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FOUR LIMIT CYCLES IN QUADRATIC NEAR-INTEGRABLE SYSTEMS*

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Abstract In this note, we report of obtaining 4 limit cycles in quadratic near-integrable polynomial systems. It is shown that when a quadratic integrable system has two centers and is perturbed by quadratic polynomials, it can generate at least 4 limit cycles with $(3,1)$ distribution. This result provides a positive answer to an open problem in this area.

Keywords Hilbert's 16th problem, quadratic near-integrable system, limit cycle, reversible system, Hopf bifurcation, Melnikov function.

MSC(2000) 34C07, 34C23.

1. Introduction

The well-known Hilbert's 16th problem is remained unsolved since Hilbert [8] proposed the 23 mathematical problems at the Second International Congress of Mathematics in 1900. Recently, a modern version of the second part of the 16th problem was formulated by Smale [20], chosen as one of the 18 challenging mathematical problems for the 21st century. To be more specific, consider the following planar system:

$$\dot{x} = P_n(x,y), \quad \dot{y} = Q_n(x,y), \tag{1}$$

where the dot denotes differentiation with respect to time, t, and $P_n(x,y)$ and $Q_n(x,y)$ represent n^{th}-degree polynomials of x and y. The second part of Hilbert's 16th problem is to find the upper bound $H(n) \le n^q$ on the number of limit cycles that the system can have, where q is a universal constant, and $H(n)$ is called Hilbert number.

If the problem is restricted to a neighborhood of an isolated fixed point, then the problem is reduced to studying degenerate Hopf bifurcations, giving rise to a weak (fine) focus point. In the past six decades, many researchers considered the local problem and obtained many results (e.g., see [2, 10, 12, 14, 15, 22]). In the last 20 years, much progress on finite cyclicity near a weak focus point or a homoclinic loop was achieved.

In this paper, we particularly consider bifurcation of limit cycles in quadratic systems. Early results can be found in a survey article by Ye [21]. Some recent

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progress has been reported in a number of papers (e.g., see [17, 18]). For general quadratic system (1) ($n = 2$), in 1952, Bautin [2] proved that there exist 3 small amplitude limit cycles around a weak focus point or a center. Later around the end of 1970's, concrete examples were given to show that general quadratic systems can have 4 limit cycles [3, 19], around two foci with $(3, 1)$ configuration. A question was naturally raised: Can near-integrable quadratic systems have 4 limit cycles? A quadratic system is called near-integrable if it is a perturbation of a quadratic integrable system by quadratic polynomials. To our best knowledge, this problem is still open.

The study of bifurcation of limit cycles for near-integrable systems is related to the so called weak Hilbert's 16th problem [1], which is transformed to finding the maximal number of isolated zeros of the Abelian integral or Melnikov function:

$$M(h, \delta) = \oint_{H(x,y)=h} Q_n \, dx - P_n \, dy, \tag{2}$$

where $H(x, y)$, P_n and Q_n are all real polynomials of x and y with $\deg H = n+1$, and $\max\{\deg P_n, \deg Q_n\} \leq n$. The weak Hilbert's 16th problem is a very important problem, closely related to the maximal number of limit cycles of the following near-Hamiltonian system [6]:

$$\dot{x} = H_y(x, y) + \varepsilon \, p_n(x, y), \quad \dot{y} = -H_x(x, y) + \varepsilon \, q_n(x, y), \tag{3}$$

where $H(x, y)$, $p_n(x, y)$ and $q_n(x, y)$ are polynomials of x and y, and $0 < \varepsilon \ll 1$ is a small perturbation.

General quadratic systems with one center have been classified by Żołądek [25] using a complex analysis on the condition of the center, as four systems: Q_3^{LV} – the Lotka-Volterra system; Q_3^H – Hamiltonian system; Q_3^R – reversible system; and Q_4 – codimension-4 system. It has been shown [9, 5] that generic quadratic Hamiltonian systems with quadratic perturbations can have maximal two limit cycles. For the Q_3^R reversible system, there have been many results published (e.g., see [4, 16, 24, 11, 13]). It has been noticed in all these papers that existence of 4 limit cycles was not reported. In this paper, we consider bifurcation of limit cycles in quadratic near-integrable systems with two centers, and show existence of 4 limit cycles. The basic idea is as follows: We first consider bifurcation of multiple small amplitude limit cycles from Hopf singularity, and then compute the global Melnikov function to serach for possible large limit cycles. In particular, we will show that perturbing a reversible, integrable quadratic system with two centers can have at least 4 limit cycles, with $(3, 1)$ distribution.

