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NUMERICAL SOLUTION OF FUZZY CAMASSA-HOLM EQUATION BY USING HOMOTOPY ANALYSIS METHODS

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Abstract In this paper, a fuzzy Camassa-Holm equation is solved by using the homotopy analysis method (HAM). The approximation solution of this equation is calculated in the form of series which its components are computed by applying a recursive relation. The existence and uniqueness of the solution and the convergence of the proposed method are proved.

Keywords Camassa-Holm equation , Homotopy analysis method, Fuzzy number.

MSC(2000) 35L05, 65H20, and 65M12.

1. Introduction

The Camassa-Holm equations are well suited to modeling the statistical of turbulent fluid flows [4]-[6]. Camassa and Holm derived a completely integrable wave equation (CH)

$$u_t + 2u_x - u_{xxt} + uu_x = 2u_x u_{xx} + uu_{xxx}. \tag{1}$$

The exact solution is $u(x,t) = e^{x+t}$, with the initial conditions:

$$u(x,0) = u_{xx}(x,0) = e^x.$$

Eq.(1) can be derived as an asymptotic model for long gravity waves at the surface of shallow water [3]. The CH equation, being a model equation for water waves, has its integrable bi-Hamiltonian structure [10]. In recent years, some works have been done in order to find the numerical solution of this equation, for example [7, 9, 11, 12, 15, 16, 17, 19, 25, 26, 27, 28, 29, 30, 31, 32].

In this work, we develop the HAM to solve the Camassa-Holm equation with the fuzzy initial conditions as follows:

$$\widetilde{u}_t \oplus 2 \odot \widetilde{u}_x \oplus (-1) \odot \widetilde{u}_{xxt} \oplus \widetilde{u} \odot \widetilde{u}_x = 2 \odot \widetilde{u}_x \odot \widetilde{u}_{xx} \oplus \widetilde{u} \odot \widetilde{u}_{xxx}. \tag{2}$$

The exact fuzzy solution is $u(x,t,\gamma) = (\underline{u}(x,t,\gamma), \overline{u}(x,t,\gamma) = (e^{x+t}(-\gamma^2+\gamma+1), e^{x+t}(\gamma^2-3\gamma+3))$, with the initial conditions:

$$u(x,0,\gamma) = (e^x(-\gamma^2+\gamma+1), e^x(\gamma^2-3\gamma+3)),\ 0 \le \gamma \le 1,$$
$$u_{xx}(x,0,\gamma) = (e^x(-\gamma^2+\gamma+1), e^x(\gamma^2-3\gamma+3)).$$

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To obtain the approximate solution of Eq.(2), by integrating one time Eq.(2) with respect to t and using the fuzzy initial conditions we obtain,

$$\begin{aligned}&\widetilde{u}(x,t)\widetilde{F}(x,t)\oplus(-2)\odot\int_0^t D(\widetilde{u}(x,t))\ dt\\&\oplus(-1)\odot\int_0^t F_1(\widetilde{u}(x,t))\ dt\oplus 2\odot\int_0^t F_2(\widetilde{u}(x,t))\ dt\oplus\int_0^t F_3(\widetilde{u}(x,t))\ dt,\end{aligned}\tag{3}$$

where,

$$\begin{aligned}D^i(\widetilde{u}(x,t)) &= \frac{\partial^i\widetilde{u}(x,t)}{\partial x^i},\quad i=1,2,3,\\ \widetilde{F}(x,t) &= \widetilde{u}(x,0)\oplus(-1)\odot\widetilde{u}_{xx}(x,0)\oplus D^2(\widetilde{u}(x,0)),\\ F_1(u(x,t)) &= \widetilde{u}(x,t)\odot D(\widetilde{u}(x,t)),\\ F_2(u(x,t)) &= D(\widetilde{u}(x,t))\odot D^2(\widetilde{u}(x,t)),v\\ F_3(u(x,t)) &= \widetilde{u}(x,t)\odot D^3(\widetilde{u}(x,t))\\ D(\widetilde{u}(x,t)) &= D^2(\widetilde{u}(x,t))=D^3(\widetilde{u}(x,t))=(\underline{u}_x(x,t,\gamma),\overline{u}_x(x,t,\gamma)\\ &= (e^{x+t}(-\gamma^2+\gamma+1),e^{x+t}(\gamma^2-3\gamma+3))\end{aligned}$$

