\tau_c\}$ as $BC=P\oplus\mathrm{Ker}\pi$. Let $\eta=(\eta_1,\bar{\eta}_1,\eta_2,\bar{\eta}_2)^{\mathrm{T}}$ and $v_t\in Q^1:=Q\cap\mathrm{C}^1\subset\mathrm{Ker}\pi$, A_{Q^1} is the restriction of A as an operator from Q^1 to the Banach space $\mathrm{Ker}\pi$. Denote $u_t=\Phi\eta+v_t$. Equation (A.5) is therefore decomposed as the system

$$\begin{cases}\dot{\eta}=B\eta+\Psi(0)\tilde{F}(\Phi\eta+v_t,\varepsilon),\\ \dot{v_t}=A_{Q^1}v_t+(\mathrm{I}-\pi)X_0\tilde{F}(\Phi\eta+v_t,\varepsilon),\end{cases} \tag{A.5}$$

where $B=\mathrm{diag}\{\mathrm{i}\omega_1,-\mathrm{i}\omega_1,\mathrm{i}\omega_2,-\mathrm{i}\omega_2\}$.

Denote $\varepsilon=(K_\varepsilon,\tau_\varepsilon)$, and let M_2 denote the operator defined in $V_2^6(\mathrm{C}^4\times Ker\pi)$, with

$$M_2^1:V_2^6(\mathrm{C}^4)\mapsto V_2^6(\mathrm{C}^4),\quad (M_2^1p)(\eta,\varepsilon)=D_\eta p(\eta,\varepsilon)B\eta-Bp(\eta,\varepsilon),$$

where $V_2^6(\mathrm{C}^4)$ denotes the linear space of the second order homogeneous polynomials in six variables $(\eta_1,\bar{\eta}_1,\eta_2,\bar{\eta}_2,\varepsilon)$ and with coefficients in C^4. Then it is easy to check that one may choose the decomposition $V_2^6(\mathrm{C}^4)=\mathrm{Im}(M_2^1)\oplus\mathrm{Im}(M_2^1)^c$ with complementary space $\mathrm{Im}(M_2^1)^c$ spanned by the elements $K_\varepsilon\eta_1e_1$, $\tau_\varepsilon\eta_1e_1$, $K_\varepsilon\bar{\eta}_1e_2$, $\tau_\varepsilon\bar{\eta}_1e_2$, $K_\varepsilon\eta_2e_3$, $\tau_\varepsilon\eta_2e_3$, $K_\varepsilon\bar{\eta}_2e_4$, $\tau_\varepsilon\bar{\eta}_2e_4$, where e_i $(i=1,2,3,4)$ are unit vectors.

Then the normal form of (A.3) on the center manifold of the equilibrium (x^*,y^*,z^*) near $K_\varepsilon=0$, $\tau_\varepsilon=0$ has the form

$$\dot{\eta}=B\eta+\frac{1}{2}g_2^1(\eta,0,\varepsilon)+\text{h.o.t.},$$

where g_2^1 is the function giving the quadratic terms in (η,ε) for $v_t=0$, and is determined by $g_2^1(\eta,0,\varepsilon)=Proj_{(\mathrm{Im}(M_2^1))^c}\times f_2^1(\eta,0,\varepsilon)$, where $f_2^1(\eta,0,\varepsilon)$ is the function giving the quadratic terms in (η,ε) for $v_t=0$ defined by the first equation of (A.5). Then the normal form, which is truncated to the quadratic order, is

$$\begin{cases}\dot{\eta_1}=\mathrm{i}\omega_1\eta_1+d_1^1K_\varepsilon\eta_1+d_2^1\tau_\varepsilon\eta_1,\\ \dot{\eta_2}=\mathrm{i}\omega_2\eta_1+d_1^2K_\varepsilon\eta_2+d_2^2\tau_\varepsilon\eta_2,\end{cases} \tag{A.6}$$

where $d_1^j=q_{j1}p_{j1}\tau_c(e^{-\mathrm{i}\omega_j\tau_c}-1)$, $d_2^j=q_{j1}a(p_{j2}-p_{j1})+q_{j1}K_cp_{j1}(e^{-\mathrm{i}\omega_j\tau_c}-1)+bq_{j2}p_{j2}-cq_{j3}p_{j3}$, and p_{jl} and q_{jl} $(j=1,2;l=1,2,3.)$ are given by (A.2).