2. Main Result

General quadratic systems with a center at the origin can be classified as four systems.

$a_3 = a_2 = 0$ (Q_3^R - Reversible system):

$$\dot{x} = y + a_1 \, x \, y, \qquad \dot{y} = -x + x^2 + a_4 \, y^2. \tag{4}$$

$a_3 = a_1 + 2\, a_4 = 0$ (Q_3^H - Hamiltonian system):

$$\dot{x} = y + a_1 \, x \, y + a_2 \, y^2, \qquad \dot{y} = -x + x^2 - \tfrac{1}{2} \, a_1 \, y^2. \tag{5}$$

$a_2 = 1 + a_4 = 0$ (Q_3^{LV} - Lokta-Volterra system):

$$\dot{x} = y + a_1\, x\, y, \qquad \dot{y} = -\,x + x^2 + a_3\, x\, y - y^2. \tag{6}$$

$a_3 - 5\,a_2 = a_1 - (5 + 3\,a_4) = a_4 + 2(1 + a_2^2) = 0$ (Q_4 - Codimension-4 system):

$$\dot{x} = y - (1 + 6\,a_2^2)\, x\, y + a_2\, y^2, \qquad \dot{y} = -\,x + x^2 + 5\,a_2\, x\, y - 2\,(1 + a_2^2)\, y^2. \tag{7}$$

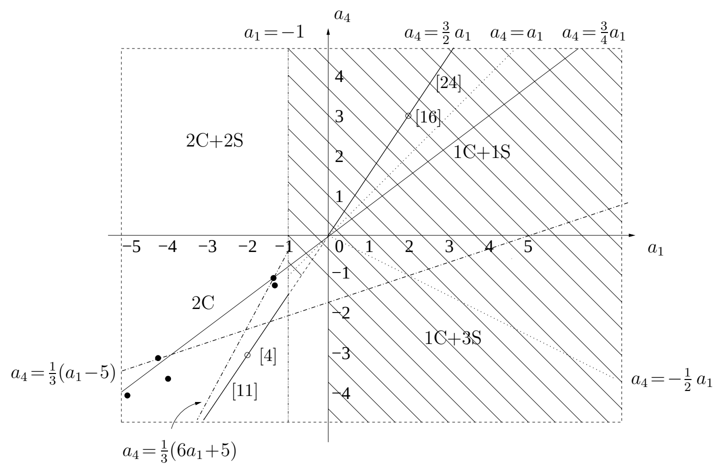

Figure 1. Case studies for the Q_3^R reversible system.

In this note, we concentrate on the reversible system (4). It is easy to see that this system is invariant under the mapping $(t,\, y) \to (-t,\, -y)$, where a_1 and a_4 can be considered as perturbation parameters. The singular point $(1, 0)$ of (4) is a center when $a_1 < -1$; but a saddle point when $a_1 > -1$. $a_1 = -1$ gives a degenerate singular point at $(1, 0)$. The distribution of singularity of the reversible system (4) is shown in Fig. 1, where 1C+1S stands for one center and one saddle point, similar meaning applies to 2C, 2C+2S and 1C+3S. The results mentioned in above references [4, 16, 24, 11, 13] are demonstrated in this figure by blank circles. The two particular dash-dotted lines: $a_4 = \frac{1}{3}\,(a_1 - 5)\ \forall\, a_1 \in (-\infty,\, -1) \cup (-1, \infty)$, and $a_4 = \frac{1}{3}\,(6\,a_1 + 5)\ \forall\, a_1 \in (-\infty,\, -1)$, as well as the five dark circles correspond to our results, presented in this note. In fact, there exist 3 small amplitude limit cycles on the two dash-dotted lines, and at least 4 limit cycles for the parameter values marked by the five dark circles.