By interval arithmetic we can write the Camassa-Holm equation (2) in the following term:

$$\begin{aligned}&[\underline{u}_t,\overline{u}_t]+2[\underline{u}_x,\overline{u}_x]-[\underline{u}_{xxt},\overline{u}_{xxt}]+[\underline{u},\overline{u}][\underline{u}_x,\overline{u}_x]\\&= 2[\underline{u}_x,\overline{u}_x][\underline{u}_{xx},\overline{u}_{xx}]+[\underline{u},\overline{u}][\underline{u}_{xxx},\overline{u}_{xxx}],\end{aligned}\tag{4}$$

by above hypothesis we can write two systems and by assuming that u ,u_t u_x,u_{xx}, u_{xxt} and u_{xxx} are positive functions, we have two following crisp systems:

$$\begin{aligned}\underline{u}(x,t,\gamma) &= \underline{F}(x,t,\gamma)-2\int_0^t D(\overline{u}(x,t,\gamma))\ dt-\int_0^t F_1(\overline{u}(x,t,\gamma))\ dt+\\ &\quad 2\int_0^t F_2(\underline{u}(x,t,\gamma))\ dt+\int_0^t F_3(\underline{u}(x,t,\gamma))\ dt,\end{aligned}\tag{5}$$

$$\begin{aligned}\overline{u}(x,t,\gamma) &= \overline{F}(x,t,\gamma)-2\int_0^t D(\underline{u}(x,t,\gamma))\ dt-\int_0^t F_1(\underline{u}(x,t,\gamma))\ dt+\\ &\quad 2\int_0^t F_2(\overline{u}(x,t,\gamma))\ dt+\int_0^t F_3(\overline{u}(x,t,\gamma))\ dt,\end{aligned}\tag{6}$$

In Eq.(3), we assume $\widetilde{F}(x,t)$ is bounded for all x,t in $J=[0,T](T\in\mathbb{R})$.

The terms $D(\widetilde{u}(x,t)),F_1(\widetilde{u}(x,t)),F_2(\widetilde{u}(x,t)),F_3(\widetilde{u}(x,t))$ are Lipschitz continuous with

$\widehat{D}(F_i(\widetilde{u}),F_i(\widetilde{u}^*))\leq L_i\ \widehat{D}(\widetilde{u},\widetilde{u}^*)(i=1,2,3)$, $\widehat{D}(D(\widetilde{u},\widetilde{u}^*)\leq L\widehat{D}(\widetilde{u},\widetilde{u}^*)$, where $\widehat{D}$ is the Hausdorff metric [18] and,

$$\alpha:=T(2L+L_1+2L_2+L_3).$$

2. Preliminaries

The basic definition of a fuzzy number is given in [14, 18] as follows:

Definition 2.1. A fuzzy number is a fuzzy set like $u : \mathbb{R} \rightarrow [0,1]$ which satisfies:

1. u is an upper semi-continuous function,
2. $u(x) = 0$ outside some interval [a,d],
3. There are real numbers b, c such as $a \leq b \leq c \leq d$ and
   3.1 $u(x)$ is a monotonic increasing function on [a, b],
   3.2 $u(x)$ is a monotonic decreasing function on [c, d],
   3.3 $u(x) = 1$ for all $x \in [b, c]$.

Definition 2.2. A fuzzy number u in parametric form is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r)$, $\overline{u}(r)$, $0 \leq r \leq 1$, which satisfy the following requirements:

1. $\underline{u}(r)$ is a bounded non-decreasing left continuous function in $(0, 1]$, and right continuous at 0,
2. $\overline{u}(r)$ is a bounded non-increasing left continuous function in $(0, 1]$, and right continuous at 0,
3. $\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.