To find the third-order normal form, let M_3 denote the operator defined in $V_3^4(\mathrm{C}^4\times Ker\pi)$, with

$$M_3^1:V_3^4(\mathrm{C}^4)\mapsto V_3^4(\mathrm{C}^4),\quad (M_3^1p)(\eta,\varepsilon)=D_\eta p(\eta,\varepsilon)B\eta-Bp(\eta,\varepsilon),$$

where $V_3^4(\mathrm{C}^4)$ denotes the linear space of the third order homogeneous polynomials in four variables $(\eta_1, \bar{\eta}_1, \eta_2, \bar{\eta}_2)$ and with coefficients in C^4. Then it is easy to check that one may choose the decomposition $V_3^4(\mathrm{C}^4) = \mathrm{Im}(M_3^1) \oplus \mathrm{Im}(M_3^1)^c$ with complementary space $\mathrm{Im}(M_3^1)^c$ spanned by the elements $\eta_1^2\bar{\eta}_1 e_1$, $\eta_1\eta_2\bar{\eta}_2 e_1$, $\eta_1\bar{\eta}_1^2 e_2$, $\bar{\eta}_1\eta_2\bar{\eta}_2 e_2$, $\eta_2^2\bar{\eta}_2 e_3$, $\eta_1\bar{\eta}_1\eta_2 e_3$, $\eta_2\bar{\eta}_2^2 e_4$, $\eta_1\bar{\eta}_1\bar{\eta}_2 e_4$, where e_i $(i = 1, 2, 3, 4)$ are unit vectors.

Then we can derive the normal form up to the third order

$$\dot{\eta} = B\eta + \frac{1}{2!}g_2^1(\eta, 0, \varepsilon) + \frac{1}{3!}g_3^1(\eta, 0, \varepsilon) + \text{h.o.t.}, \tag{A.7}$$

where

$$\frac{1}{3!}g_3^1(\eta, 0, 0) = \frac{1}{3!}(I - P_{I,3}^1)f_3^1(\eta, 0, 0),$$

and $f_3^1(\eta, 0, 0)$ is the function giving the cubic terms in (η, ε, v_t) for $\varepsilon = 0$, $v_t = 0$ defined by the first equation of (A.5), therefore, the normal on the manifold form arising from (A.5) becomes the following form

$$\begin{cases} \dot{\eta}_1 = \mathrm{i}\omega_1\eta_1 + d_1^1 K_\varepsilon \eta_1 + d_2^1 \tau_\varepsilon \eta_1 + P_{11}\eta_1|\eta_1|^2 + P_{12}\eta_1|\eta_2|^2, \\ \dot{\eta}_2 = \mathrm{i}\omega_2\eta_2 + d_1^2 K_\varepsilon \eta_2 + d_2^2 \tau_\varepsilon \eta_2 + P_{21}\eta_2|\eta_1|^2 + P_{22}\eta_2|\eta_2|^2, \end{cases} \tag{A.8}$$

where d_i^j $(i, j = 1, 2)$ are given by (A.6),

$$\begin{aligned} P_{11} =& \frac{2\tau_c(h\bar{q}_{13}p_{11}^2 - k\bar{q}_{12}p_{11}p_{13})(hq_{13}\bar{p}_{11}^2 - kq_{12}\bar{p}_{11}\bar{p}_{13})}{3\mathrm{i}\omega_1} \\ &- \frac{\tau_c(hq_{13}p_{11}^2 - kq_{12}p_{11}p_{13})[2hq_{13}p_{11}\bar{p}_{11} - kq_{12}(\bar{p}_{11}p_{13} + p_{11}\bar{p}_{13})]}{\mathrm{i}\omega_1} \\ &+ \frac{\tau_c[2h\bar{q}_{13}p_{11}\bar{p}_{11} - k\bar{q}_{12}(\bar{p}_{11}p_{13} + p_{11}\bar{p}_{13})][2hq_{13}p_{11}\bar{p}_{11} - kq_{12}(\bar{p}_{11}p_{13} + p_{11}\bar{p}_{13})]}{\mathrm{i}\omega_1} \\ &+ \frac{\tau_c(hq_{23}p_{11}^2 - kq_{22}p_{11}p_{13})[2hq_{13}\bar{p}_{11}p_{21} - kq_{12}(p_{21}\bar{p}_{13} + \bar{p}_{11}p_{23})]}{2\mathrm{i}\omega_1 - \mathrm{i}\omega_2} \\ &- \frac{\tau_c[2hq_{23}p_{11}\bar{p}_{11} - kq_{22}(\bar{p}_{11}p_{13} + \bar{p}_{11}p_{13})][2hq_{13}p_{11}p_{21} - kq_{12}(p_{21}p_{13} + p_{11}p_{23})]}{\mathrm{i}\omega_2} \\ &+ \frac{\tau_c(h\bar{q}_{23}p_{11}^2 - k\bar{q}_{22}p_{11}p_{13})[2hq_{13}\bar{p}_{11}\bar{p}_{21} - kq_{12}(\bar{p}_{21}\bar{p}_{13} + \bar{p}_{11}\bar{p}_{23})]}{2\mathrm{i}\omega_1 + \mathrm{i}\omega_2} \\ &+ \frac{\tau_c[2h\bar{q}_{23}p_{11}\bar{p}_{11} - k\bar{q}_{22}(\bar{p}_{11}p_{13} + p_{11}\bar{p}_{13})][2hq_{13}p_{11}\bar{p}_{21} - kq_{12}(\bar{p}_{21}p_{13} + p_{11}\bar{p}_{23})]}{\mathrm{i}\omega_2} \\ &+ \tau_c(2hq_{13}p_{11} - q_{12}kp_{13})h_{1100}^1 + \tau_c(2hq_{13}\bar{p}_{11} - q_{12}k\bar{p}_{13})h_{2000}^1 \\ &- \tau_c q_{12}k(p_{11}h_{1100}^3 + \bar{p}_{11}h_{2000}^3), \end{aligned}$$