In the following, we consider perturbing the reversible system (4). Without loss of generality, we may assume the perturbed system is given by

$$\dot{x} = y\,(1 + a_1\, x) + \varepsilon\, a_{10}\, x, \qquad \dot{y} = -\,x + x^2 + a_4\, y^2 + \varepsilon\,(b_{01}\, y + b_{11}\, x\, y), \tag{8}$$

where $a_1 < -1$ and $0 < \varepsilon \ll 1$.

Consider system (8) for $a_1 < -1$. The system $(8)_{\varepsilon=0}$ is a reversible integrable system. Multiplying $\gamma = |1 + a_1 x|^{-\frac{a_1 + 2a_4}{a_1}}$ on both sides of (8) yields the perturbed Hamiltonian system:

$$\begin{aligned} \frac{dx}{d\tau} &= \gamma\,(y + a_1\, x\, y) + \varepsilon\, \gamma\, a_{10}\, x, \\ \frac{dy}{d\tau} &= \gamma\,(-\,x + x^2 + a_4\, y^2) + \varepsilon\, \gamma\,(b_{01}\, y + b_{11}\, x\, y), \end{aligned} \tag{9}$$

with the Hamiltonian of $(9)_{\varepsilon=0}$, given by

$$H(x,y) = \frac{1}{2}\,\mathrm{sign}(1+a_1x)\,|1+a_1x|^{-\frac{2a_4}{a_1}}\left[y^2 + \tfrac{(1+a_1-a_4)\,(1+2\,a_4\,x)}{a_4\,(a_1-a_4)\,(a_1-2\,a_4)} - \tfrac{x^2}{a_1-a_4}\right], \quad (10)$$

for $a_4 \neq 0,\ a_1 \neq a_4,\ a_1 \neq 2\,a_4$. Now introduce

$$L_h : H(x,y) = h \begin{cases} h \in (h_{00}, \infty), & \text{for } 1 + a_1\,x > 0, \quad (h_{00} = H(0,0)) \\ h \in (-\infty, h_{10}), & \text{for } 1 + a_1\,x < 0, \quad (h_{10} = H(1,0)), \end{cases} \quad (11)$$

and define the Melnikov function:

$$M(h, a_{ij}, b_{ij}) = \oint_{L_h} q(x,y,b_{ij})\,dx - p(x,y,a_{ij})\,dy, \quad (12)$$

where $p(x,y,a_{ij}) = \gamma\,a_{10}\,x$ and $q(x,y,b_{ij}) = \gamma\,(b_{01} + b_{11}\,x)\,y$. Using the results in [6], we can expand M near $h = h_{00}$ and $h = h_{10}$ as

$$\begin{aligned} M_0(h,a_{ij},b_{ij}) &= \sum_{k=0} \mu_{0k}\,(h-h_{00})^k, \quad \text{for } 0 < h - h_{00} \ll 1, \\ M_1(h,a_{ij},b_{ij}) &= \sum_{k=0} \mu_{k0}\,(h_{10}-h)^k, \quad \text{for } 0 < h_{10} - h \ll 1, \end{aligned} \quad (13)$$

where the coefficients $\mu_{ij},\ i = 0,\,1;\ j = 0,\,1,\,2,\,\cdots$ can be obtained by using the Maple programs developed in [7].

In general, we can use the perturbation coefficients a_{10}, b_{10} and b_{11} to obtain $\mu_{00} = \mu_{01} = \mu_{02} = 0$, but $\mu_{03} \neq 3$, implying that proper perturbations on these parameters can generate maximal 3 small amplitude limit cycles around $(0,0)$. Similarly, we can have maximal 3 small amplitude limit cycles around $(1,0)$. This shows that for small amplitude limit cycles, it has $(3,0)$ or $(0,3)$ distribution. Other distributions such as $(2,0)$, $(0,2)$ and $(1,1)$ exit, but no $(2,1)$ or $(1,2)$ distributions.