Definition 2.3. For arbitrary $\tilde{u} = (\underline{u}(r), \overline{u}(r))$ and $\tilde{v} = (\underline{v}(r), \overline{v}(r))$ and scalar k, we define addition, subtraction and scalar multiplication by k are respectively as following:

$$\underline{u+v}(r) = \underline{u}(r) + \underline{v}(r), \qquad \overline{u+v}(r) = \overline{u}(r) + \overline{v}(r)$$

$$\underline{u-v}(r) = \underline{u}(r) - \overline{v}(r), \qquad \overline{u-v}(r) = \overline{u}(r) - \underline{v}(r)$$

$$\tilde{ku} = \begin{cases} (k\underline{u}(r), k\overline{u}(r)), & k \geq 0 \\ (k\overline{u}(r), k\underline{u}(r)), & k < 0 \end{cases}$$

$$\underline{u} = \sum_{i=0}^{\infty} \underline{u}_i,$$

$$\overline{u} = \sum_{i=0}^{\infty} \overline{u}_i.$$

Now we can solve this two crisp equations.

3. Description of the HAM

Consider

$$N[\overline{u}] = 0,$$

where N is a nonlinear operator, $\overline{u}(x,t)$ is unknown function and x is an independent variable. let $\overline{u}_0(x,t)$ denote an initial guess of the exact solution $\overline{u}(x,t)$, $h \neq 0$ an auxiliary parameter, $H_1(x,t) \neq 0$ an auxiliary function, and L an auxiliary linear operator with the property $L[s(x,t)] = 0$ when $s(x,t) = 0$. Then using $q \in [0,1]$ as an embedding parameter, we construct a homotopy as follows:

$$\begin{aligned} & (1-q)L[\phi(x,t;q)-\overline{u}_0(x,t)] - qhH_1(x,t)N[\phi(x,t;q)] \\ = \;& \hat{H}[\phi(x,t;q);\overline{u}_0(x,t),H_1(x,t),h,q]. \end{aligned} \tag{7}$$

It should be emphasized that we have great freedom to choose the initial guess $\overline{u}_0(x,t)$, the auxiliary linear operator L, the non-zero auxiliary parameter h, and the auxiliary function $H_1(x,t)$.

Enforcing the homotopy (7) to be zero, i.e.,

$$\hat{H}_1[\phi(x,t;q);\overline{u}_0(x,t),H_1(x,t),h,q] = 0, \tag{8}$$

we have the so-called zero-order deformation equation

$$(1-q)L[\phi(x,t;q)-\overline{u}_0(x,t)] = qhH_1(x,t)N[\phi(x,t;q)]. \tag{9}$$

When $q=0$, the zero-order deformation Eq.(9) becomes

$$\phi(x;0) = \overline{u}_0(x,t), \tag{10}$$

and when $q=1$, since $h \neq 0$ and $H_1(x,t) \neq 0$, the zero-order deformation Eq.(9) is equivalent to

$$\phi(x,t;1) = \overline{u}(x,t). \tag{11}$$

Thus, according to (10) and (11), as the embedding parameter q increases from 0 to 1, $\phi(x,t;q)$ varies continuously from the initial approximation $\overline{u}_0(x,t)$ to the exact solution $\overline{u}(x,t)$. Such a kind of continuous variation is called deformation in homotopy [20, 21, 8, 22, 23, 24].

Due to Taylor's theorem, $\phi(x,t;q)$ can be expanded in a power series of q as follows

$$\phi(x,t;q) = \overline{u}_0(x,t) + \sum_{m=1}^{\infty} \overline{u}_m(x,t)q^m, \tag{12}$$

where,

$$\overline{u}_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t;q)}{\partial q^m} \mid_{q=0} .$$

Let the initial guess $\overline{u}_0(x,t)$, the auxiliary linear parameter L, the nonzero auxiliary parameter h and the auxiliary function $H_1(x,t)$ be properly chosen so that the power series (12) of $\phi(x,t;q)$ converges at $q=1$, then, we have under these assumptions the solution series

$$\overline{u}(x,t) = \phi(x,t;1) = \overline{u}_0(x,t) + \sum_{m=1}^{\infty} \overline{u}_m(x,t). \tag{13}$$