$$\begin{aligned} P_{22} =& \frac{\tau_c(hq_{13}p_{21}^2 - kq_{12}p_{21}p_{23})[2hq_{23}p_{11}\bar{p}_{21} - kq_{22}(\bar{p}_{21}p_{13} + p_{11}\bar{p}_{23})]}{2\mathrm{i}\omega_2 - \mathrm{i}\omega_1} \\ &- \frac{\tau_c[2hq_{13}p_{21}\bar{p}_{21} - kq_{12}(\bar{p}_{21}p_{23} + p_{21}\bar{p}_{23})][2hq_{23}p_{11}p_{21} - kq_{22}(p_{21}p_{13} + p_{11}p_{23})]}{\mathrm{i}\omega_1} \\ &+ \frac{\tau_c(h\bar{q}_{13}p_{21}^2 - k\bar{q}_{12}p_{21}p_{23})[2hq_{23}\bar{p}_{11}\bar{p}_{21} - kq_{22}(\bar{p}_{21}\bar{p}_{13} + \bar{p}_{11}\bar{p}_{23})]}{2\mathrm{i}\omega_2 + \mathrm{i}\omega_1} \\ &+ \frac{\tau_c[2h\bar{q}_{13}p_{21}\bar{p}_{21} - k\bar{q}_{12}(\bar{p}_{21}p_{23} + p_{21}\bar{p}_{23})][2hq_{23}\bar{p}_{11}p_{21} - kq_{22}(p_{21}\bar{p}_{13} + \bar{p}_{11}p_{23})]}{\mathrm{i}\omega_1} \end{aligned}$$

$$
\begin{aligned}
&- \frac{\tau_c(hq_{23}p_{21}^2 - kq_{22}p_{21}p_{23})[2hq_{23}p_{21}\bar{p}_{21} - kq_{22}(p_{23}\bar{p}_{21} + p_{21}\bar{p}_{23})]}{\mathrm{i}\omega_2} \\
&+ \frac{2\tau_c(h\bar{q}_{23}p_{21}^2 - k\bar{q}_{22}p_{21}p_{23})(hq_{23}\bar{p}_{21}^2 - kq_{22}\bar{p}_{21}\bar{p}_{23})}{3\mathrm{i}\omega_2} \\
&+ \frac{\tau_c[2h\bar{q}_{23}p_{21}\bar{p}_{21} - k\bar{q}_{22}(\bar{p}_{21}p_{23} + p_{21}\bar{p}_{23})][2hq_{23}p_{21}\bar{p}_{21} - kq_{22}(\bar{p}_{21}p_{23} + p_{21}\bar{p}_{23})]}{\mathrm{i}\omega_2} \\
&+ \tau_c(2hq_{23}p_{21} - q_{22}kp_{23})h_{0011}^1 + \tau_c(2hq_{23}\bar{p}_{21} - q_{22}k\bar{p}_{23})h_{0020}^1 \\
&- \tau_c q_{22}k(p_{21}h_{0011}^3 + \bar{p}_{21}h_{0020}^3),
\end{aligned}
$$