Next, based on the above distribution of small amplitude limit cycles, we investigate the possibility of large limit cycles by applying the Melnikov function. Since it is not possible to find the closed form of the Melnikov function, we shall choose proper values for a_1 to find more limit cycles, and only prove the case of $(3,0)$-distribution (other cases can be similarly proved).

For the $(3,0)$-distribution, we obtain

$$a_4 = \tfrac{1}{3}\,(a_1 - 5), \quad b_{01} = -\,a_{10}, \quad b_{11} = -\,10\,(1+a_1)\,a_{10}.$$

Taking $a_1 = -\frac{30}{7}$ yields $a_4 = -\frac{65}{21}$, which denotes a point (a blank circle) on the line $a_4 = \frac{1}{3}(a_1 - 5)$ in the a_1-a_4 parameter plane (see Fig. 1). Further, we have $b_{11} = \frac{230}{7}\,a_{10}$, and $\gamma = (1 - \frac{30}{7}\,x)^{-\frac{22}{9}}$ $(x \neq \frac{7}{30})$. Then, the Hamiltonian (10) becomes

$$H(x,y) = \tfrac{16250\,y^2 + 13650\,x^2 + 2730\,x - 441}{32500\,(1-\frac{30}{7}\,x)^{13/9}} \qquad \text{for } x \neq \tfrac{7}{30},$$

with

$$h_{00} = -\tfrac{441}{32500} > h_{10} = -\tfrac{15939}{32500}(\tfrac{7}{23})^{13/9}.$$

The Melnikov functions $M_i(h,a_{10})$ can be expressed as $M_i(h,a_{10}) = M_{i0}(h)\,a_{10}$, $i = 0,\,1$. Without loss of generality, we may assume $a_{10} > 0$, and thus $M_i(h,a_{10})$

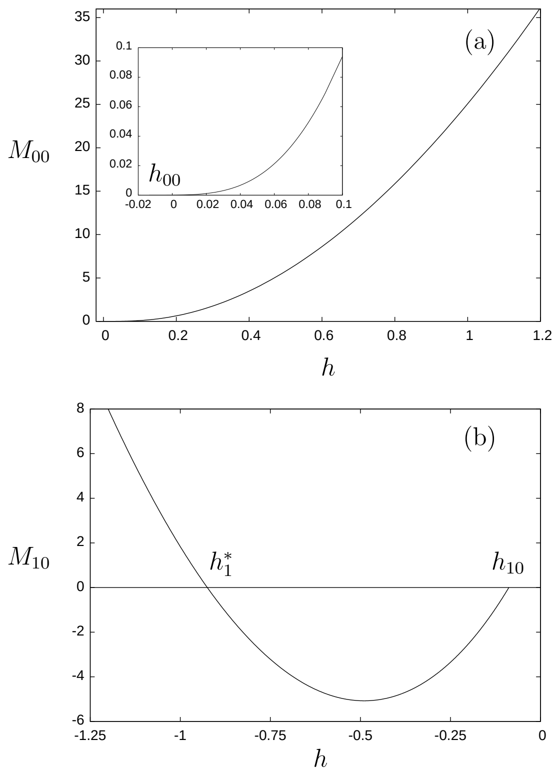

Figure 2. Functions $M_{00}(h)$ and $M_{10}(h)$ under the conditions $\mu_{00} = \mu_{01} = \mu_{02} = 0$, $\mu_{03} \neq 0$ and $\mu_{10} \neq 0$, for $a_1 = -\frac{30}{7}$ and $a_4 = \frac{1}{3}(a_1 - 5) = -\frac{65}{21}$: (a) $M_{00}(h) > 0$ for $h \in [h_{00}, +\infty)$, with $h_0 = -\frac{441}{32500} \approx -0.01357$; and (b) $M_{10}(h)$ for $h \in (-\infty, h_1]$, with $h_{10} = -\frac{15939}{32500}(\frac{7}{23})^{13/9} \approx -0.08797$, crossing the h-axis at $h = h_1^* \in (-0.9250363254, -0.9250363253)$.