From Eq.(12), we can write Eq.(9) as follows

$$\begin{aligned} & (1-q)L[\phi(x,t,q)-\overline{u}_0(x,t)] \\ = \;& (1-q)L[\textstyle\sum_{m=1}^{\infty}\overline{u}_m(x,t)\ q^m] \\ = \;& q\ h\ H_1(x,t)N[\phi(x,t,q)] \\ \Rightarrow \;& L[\textstyle\sum_{m=1}^{\infty}\overline{u}_m(x,t)\ q^m] - q\ L[\textstyle\sum_{m=1}^{\infty}\overline{u}_m(x,t)q^m] \\ = \;& q\ h\ H_1(x,t)N[\phi(x,t,q)] \end{aligned} \tag{14}$$

By differentiating (14) m times with respect to q, we obtain

$$\begin{aligned}
& \{L[\textstyle\sum_{m=1}^{\infty} \overline{u}_m(x,t)\ q^m] - q\ L[\sum_{m=1}^{\infty} \overline{u}_m(x,t) q^m]\}^{(m)} \\
= & \ \{q\ h\ H_1(x,t) N[\phi(x,t,q)]\}^{(m)} \\
= & \ m!\ L[\overline{u}_m(x,t) - \overline{u}_{m-1}(x,t)] \\
= & \ h\ H_1(x,t)\ m\ \tfrac{\partial^{m-1} N[\phi(x,t;q)]}{\partial q^{m-1}}\ |_{q=0}\ .
\end{aligned}$$

Therefore,

$$L[\overline{u}_m(x,t) - \chi_m \overline{u}_{m-1}(x,t)] = h H_1(x,t) \Re_m(\overline{u}_{m-1}(x,t)), \tag{15}$$

where,

$$\Re_m(\overline{u}_{m-1}(x,t)) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(x,t;q)]}{\partial q^{m-1}}\ |_{q=0}, \tag{16}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}$$

Note that the high-order deformation Eq.(15) is governing the linear operator L, and the term $\Re_m(\overline{u}_{m-1}(x,t))$ can be expressed simply by (16) for any nonlinear operator N.

To obtain the approximation solution of Eq.(5), according to HAM, let

$$\begin{aligned}
N[\overline{u}(x,t)] \ = \ & \overline{u}(x,t) - \overline{F}(x,t) + 2\textstyle\int_0^t D(\underline{u}(x,t))\ dt + \int_0^t F_1(\underline{u}(x,t))\ dt \\
& -2\textstyle\int_0^t F_2(\overline{u}(x,t))\ dt - \int_0^t F_3(\overline{u}(x,t))\ dt,
\end{aligned}$$

so,

$$\begin{aligned}
& \Re_m(\overline{u}_{m-1}(x,t)) \\
= \ & \overline{u}_{m-1}(x,t) - \overline{F}(x,t) + 2\textstyle\int_0^t D(\underline{u}_{m-1}(x,t))\ dt + \int_0^t F_1(\underline{u}_{m-1}(x,t))\ dt \\
& -2\textstyle\int_0^t F_2(\overline{u}_{m-1}(x,t))\ dt - \int_0^t F_3(\overline{u}_{m-1}(x,t))\ dt,
\end{aligned} \tag{17}$$

Substituting (17) into (15)

$$\begin{aligned}
& L[\overline{u}_m(x,t) - \chi_m \overline{u}_{m-1}(x,t)] \\
= \ & h H_1(x,t)[\overline{u}_{m-1}(x,t) + 2\textstyle\int_0^t D(\underline{u}_{m-1}(x,t))\ dt + \int_0^t F_1(\underline{u}_{m-1}(x,t))\ dt \\
& -2\textstyle\int_0^t F_2(\overline{u}_{m-1}(x,t))\ dt - \int_0^t F_3(\overline{u}_{m-1}(x,t))\ dt + (1-\chi_m)\overline{F}(x,t)].
\end{aligned} \tag{18}$$