$$
\begin{aligned}
P_{12} =& \frac{\tau_c[2h\bar{q}_{13}p_{11}p_{21} - k\bar{q}_{12}(p_{21}p_{13} + p_{11}p_{23})][2hq_{13}\bar{p}_{11}\bar{p}_{21} - kq_{12}(\bar{p}_{21}\bar{p}_{13} + \bar{p}_{11}\bar{p}_{23})]}{2\mathrm{i}\omega_1 + \mathrm{i}\omega_2} \\
&- \frac{2\tau_c[2hq_{13}p_{21}\bar{p}_{21} - kq_{12}(\bar{p}_{21}p_{23} + p_{21}\bar{p}_{23})](hq_{13}p_{11}^2 - kq_{12}p_{11}p_{13})}{\mathrm{i}\omega_1} \\
&+ \frac{\tau_c[2h\bar{q}_{13}p_{11}\bar{p}_{21} - k\bar{q}_{12}(\bar{p}_{21}p_{13} + p_{11}\bar{p}_{23})][2hq_{13}\bar{p}_{11}p_{21} - kq_{12}(p_{21}\bar{p}_{13} + \bar{p}_{11}p_{23})]}{2\mathrm{i}\omega_1 - \mathrm{i}\omega_2} \\
&+ \frac{\tau_c[2h\bar{q}_{13}p_{21}\bar{p}_{21} - k\bar{q}_{12}(\bar{p}_{21}p_{23} + p_{21}\bar{p}_{23})][2hq_{13}\bar{p}_{11}p_{11} - kq_{12}(\bar{p}_{11}p_{13} + p_{11}\bar{p}_{13})]}{\mathrm{i}\omega_1} \\
&+ \frac{\tau_c[2hq_{23}p_{11}p_{21} - kq_{22}(p_{21}p_{13} + p_{11}p_{23})][2hq_{13}p_{21}\bar{p}_{21} - kq_{12}(\bar{p}_{21}p_{23} + p_{21}\bar{p}_{23})]}{\mathrm{i}\omega_1} \\
&+ \frac{2\tau_c[2hq_{23}p_{11}\bar{p}_{21} - kq_{22}(\bar{p}_{21}p_{13} + p_{11}\bar{p}_{23})](hq_{13}p_{21}^2 - kq_{12}p_{21}p_{23})}{\mathrm{i}\omega_1 - 2\mathrm{i}\omega_2} \\
&- \frac{\tau_c[2hq_{23}p_{21}\bar{p}_{21} - kq_{22}(\bar{p}_{21}p_{23} + p_{21}\bar{p}_{23})][2hq_{13}p_{11}p_{21} - kq_{12}(p_{21}p_{13} + p_{11}p_{23})]}{\mathrm{i}\omega_2} \\
&+ \frac{2\tau_c[2h\bar{q}_{23}p_{11}p_{21} - k\bar{q}_{22}(p_{21}p_{13} + p_{11}p_{23})](hq_{13}\bar{p}_{21}^2 - kq_{12}\bar{p}_{21}\bar{p}_{23})}{\mathrm{i}\omega_1 + 2\mathrm{i}\omega_2} \\
&+ \frac{\tau_c[2h\bar{q}_{23}p_{11}\bar{p}_{21} - k\bar{q}_{22}(\bar{p}_{21}p_{13} + p_{11}\bar{p}_{23})][2hq_{13}p_{21}\bar{p}_{21} - kq_{12}(\bar{p}_{21}p_{23} + p_{21}\bar{p}_{23})]}{\mathrm{i}\omega_1} \\
&+ \frac{\tau_c[2h\bar{q}_{23}p_{21}\bar{p}_{21} - k\bar{q}_{22}(\bar{p}_{21}p_{23} + p_{21}\bar{p}_{23})][2hq_{13}p_{11}\bar{p}_{21} - kq_{12}(\bar{p}_{21}p_{13} + p_{11}\bar{p}_{23})]}{\mathrm{i}\omega_2} \\
&+ \tau_c(2hq_{13}p_{11} - q_{12}kp_{13})h_{0011}^1 - \tau_c q_{12}k(p_{11}h_{0011}^3 + p_{21}h_{1001}^3 + \bar{p}_{21}h_{1010}^3) \\
&+ \tau_c(2hq_{13}p_{21} - q_{12}kp_{23})h_{1001}^1 + \tau_c(2hq_{13}\bar{p}_{21} - q_{12}k\bar{p}_{23})h_{1010}^1,
\end{aligned}
$$

$$
\begin{aligned}
P_{21} =& \frac{\tau_c[2hq_{13}p_{11}p_{21} - kq_{12}(p_{21}p_{13} + p_{11}p_{23})][2hq_{23}p_{11}\bar{p}_{11} - kq_{22}(\bar{p}_{11}p_{13} + p_{11}\bar{p}_{13})]}{\mathrm{i}\omega_2} \\
&- \frac{\tau_c[2hq_{13}p_{11}\bar{p}_{11} - kq_{12}(\bar{p}_{11}p_{13} + p_{11}\bar{p}_{13})][2hq_{23}p_{11}p_{21} - kq_{22}(p_{21}p_{13} + p_{11}p_{23})]}{\mathrm{i}\omega_1} \\
&+ \frac{2\tau_c[2hq_{13}\bar{p}_{11}p_{21} - kq_{12}(p_{21}\bar{p}_{13} + \bar{p}_{11}p_{23})](hq_{23}p_{11}^2 - kq_{22}p_{11}p_{13})}{\mathrm{i}\omega_2 - 2\mathrm{i}\omega_1} \\
&+ \frac{\tau_c[2h\bar{q}_{13}p_{11}\bar{p}_{11} - k\bar{q}_{12}(\bar{p}_{11}p_{13} + p_{11}\bar{p}_{13})][2hq_{23}\bar{p}_{11}p_{21} - kq_{22}(p_{21}\bar{p}_{13} + \bar{p}_{11}p_{23})]}{\mathrm{i}\omega_1} \\
&+ \frac{2\tau_c[2h\bar{q}_{13}p_{11}p_{21} - k\bar{q}_{12}(p_{21}p_{13} + p_{11}p_{23})](hq_{23}\bar{p}_{11}^2 - kq_{22}\bar{p}_{11}\bar{p}_{13})}{2\mathrm{i}\omega_1 + \mathrm{i}\omega_2} \\
&+ \frac{\tau_c[2h\bar{q}_{13}\bar{p}_{11}p_{21} - k\bar{q}_{12}(p_{21}\bar{p}_{13} + \bar{p}_{11}p_{23})][2hq_{23}p_{11}\bar{p}_{11} - kq_{22}(\bar{p}_{11}p_{13} + p_{11}\bar{p}_{13})]}{\mathrm{i}\omega_2} \\
&- \frac{2\tau_c[2hq_{23}p_{11}\bar{p}_{11} - kq_{22}(\bar{p}_{11}p_{13} + p_{11}\bar{p}_{13})](hq_{23}p_{21}^2 - kq_{22}p_{21}p_{23})}{\mathrm{i}\omega_2}
\end{aligned}
$$