and $M_{i0}(h)$ have the same sign. It is noted that for the above chosen parameter values, we have

$$\mu_{03} = \frac{139150000\,\pi}{453789}\, a_{10} > 0 \quad \text{and} \quad \mu_{10} = -\frac{2500\sqrt{161}\,\pi}{3703}\, a_{10} < 0.$$

The computation results of $M_{00}(h)$ for $h \in (h_{00}, \infty)$ and $M_{10}(h)$ for $h \in (-\infty, h_{10})$ are shown, respectively, in Figs. 2(a) and 2(b). Figure 2(a) shows that $M_{00}(h) > 0$ for $h \in (h_{00}, \infty)$, and its sign agrees with that of $\mu_{03} > 0$ for $0 < h - h_{00} \ll 1$, as expected. It is also noted, as shown in Fig. 2(b), that the sign of $M_{10}(h)$ agrees with that of $\mu_{10} < 0$ for $0 < h_{10} - h \ll 1$. However, this interval contains a critical value $h = h_1^* \in (-0.9250363254, -0.9250363253)$ at which $M_{10}(h_1^*) = 0$ and the function $M_{10}(h)$ changes its sign as h crosses this critical point. Thus, for this case, besides the 3 small amplitude limit cycles, there exists at least one large limit cycle bifurcating from the closed orbit $L_{h_1^*}$ of (11). This large limit cycle is shown in Fig. 3(a), which encloses the center $(1, 0)$; and

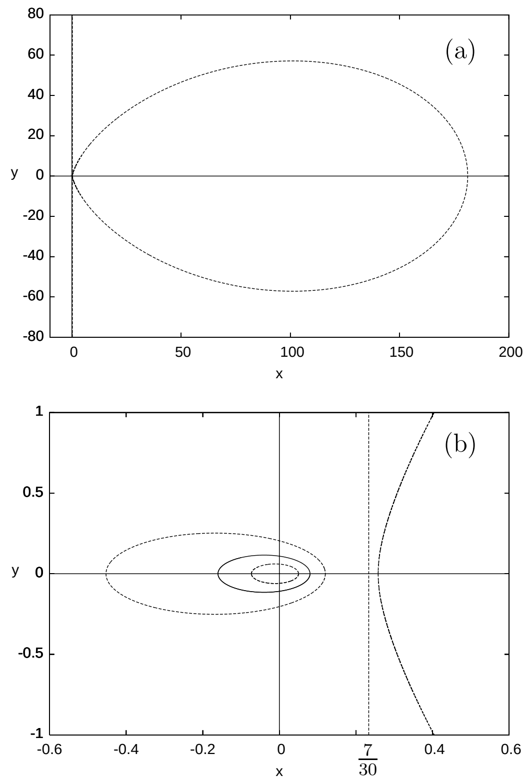

Figure 3. Illustration of the existence of 4 limit cycles when $a_1 = -\frac{30}{7}$, $a_4 = \frac{1}{3}(a_1 - 5) = -\frac{65}{21} - \varepsilon_1$, and $a_{10} = \frac{1}{2000}$, $b_{11} = \frac{230}{21}\, a_{10} - \varepsilon_2$, $b_{01} = -\, a_{10} - \varepsilon_3$, where $0 < \varepsilon_3 \ll \varepsilon_2 \ll \varepsilon_1 \ll \varepsilon$: (a) An unstable large limit cycle enclosing the center $(1, 0)$; and (b) Zoomed area around the center $(0, 0)$ showing the existence of 3 small amplitude limit cycles.

Fig. 3(b) illustrates the existence of 3 small amplitude limit cycles around the center $(0, 0)$.

The above result shows that a quadratic non-Hamiltonian integrable system with two centers can have at least 4 limit cycles under quadratic perturbations, with distributions either $(3, 1)$ or $(1, 3)$. This result gives a new record, answering the open problem of the existence of limit cycles in near-integrable quadratic systems.

References

[1] V. I. Arnold, *Loss of stability of self-oscillations close to resonance and versal deformations of equivariant vector fields*, Funct. Anal. Appl., 11(1977), 85-92.

[2] N. N. Bautin, *On the number of limit cycles which appear with the variation of*

coefficients from an equilibrium position of focus or center type, Mat. Sbornik (N.S.), 30:72(1952), 181-196.