We take an initial guess $\overline{u}_0(x,t) = \overline{F}(x,t)$, an auxiliary linear operator $L\overline{u} = \overline{u}$, a nonzero auxiliary parameter $h = -1$, and auxiliary function $H_1(x,t) = 1$. This is substituted into (18) to give the recurrence relation

$$\begin{aligned}
\overline{u}_0(x,t) \ = \ & \overline{F}(x,t), \\
\overline{u}_{n+1}(x,t) \ = \ & -2\textstyle\int_0^t D(\underline{u}_n(x,t))\ dt - \int_0^t F_1(\underline{u}_n(x,t))\ dt \\
& +2\textstyle\int_0^t F_2(\overline{u}_n(x,t))\ dt + \int_0^t F_3(\overline{u}_n(x,t))\ dt, \quad n \geq 1.
\end{aligned} \tag{19}$$

Also, we can write

$$\begin{aligned} \underline{u}_0(x,t) &= \underline{F}(x,t), \\ \underline{u}_{n+1}(x,t) &= -2\int_0^t D(\overline{u}_n(x,t))\ dt - \int_0^t F_1(\overline{u}_n(x,t))\ dt \\ &\quad +2\int_0^t F_2(\underline{u}_n(x,t))\ dt + \int_0^t F_3(\underline{u}_n(x,t))\ dt, \quad n \geq 1. \end{aligned} \tag{20}$$

(x,t)	Errors ($\widehat{D}$) ($\gamma = 0.3$, n=4))	Errors ($\widehat{D}$) ($\gamma = 0.5$,n=4)
$(0.3, 0.15)$	0.030281	0.032267
$(0.35, 0.20)$	0.034184	0.032267
$(0.4, .25)$	0.038754	0.036754
$(0.45, 0.30)$	0.032683	0.038867
$(0.5, 0.37)$	0.045375	0.043578
$(0.55, 0.40)$	0.047284	0.045638
$(0.6, 0.45)$	0.049881	0.047245
$(0.65, 0.48)$	0.052674	0.051257
$(0.7, 0.50)$	0.055843	0.053897
$(0.75, 0.54)$	0.057698	0.056245
$(0.8, 0.62)$	0.059675	0.057895

4. Existence and convergency of homotopy analysis method

Theorem 4.1. *Let $0 < \alpha < 1$, then equation (3), has a unique solution.*

Proof. Let $\widetilde{u}$ and $\widetilde{u}^*$ be two different solutions of (3) then

$$\begin{aligned} & D(\widetilde{u}, \widetilde{u}^*) \\ = \ & D(\widetilde{F}(x,t) \oplus (-2) \odot \int_0^t D(\widetilde{u}(x,t))\ dt \\ & \oplus(-1) \odot \int_0^t F_1(\widetilde{u}(x,t))\ dt \oplus 2 \odot \int_0^t F_2(\widetilde{u}(x,t))\ dt \oplus \int_0^t F_3(\widetilde{u}(x,t))\ dt, \\ & \widetilde{F}(x,t) \oplus (-2) \odot \int_0^t D(\widetilde{u}^*(x,t))\ dt \oplus (-1) \odot \int_0^t F_1(\widetilde{u}^*(x,t))\ dt \\ & \oplus 2 \odot \int_0^t F_2(\widetilde{u}^*(x,t))\ dt \oplus \int_0^t F_3(\widetilde{u}^*(x,t))\ dt) \\ & \leq T(2L + L_1 + 2L_2 + L_3)\ D(\widetilde{u}, \widetilde{u}^*) = \alpha\ D(\widetilde{u}, \widetilde{u}^*). \end{aligned}$$

From which we get $(1-\alpha)D(\widetilde{u}, \widetilde{u}^*) \leq 0$. Since $0 < \alpha < 1$, then $D(\widetilde{u}, \widetilde{u}^*) = 0$. Implies $\widetilde{u} = \widetilde{u}^*$ and completes the proof. □

Theorem 4.2. *If the series solutions (19) and (20) of problem (3) using HAM convergent then it converges to the exact solution of the problem (3).*