$$+\frac{\tau_c[2hq_{23}p_{11}p_{21}-kq_{22}(p_{21}p_{13}+p_{11}p_{23})][2hq_{23}\bar{p}_{11}p_{21}-kq_{22}(p_{21}\bar{p}_{13}+\bar{p}_{11}p_{23})]}{\mathrm{i}\omega_1}$$
$$-\frac{\tau_c[2hq_{23}\bar{p}_{11}p_{21}-kq_{22}(p_{21}\bar{p}_{13}+\bar{p}_{11}p_{23})][2hq_{23}p_{11}p_{21}-kq_{22}(p_{21}p_{13}+p_{11}p_{23})]}{\mathrm{i}\omega_1}$$
$$+\frac{\tau_c[2h\bar{q}_{23}p_{11}\bar{p}_{11}-k\bar{q}_{22}(\bar{p}_{11}p_{13}+p_{11}\bar{p}_{13})][2hq_{23}p_{21}\bar{p}_{21}-kq_{22}(\bar{p}_{21}p_{23}+p_{21}\bar{p}_{23})]}{\mathrm{i}\omega_2}$$
$$+\frac{\tau_c[2h\bar{q}_{23}p_{11}p_{21}-k\bar{q}_{22}(p_{21}p_{13}+p_{11}p_{23})][2hq_{23}\bar{p}_{11}\bar{p}_{21}-kq_{22}(\bar{p}_{21}\bar{p}_{13}+\bar{p}_{11}\bar{p}_{23})]}{\mathrm{i}\omega_1+2\mathrm{i}\omega_2}$$
$$+\frac{\tau_c[2h\bar{q}_{23}\bar{p}_{11}p_{21}-k\bar{q}_{22}(p_{21}\bar{p}_{13}+\bar{p}_{11}p_{23})][2hq_{23}p_{11}\bar{p}_{21}-kq_{22}(\bar{p}_{21}p_{13}+p_{11}\bar{p}_{23})]}{2\mathrm{i}\omega_2-\mathrm{i}\omega_1}$$
$$+\tau_c(2hq_{23}p_{11}-q_{22}kp_{13})h^1_{0110}+\tau_c(2hq_{23}\bar{p}_{11}-q_{22}k\bar{p}_{13})h^1_{1010}$$
$$+\tau_c(2hq_{23}p_{21}-q_{22}kp_{23})h^1_{1100}-\tau_c q_{22}k(p_{11}h^3_{0110}+\bar{p}_{11}h^3_{1010}+p_{21}h^3_{1100}),$$

