

# ON UNIQUENESS AND ANALYTICITY IN THERMOVISCOELASTIC SOLIDS WITH VOIDS\*

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**Abstract** In this paper we consider the most general system proposed to describe the thermoviscoelasticity with voids. We study two qualitative properties of the solutions of this theory. First, we obtain a uniqueness result when we do not assume any sign to the internal energy. Second we extend some previous results and prove the analyticity of the solutions. The impossibility of localization in time of the solutions is a consequence. Last result we present corresponds to the analyticity of solutions in case that the dissipation is not very strong, but with suitable coupling terms.

**Keywords** Thermoviscoelasticity, uniqueness, analyticity, exponential decay.

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## 1. Introduction

Elastic solids with voids is one of the simple extensions of the theory of the classical elasticity. It allows the treatment of porous solids in which the matrix material is elastic and the interstices are void of material. In this paper, we deal with the theory established by Cowin and Nunziato [3, 4, 14]. Besides the usual elastic effects, these materials have a microstructure with an important property: the mass in each point can be obtained as the product of the mass density of the material matrix by the volume fraction. That is materials where the skeletal or matrix material is elastic and the interstices are void of material. These kind of materials have been widely discussed in the book by Ieşan [7]. The significance of elastic materials with microstructure has been demonstrated amply by the huge quantity of articles published in the last four decades covering applications to different fields of physics and engineering (such as petroleum industry, material science, biology, etc) and the theory of elasticity with voids applies to solids characterized by small distributed pores, such as rocks, soils, wood, ceramics, pressed powders or biological materials such as bones.

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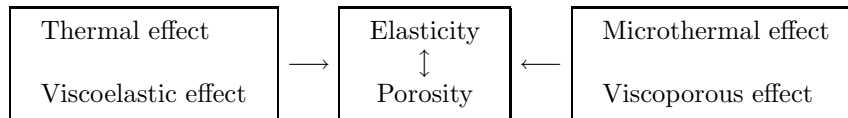
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Elasticity problems have attracted the attention of researchers from different fields interested in the temporal decay behavior of the solutions. As the elastic materials with voids have macroscopic and microscopic structures, it is relevant to clarify the interactions between both structures. One would like to know if the coupling is strong or weak. One aspect to clarify of the coupling could be to consider dissipation mechanisms at macroscopic (and/or microscopic) level and to study the kind of longtime behavior of solutions. Many papers has been published where the authors try to clarify the rate of decay of solutions in elasticity with voids. The first contribution in this line was proposed by Quintanilla [17]. There, the author showed that this coupling is generically weak in the sense that the dissipation at the level of the microstructure is not able to bring all of the system to an exponential decay. That is the decay of the solutions can be very slow. Since this contribution many people have tried to see how the different mechanisms we consider bring all of the system to a exponential decay or a slow decay. Some different dissipation mechanisms as rate type viscoelasticity, rate type porous viscosity, thermal effects, microthermal effects, boundary effects etc. have been considered. It is not very difficult to see that every one of these mechanisms is able to bring the macroscopic (or the microscopic) components if we only consider them separately. However we generically need at least two dissipation mechanisms to obtain exponential decay of solutions. To be precise Casas and Quintanilla [1, 2] proved the exponential decay if we combine porous dissipation (or microtemperatures) with temperatures, Glowinski and Lada [5], Lazzari and Nibbi [9] did a similar thing when they propose several dissipation mechanisms on the boundary, Magaña and Quintanilla [12] developed a very systematic study in case that we consider rate type viscoelasticity, rate type porous viscosity, thermal effects and microthermal effects as well as hyperbolic heat conduction. The main conclusions can be recalled with the help of a scheme:



If we take simultaneously one effect from the right square and another one from the left square, then we get exponential stability. However, if we consider two simultaneous effects from one square only, then we get slow decay.

We also mention the general study developed at [16] when the dissipation mechanisms are of memory type.

Pamplona *et al.* [15] also proved that for isotropic bodies the solutions are analytic if the dissipation mechanisms are of rate type [8]. All these contributions apply to centrosymmetric and isotropic one dimensional materials. Here, we want to consider the most general system of equations proposed to describe thermoelastic solids with voids. The system of equations proposed by Ieşan [6] is an extension from two (at least) view points. On one side we assume that the material is not centrosymmetric neither isotropic and on the other side the dissipation mechanisms is the most general has been considered until this moment (see the recent contribution of Ieşan [6]). Apart to include a rate effect on the volumetric response (as in [3, 4]), we assume that the time derivative of the strain tensor and the time derivative of the gradient of the volumetric fraction field are also included in the set of

constitutive variables. It is worth noting that all the studies concerning time decay previously mentioned in poro-elasticity is concerning isotropic materials. However, in the last time a big interest has been developed to understand the chiral materials. Our contribution can be considered in this line. It is motivated by the desire to know what kind of behavior we can expect for these materials.