Proof. We assume:

$$\begin{aligned} \phi_{k+1}(x,t) &= F(x,t) \oplus \sum_{i=1}^{k+1} [(-2) \odot \int_0^t D(\widetilde{u}(x,t))\ dt \\ &\quad \oplus(-1) \odot \int_0^t F_1(\widetilde{u}(x,t))\ dt \oplus 2 \odot \int_0^t F_2(\widetilde{u}(x,t))\ dt \\ &\quad \oplus \int_0^t F_3(\widetilde{u}(x,t))\ dt], \quad k \geq 0. \end{aligned}$$

$$
\begin{aligned}
& D(\phi_{k+1}(x,t),\phi_k(x,t)) \\
= \quad & D(F(x,t) \oplus \sum_{i=1}^{k+1}[(-2)\odot\int_0^t D(\widetilde{u}_i(x,t))\ dt \\
& \oplus(-1)\odot\int_0^t F_1(\widetilde{u}_i(x,t))\ dt \oplus 2\odot\int_0^t F_2(\widetilde{u}_i(x,t))\ dt \oplus \int_0^t F_3(\widetilde{u}_i(x,t))\ dt], \\
& F(x,t)\oplus\sum_{i=1}^{k+1}[(-2)\odot\int_0^t D(\widetilde{u}_{i-1}(x,t))\ dt \oplus (-1)\odot\int_0^t F_1(\widetilde{u}_{i-1}(x,t))\ dt \\
& \oplus 2\odot\int_0^t F_2(\widetilde{u}_{i-1}(x,t))\ dt \oplus \int_0^t F_3(\widetilde{u}_{i-1}(x,t))\ dt]) \\
= \quad & D(\phi_k(x,t)\oplus(-2)\odot\int_0^t\int_a^x (x-t)\ \odot F_1(u_k(x,t))\ dt\ dx \\
& \oplus(-1)\odot\int_0^t\int_a^x (x-t)\ \odot F_2(u_k(x,t))\ dt\ dx, \phi_k(x,t)) \\
= \quad & D((-2)\odot\int_0^t D(\widetilde{u}(x,t))\ dt \oplus (-1)\odot\int_0^t F_1(\widetilde{u}(x,t))\ dt \\
& \oplus 2\odot\int_0^t F_2(\widetilde{u}(x,t))\ dt \oplus \int_0^t F_3(\widetilde{u}(x,t))\ dt, \widetilde{0}) \\
\leq \quad & D(\widetilde{u}_k(x,t),\widetilde{0}) \\
& D(\widetilde{u}_k(x,t),\widetilde{0}) \leq \alpha^k D(F,\widetilde{0}) \\
\Longrightarrow \quad & D(\phi_{k+1}(x,t),\phi_k(x,t)) \leq \alpha^{k+1} D(F,\widetilde{0}) \\
\Longrightarrow \quad & \sum_{k=0}^{\infty} D(\phi_{k+1}(x,t),\phi_k(x,t)) \leq \alpha^{k+1} D(F,\widetilde{0}) \sum_{k=0}^{\infty}\alpha^k.
\end{aligned}
$$

Algorithm:
Step 1. Set $n \leftarrow 0$.
Step 2. Calculate the recursive relations (19) and (20).
Step 3. If $\mid u_{n+1} - u_n \mid < \varepsilon$ then go to step 4,
else $n \leftarrow n+1$ and go to step 2.
Step 4. Print $u(x,t) = \sum_{i=0}^{n} u_i(x,t)$ as the approximate of the exact solution.

□

Lemma 4.1. *The computational complexity of the HAM is* $O(n)$.

Proof. The number of computations including division, production, sum and subtraction.
$\overline{u}_0, \underline{u}_0 : 6.$
$\overline{u}_1, \underline{u}_1 : 22.$
.
.
$\overline{u}_{n+1}, \underline{u}_{n+1} : 22.$
The total number of the computations is equal to
$\sum_{i=0}^{n+1} \overline{u}_i(x,t) + \sum_{i=0}^{n+1} \underline{u}_i(x,t) = O(n).$

□

5. Conclusion

The HAM has been shown to solve effectively, easily and accurately a large class of nonlinear problems with the approximations which convergent are rapidly to the exact solutions. In this work, the HAM has been successfully employed to obtain the approximate analytical solution of the fuzzy Camassa-Holm equation.

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