where $p_{ij},\ q_{ij}\ (i=1,2;j=1,2,3)$ are given by (A.2),

$$h^1_{2000}=\frac{(c+2\omega_1\mathrm{i})[a(a_{23}hp_{11}^2-ka_{22}p_{11}p_{13})-2\omega_1\mathrm{i}(ka_{12}p_{11}p_{13}-a_{13}hp_{11}^2)]}{(c+2\omega_1\mathrm{i})[2\omega_1\mathrm{i}(2\omega_1\mathrm{i}+K_c+a)+a(kz^*-b)-2\mathrm{i}\omega_1K_ce^{-2\mathrm{i}\omega_1\tau_c}]+2ahkx^{*2}}$$
$$+\frac{akx^*p_{11}(ka_{32}p_{13}-ha_{33}p_{11})+2\mathrm{i}\omega_1(2\mathrm{i}\omega_1+c)K_ce^{-2\mathrm{i}\omega_1\tau_c}m_{2000}}{(c+2\omega_1\mathrm{i})[2\omega_1\mathrm{i}(2\omega_1\mathrm{i}+K_c+a)+a(kz^*-b)-2\mathrm{i}\omega_1K_ce^{-2\mathrm{i}\omega_1\tau_c}]+2ahkx^{*2}},$$
$$h^3_{2000}=\frac{2hx^*h^1_{2000}-ka_{32}p_{11}p_{13}+ha_{33}p_{11}^2}{c+2\omega_1\mathrm{i}},$$
$$h^1_{1100}=\frac{2hp_{11}\bar{p}_{11}(kx^*a_{33}-ca_{23})+k(ca_{22}-kx^*a_{32})(\bar{p}_{11}p_{13}+p_{11}\bar{p}_{13})}{c(b-kz^*)-2hkx^{*2}},$$
$$h^3_{1100}=\frac{2ha_{23}p_{11}\bar{p}_{11}-a_{22}k(\bar{p}_{11}p_{13}+p_{11}\bar{p}_{13})-(kz^*-b)h^1_{1100}}{kx^*},$$
$$h^1_{0110}=\frac{2h[a_{13}(\omega_2\mathrm{i}-\omega_1\mathrm{i})+aa_{23}]\bar{p}_{11}p_{21}(c+\omega_2\mathrm{i}-\omega_1\mathrm{i})-2ahkx^*a_{33}\bar{p}_{11}p_{21}}{R_1}$$
$$-\frac{k[(aa_{22}+a_{12}\omega_2\mathrm{i}-a_{12}\omega_1\mathrm{i})(c+\omega_2\mathrm{i}-\omega_1\mathrm{i})-akx^*a_{32}](p_{21}\bar{p}_{13}+\bar{p}_{11}p_{23})}{R_1}$$
$$+\frac{K_ce^{\mathrm{i}\omega_1\tau_c-\mathrm{i}\omega_2\tau_c}(\mathrm{i}\omega_2-\mathrm{i}\omega_1)(\mathrm{i}\omega_2-\mathrm{i}\omega_1)m_{0110}}{R_1},$$
$$h^3_{0110}=\frac{2ha_{33}\bar{p}_{11}p_{21}-ka_{32}(p_{21}\bar{p}_{13}+\bar{p}_{11}p_{23})+2hx^*h^1_{0110}}{\mathrm{i}\omega_2-\mathrm{i}\omega_1+\mathrm{c}},$$
$$h^1_{1010}=\frac{2hp_{11}p_{21}[aa_{23}(c+\omega_2\mathrm{i}+\omega_1\mathrm{i})-akx^*a_{33}+a_{13}(\omega_2\mathrm{i}+\omega_1\mathrm{i})(c+\omega_2\mathrm{i}+\omega_1\mathrm{i})]}{R_2}$$
$$+\frac{k(p_{21}p_{13}+p_{11}p_{23})[akx^*a_{32}-(aa_{22}+a_{12}\omega_2\mathrm{i}+a_{12}\omega_1\mathrm{i})(\omega_2\mathrm{i}+\omega_1\mathrm{i}+c)]}{R_2}$$
$$+\frac{K_ce^{-\mathrm{i}\omega_1\tau_c-\mathrm{i}\omega_2\tau_c}(\mathrm{i}\omega_1+\mathrm{i}\omega_2)(\mathrm{i}\omega_1+\mathrm{i}\omega_2+c)m_{1010}}{R_2},$$
$$h^3_{1010}=\frac{2ha_{33}p_{11}p_{21}-ka_{32}(p_{21}p_{13}+p_{23}p_{11})+2hx^*h^1_{1010}}{\omega_2\mathrm{i}+\omega_1\mathrm{i}+c},$$
$$h^1_{0020}=\frac{(2\omega_2\mathrm{i}+c)[aha_{23}p_{21}^2-aka_{22}p_{21}p_{23}-2\omega_2\mathrm{i}(ka_{12}p_{21}p_{23}-ha_{13}p_{21}^2)]}{(c+2\omega_2\mathrm{i})[2\omega_2\mathrm{i}(2\omega_2\mathrm{i}+K_c+a-K_ce^{-2\mathrm{i}\omega_2\tau_c})+a(kz^*-b)]+2ahkx^{*2}}$$
$$-\frac{akx^*(ha_{33}p_{21}^2-ka_{32}p_{21}p_{23})+2K_ce^{-2\mathrm{i}\omega_2\tau_c}\mathrm{i}\omega_2(2\mathrm{i}\omega_2+c)m_{0020}}{(c+2\omega_2\mathrm{i})[2\omega_2\mathrm{i}(2\omega_2\mathrm{i}+K_c+a-K_ce^{-2\mathrm{i}\omega_2\tau_c})+a(kz^*-b)]+2ahkx^{*2}},$$
$$h^3_{0020}=\frac{a_{33}hp_{21}^2-ka_{32}p_{21}p_{23}+2hx^*h^1_{0020}}{c+2\mathrm{i}\omega_2},$$

$$h^1_{0011} = \frac{k(kx^*a_{32} - ca_{22})(\bar{p}_{21}p_{23} + p_{21}\bar{p}_{23}) + 2hp_{21}\bar{p}_{21}(ca_{23} - kx^*a_{33})}{c(kz^* - b) + 2hkx^{*2}},$$

$$h^3_{0011} = \frac{2ha_{23}p_{21}\bar{p}_{21} - ka_{22}(\bar{p}_{21}p_{23} + p_{21}\bar{p}_{23}) - (kz^* - b)h^1_{0011}}{kx^*},$$