[3] L. S. Chen and M. S. Wang, *The relative position, and the number, of limit cycles of a quadratic differential system*, Acta. Math. Sinica, 22(1979), 751-758.

[4] F. Dumortier, C. Li and Z. Zhang, *Unfolding of a quadratic integrable system with two centers and two unbounded heteroclinic loop*, J. Diff. Eqns., 139(1997), 146-193.

[5] L. Gavrilov, *The infinitesimal 16th Hilbert problem in the quadratic case*, Inventiones Mathematicae, 143(2001), 449-497.

[6] M. Han, *Bifurcation of limit cycles of planar systems*, Handbook of Differential Equations, Ordinary Differential Equations, Vol. 3 (Eds. A. Canada, P. Drabek and A. Fonda), Elsevier, 2006.

[7] M. Han, J. Yang and P. Yu, *Hopf Bifurcations for near-Hamiltonian Systems*, Int. J. Bifurcation and Chaos, 19:12(2009), 4117-4130.

[8] D. Hilbert, *Mathematical problems*, (M. Newton, Transl.) Bull. Amer. Math., 8(1902), 437-479.

[9] E. Horozov and I. D. Iliev, *On the number of limit cycles in perturbations of quadratic Hamiltonian systems*, J. Diff. Eqns., 113(1994), 198-224.

[10] I. S. Kukles, *Necessary and sufficient conditions for the existence of center*, Dokl. Akad. Nauk, 42(1944), 160-163.

[11] I. D. Iliev, C. Z. Li and J. Yu, *Bifurcation of limit cycles from quadratic non-Hamiltonian systems with two centers and two unbounded heteroclinic loops*, Nonlinearity, 18(2005), 305-330.

[12] J. Li and Z. Liu, *Bifurcation set and limit cycles forming compound eyes in a perturbed Hamiltonian system*, Publications Mathematiques, 35(1991), 487-506.

[13] C. Li and J. Llibre, *Quadratic perturbations of a quadratic reversible Lotka-Volterra system*, Qual. Theory Dyn. syst., 9(2010), 235-249.

[14] Y. Liu and J. Li, *On the singularity values of complex autonomous differential systems*, Sci. China (Ser. A), 3(1989), 2450-2555.

[15] K. E. Malkin, *Criteria for center of a differential equation*, Volg. Matem. Sbornik, 2(1964), 87-91.

[16] L. Peng, *Unfolding of a quadratic integrable system with a homoclinic loop*, Acta Math. Sinica, 18(2002), 737–754.

[17] R. Roussarie, *Bifurcation of planar vector fields and Hilbert's 16th problem*, Progress in Mathematics, 164, Birkauser Verlag, Basel, 1998.

[18] R. Roussarie and D. Schlomiuk, *On the geometric structure of the class of planar quadratic differential systems*, Qualitative Theory of Dynamical Systems, 3(2002), 93-122.

[19] S. Shi, *A concrete example of the existence of four limit cycles for plane quadratic systems*, Sci. Sinica, 11(1979), 1051-1056(Chinese); 23(1980), 153-158(English).

[20] S. Smale, *Mathematical problems for the next century*, The Math. Intell., 20(1998), 7-15.

[21] Y. Q. Ye, *Some problems in the qualitative theory of ordinary differential equations*, J. Diff. Eqns., 46(1982), 153-164.

[22] P. Yu and M. Han, *Twelve limit cycles in a cubic case of the 16th Hilbert problem*, Int. J. Bifurcation & Chaos, 15:7(2005), 2191-2205.

[23] P. Yu and M. Han, *Four limit cycles from perturbing quadratic integrable systems by quadratic polynomials*, arXiv.org No. 1002.1005v1, posted since February 4, 2010.

[24] J. Yu and C. Z. Li, *Bifurcation of a class of planar non-Hamiltonian integrable systems with one center and one homoclinic loop*, J. Math. Anal. Appl., 269(2002), 227-243.

[25] H. Żołądek, *Quadratic systems with center and their perturbations*, J. Diff. Eqns., 109(1994), 223-273.