Our intention is to show how the dissipation mechanisms implies several qualitative properties of the solutions. In the first part of the paper we give a new result of uniqueness of solutions for the problem which is different from the one proposed by Ieşan [6]. In fact we do not make any assumption on the sign of the internal energy, but we assume the positivity of the dissipation. The other main aim of this paper is to extend the arguments proposed by Pamplona *et al.* [15] to obtain the analyticity of solutions to the general system for thermoviscoelastic materials with voids. We also prove the analyticity of the solutions a case where the coupling mechanisms play a fundamental role and the dissipation mechanisms are weaker.

This paper is structured as follows. In section 2 we state the general system of equations for the thermoviscoelastic solids with voids. A uniqueness result is proved in section 3. This uniqueness result uses in a strong way the positivity of the dissipation mechanism, but we do not impose any assumption on the internal energy. In section 4 we show the well-posedness of the problem in case we also assume that the internal energy and the dissipation are positive. However, to do that we restrict our attention to the one-dimensional case. In section 5 we show that the semigroup is analytic, which in particular implies the exponential decay. In the last section we show how the coupling mechanisms can be used to obtain the analyticity of solution in case that the dissipation is less strong than in the general case.

## 2. Basic Equations

We will denote by  $\Omega$  a bounded domain smooth enough to guarantee the use of the divergence theorem.

The evolution equations for the theory of elastic solids with voids are

$$\rho \ddot{u}_i = t_{ji,j}, \quad \rho \kappa \dot{\varphi} = H_{j,j} + g, \quad \rho T_0 \dot{\Xi} = Q_{j,j}. \quad (2.1)$$

Here,  $t_{ji}$  is the stress,  $H_i$  is the equilibrated stress,  $g$  is the equilibrated body force,  $Q_i$  is the heat flux and  $T_0$  is the absolute temperature in the reference configuration which is assumed positive. The variables  $u_i, \varphi$  and  $\Xi$  are the displacement, the volume fraction and the entropy respectively. We assume that  $\rho$  and  $\kappa$  are positive constants whose physical meaning is well known. In general, we can consider several dissipation mechanisms in this theory. We here, restrict our attention to the case that the viscoelasticity is present and the viscosity at the microstructure is also present apart the temperature effect. That is in our case, we assume the following constitutive equations (see [6])

$$\begin{aligned} t_{ij} &= C_{ijrs} e_{rs} + B_{ij} \varphi + D_{ijk} \varphi_{,k} - \beta_{ij} \theta + S_{ij}^*, \\ H_i &= A_{ij} \varphi_{,j} + D_{rsi} e_{rs} + d_i \varphi - a_i \theta + H_i^*, \\ g &= -B_{ij} e_{ij} - \xi \varphi - d_i \varphi_{,i} + m \theta + g^*, \\ \rho \Xi &= \beta_{ij} e_{ij} + a \theta + m \varphi + a_i \varphi_{,i}, \\ Q_i &= k_{ij} \theta_{,j} + f_{irs} \dot{e}_{rs} + b_i \dot{\varphi} + a_{ij} \dot{\varphi}_{,j}. \end{aligned}$$

where

$$\begin{aligned} S_{ij}^* &= C_{ijrs}^* \dot{e}_{rs} + B_{ij}^* \dot{\varphi} + D_{ijk}^* \dot{\varphi}_{,k} + M_{ijk} \theta_{,k}, \\ H_i^* &= A_{ij}^* \dot{\varphi}_{,j} + G_{rsi}^* \dot{e}_{rs} + d_i^* \dot{\varphi} + P_{ij} \theta_{,j}, \\ g^* &= -F_{ij}^* \dot{e}_{ij} - \xi^* \dot{\varphi} - \gamma_i^* \dot{\varphi}_{,i} - R_j \theta_{,j}. \end{aligned}$$

Here

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}).$$

The constitutive tensors satisfy the symmetries

$$C_{ijrs} = C_{rsij} = C_{jirs}, \quad \beta_{ij} = \beta_{ji}, \quad D_{ijk} = D_{jik}, \quad A_{ij} = A_{ji}, \quad B_{ij} = B_{ji},$$

and

$$\begin{aligned} C_{ijrs}^* &= C_{rsij}^* = C_{jirs}^*, \quad B_{ij}^* = B_{ji}^*, \quad D_{ijk}^* = D_{jik}^*, \quad M_{ij} = M_{ji}, \quad A_{ij}^* = A_{ji}^*, \\ G_{ijk}^* &= G_{jik}^*, \quad P_{ij} = P_{ji}, \quad F_{ij}^* = F_{ji}^*, \quad k_{ij} = k_{ji}, \quad f_{irs} = f_{isr}. \end{aligned}$$