$$\begin{aligned} h^1_{1001} =& \frac{2hp_{11}\bar{p}_{21}[aa_{23}(\omega_1\mathrm{i} - \omega_2\mathrm{i} + c) - akx^*a_{33} + a_{13}(\omega_1\mathrm{i} - \omega_2\mathrm{i})(\omega_1\mathrm{i} - \omega_2\mathrm{i} + c)]}{R_3} \\ &+ \frac{k(\bar{p}_{21}p_{13} + p_{11}\bar{p}_{23})[akx^*a_{32} - (aa_{22} + a_{12}\omega_1\mathrm{i} - a_{12}\omega_2\mathrm{i})(\omega_1\mathrm{i} - \omega_2\mathrm{i} + c)]}{R_3} \\ &+ \frac{K_c e^{\mathrm{i}\omega_2\tau_c - \mathrm{i}\omega_1\tau_c}(\mathrm{i}\omega_1 - \mathrm{i}\omega_2)(\mathrm{i}\omega_1 - \mathrm{i}\omega_2 + c)m_{1001}}{R_3}, \end{aligned}$$

$$h^3_{1001} = \frac{2ha_{33}p_{11}\bar{p}_{21} - ka_{32}(\bar{p}_{21}p_{13} + p_{11}\bar{p}_{23}) + 2hx^*h^1_{1001}}{\omega_1\mathrm{i} - \omega_2\mathrm{i} + c},$$

where $a_{ij} = \begin{cases} 1 - p_{1i}q_{1j} - \bar{p}_{1i}\bar{q}_{1j} - p_{2i}q_{2j} - \bar{p}_{2i}\bar{q}_{2j}, & i = j, \\ p_{1i}q_{1j} + \bar{p}_{1i}\bar{q}_{1j} + p_{2i}q_{2j} + \bar{p}_{2i}\bar{q}_{2j}, & i \neq j, \end{cases} \quad (i = 1, 2, 3;\ j = 1, 2, 3).$

$$\begin{aligned} m_{0110} =& \frac{[2h\bar{p}_{11}p_{21}p_{11}q_{13} - kp_{11}q_{12}(p_{21}\bar{p}_{13} + \bar{p}_{11}p_{23})](e^{\mathrm{i}\omega_2\tau_c - 2\mathrm{i}\omega_1\tau_c} - 1)}{2\mathrm{i}\omega_1 - \mathrm{i}\omega_2} \\ &- \frac{[2h\bar{p}^2_{11}p_{21}\bar{q}_{13} - k\bar{p}_{11}\bar{q}_{12}(p_{21}\bar{p}_{13} + \bar{p}_{11}p_{23})](e^{\mathrm{i}\omega_2\tau_c} - 1)}{\mathrm{i}\omega_2} \\ &+ \frac{[2h\bar{p}_{11}p^2_{21}q_{23} - kp_{21}q_{22}(p_{21}\bar{p}_{13} + \bar{p}_{11}p_{23})](e^{-\mathrm{i}\omega_1\tau_c} - 1)}{\mathrm{i}\omega_1} \\ &+ \frac{[2h\bar{p}_{11}p_{21}\bar{p}_{21}\bar{q}_{23} - k\bar{p}_{21}\bar{q}_{22}(p_{21}\bar{p}_{13} + \bar{p}_{11}p_{23})](e^{2\mathrm{i}\omega_2\tau_c - \mathrm{i}\omega_1\tau_c} - 1)}{\mathrm{i}\omega_1 - 2\mathrm{i}\omega_2}, \end{aligned}$$

$$\begin{aligned} m_{2000} =& \frac{(hp^3_{11}q_{13} - kp^2_{11}p_{13}q_{12})(e^{\mathrm{i}\omega_1\tau_c} - 1)}{-\mathrm{i}\omega_1} \\ &- \frac{(hp^2_{11}\bar{p}_{11}\bar{q}_{13} - kp_{11}p_{13}\bar{p}_{11}\bar{q}_{12})(e^{3\mathrm{i}\omega_1\tau_c} - 1)}{3\mathrm{i}\omega_1} \\ &+ \frac{(hp^2_{11}p_{21}q_{23} - kp_{11}p_{13}p_{21}q_{22})(e^{2\mathrm{i}\omega_1\tau_c - \mathrm{i}\omega_2\tau_c} - 1)}{\mathrm{i}\omega_2 - 2\mathrm{i}\omega_1} \\ &- \frac{(hp^2_{11}\bar{p}_{21}\bar{q}_{23} - kp_{11}p_{13}\bar{p}_{21}\bar{q}_{22})(e^{2\mathrm{i}\omega_1\tau_c + \mathrm{i}\omega_2\tau_c} - 1)}{\mathrm{i}\omega_2 + 2\mathrm{i}\omega_1}, \end{aligned}$$

$$\begin{aligned} m_{1010} =& \frac{[2hp^2_{11}p_{21}q_{13} - kp_{11}q_{12}(p_{21}p_{13} + p_{11}p_{23})](1 - e^{\mathrm{i}\omega_2\tau_c})}{\mathrm{i}\omega_2} \\ &- \frac{[2hp_{11}p_{21}\bar{p}_{11}\bar{q}_{13} - k\bar{p}_{11}\bar{q}_{12}(p_{21}p_{13} + p_{11}p_{23})](e^{2\mathrm{i}\omega_1\tau_c + \mathrm{i}\omega_2\tau_c} - 1)}{2\mathrm{i}\omega_1 + \mathrm{i}\omega_2} \\ &- \frac{[2hp_{11}p^2_{21}q_{23} - kp_{21}q_{22}(p_{21}p_{13} + p_{11}p_{23})](e^{\mathrm{i}\omega_1\tau_c} - 1)}{\mathrm{i}\omega_1} \\ &- \frac{[2hp_{11}p_{21}\bar{p}_{21}\bar{q}_{23} - k\bar{p}_{21}\bar{q}_{22}(p_{21}p_{13} + p_{11}p_{23})](e^{\mathrm{i}\omega_1\tau_c + 2\mathrm{i}\omega_2\tau_c} - 1)}{\mathrm{i}\omega_1 + 2\mathrm{i}\omega_2}, \end{aligned}$$

$$\begin{aligned} m_{0020} =& \frac{(hp^2_{21}p_{11}q_{13} - kp_{21}p_{23}p_{11}q_{12})(e^{2\mathrm{i}\omega_2\tau_c - \mathrm{i}\omega_1\tau_c} - 1)}{\mathrm{i}\omega_1 - 2\mathrm{i}\omega_2} \\ &- \frac{(hp^3_{21}q_{23} - kp^2_{21}p_{23}q_{22})(e^{\mathrm{i}\omega_2\tau_c} - 1)}{\mathrm{i}\omega_2} \end{aligned}$$

$$
\begin{aligned}
&- \frac{(hp_{21}^2\bar{p}_{11}\bar{q}_{13} - kp_{21}p_{23}\bar{p}_{11}\bar{q}_{12})(e^{2\mathrm{i}\omega_2\tau_c+\mathrm{i}\omega_1\tau_c} - 1)}{\mathrm{i}\omega_1 + 2\mathrm{i}\omega_2} \\
&- \frac{(hp_{21}^2\bar{p}_{21}\bar{q}_{23} - kp_{21}p_{23}\bar{p}_{21}\bar{q}_{22})(e^{3\mathrm{i}\omega_2\tau_c} - 1)}{3\mathrm{i}\omega_2},
\end{aligned}
$$

$$
\begin{aligned}
m_{1001} =& \frac{[2hp_{11}^2\bar{p}_{21}q_{13} - kp_{11}q_{12}(\bar{p}_{21}p_{13} + p_{11}\bar{p}_{23})](e^{-\mathrm{i}\omega_2\tau_c} - 1)}{\mathrm{i}\omega_2} \\
&+ \frac{[2hp_{11}\bar{p}_{21}\bar{p}_{11}\bar{q}_{13} - k\bar{p}_{11}\bar{q}_{12}(\bar{p}_{21}p_{13} + p_{11}\bar{p}_{23})](e^{2\mathrm{i}\omega_1\tau_c-\mathrm{i}\omega_2\tau_c} - 1)}{\mathrm{i}\omega_2 - 2\mathrm{i}\omega_1} \\
&+ \frac{[2hp_{11}\bar{p}_{21}p_{21}q_{23} - kp_{21}q_{22}(\bar{p}_{21}p_{13} + p_{11}\bar{p}_{23})](e^{\mathrm{i}\omega_1\tau_c-2\mathrm{i}\omega_2\tau_c} - 1)}{2\mathrm{i}\omega_2 - \omega_1} \\
&- \frac{[2hp_{11}\bar{p}_{21}^2\bar{q}_{23} - k\bar{p}_{21}\bar{q}_{22}(\bar{p}_{21}p_{13} + p_{11}\bar{p}_{23})](e^{\mathrm{i}\omega_1\tau_c} - 1)}{\mathrm{i}\omega_1}.
\end{aligned}
$$

$$
\begin{aligned}
R_1 =& (c + \omega_2\mathrm{i} - \omega_1\mathrm{i})[\mathrm{i}(\omega_2 - \omega_1)(\omega_2\mathrm{i} - \omega_1\mathrm{i} + K_c + a - K_c e^{(\omega_1-\omega_2)\mathrm{i}\tau_c}) + a(kz^* - b)] \\
&+ 2ahkx^{*2}, \\
R_2 =& (c + \omega_2\mathrm{i} + \omega_1\mathrm{i})[\mathrm{i}(\omega_2 + \omega_1)(K_c + a + \omega_2\mathrm{i} + \omega_1\mathrm{i} - K_c e^{-(\omega_1+\omega_2)\mathrm{i}\tau_c}) + a(kz^* - b)] \\
&+ 2ahkx^{*2}, \\
R_3 =& (c + \omega_1\mathrm{i} - \omega_2\mathrm{i})[\mathrm{i}(\omega_1 - \omega_2)(K_c + a + \omega_1\mathrm{i} - \omega_2\mathrm{i} - K_c e^{(\omega_2-\omega_1)\mathrm{i}\tau_c}) + a(kz^* - b)] \\
&+ 2ahkx^{*2}.
\end{aligned}
$$

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