If we substitute the constitutive equations into the evolution equations we obtain the system of field equations

$$\begin{aligned} \rho \ddot{u}_i &= (C_{ijrs} e_{rs} + B_{ij} \varphi + D_{ijk} \varphi_{,k} - \beta_{ij} \theta)_{,j} \\ &\quad + (C_{ijrs}^* \dot{e}_{rs} + B_{ij}^* \dot{\varphi} + D_{ijk}^* \dot{\varphi}_{,k} + M_{ijk} \theta_{,k})_{,j} \end{aligned} \quad (2.2)$$

$$\begin{aligned} \rho \kappa \ddot{\varphi} &= (A_{ij} \varphi_{,j} + D_{rsi} e_{rs} + d_i \varphi - a_i \theta)_{,i} \\ &\quad + (A_{ij}^* \dot{\varphi}_{,j} + G_{rsi}^* \dot{e}_{rs} + d_i^* \dot{\varphi} + P_{ij} \theta_{,j})_{,i} \\ &\quad - B_{ij} e_{ij} - \xi \varphi - d_i \varphi_{,i} + m \theta - F_{ij}^* \dot{e}_{ij} - \xi^* \dot{\varphi} - \gamma_i^* \dot{\varphi}_{,i} - R_j \theta_{,j}, \end{aligned} \quad (2.3)$$

$$T_0(\beta_{ij} \dot{e}_{ij} + a \dot{\theta} + m \dot{\varphi} + a_i \dot{\varphi}_{,i}) = (k_{ij} \theta_{,j} + f_{irs} \dot{e}_{rs} + b_i \dot{\varphi} + a_{ij} \dot{\varphi}_{,j})_{,i}. \quad (2.4)$$

In order to determine a problem we need to impose the boundary conditions and initial conditions. From now on, we assume

$$u_i(\mathbf{x}, t) = \varphi(\mathbf{x}, t) = \theta(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \partial\Omega, \quad t \geq 0, \quad (2.5)$$

and the initial conditions

$$\begin{aligned} u_i(\mathbf{x}, 0) &= u_i^0(\mathbf{x}), \quad \dot{u}_i(\mathbf{x}, 0) = u_i^1(\mathbf{x}), \quad \varphi(\mathbf{x}, 0) = \varphi_0(\mathbf{x}), \\ \dot{\varphi}(\mathbf{x}, 0) &= \varphi_1(\mathbf{x}), \quad \theta(\mathbf{x}, 0) = \theta_0(\mathbf{x}), \quad \mathbf{x} \in \Omega. \end{aligned} \quad (2.6)$$

The internal energy of the system is given by

$$2U = C_{ijrs} e_{ij} e_{rs} + A_{ij} \varphi_{,i} \varphi_{,j} + \xi \varphi^2 + 2B_{ij} \varphi e_{ij} + 2D_{ijk} e_{ij} \varphi_{,k} + 2d_i \varphi \varphi_{,i}.$$

The dissipation of the system is defined by the function

$$\begin{aligned} \Pi &= C_{ijrs}^* \dot{e}_{ij} \dot{e}_{rs} + A_{ij}^* \dot{\varphi}_{,i} \dot{\varphi}_{,j} + \xi^* |\dot{\varphi}|^2 + \frac{1}{T_0} k_{ij} \theta_{,i} \theta_{,j} + (B_{ij}^* + F_{ij}^*) \dot{e}_{ij} \dot{\varphi} \\ &\quad + (D_{ijk}^* + G_{ijk}^*) \dot{e}_{ij} \dot{\varphi}_{,k} + (M_{ijk} + \frac{f_{kij}}{T_0}) \dot{e}_{ij} \theta_{,k} + (d_i^* + \gamma_i^*) \dot{\varphi} \dot{\varphi}_{,i} \\ &\quad + (R_j + \frac{b_j}{T_0}) \dot{\varphi} \theta_{,j} + (P_{ij} + \frac{a_{ij}}{T_0}) \dot{\varphi}_{,i} \theta_{,j}. \end{aligned}$$

When the internal energy is assumed positive we have that the inequality

$$U \geq C_1 (e_{ij}e_{ij} + \varphi_{,i}\varphi_{,i} + \varphi^2),$$

is satisfied for a positive constant  $C_1$ , meanwhile when the dissipation is assumed to be positive we have that the inequality

$$\Pi \geq C_2 (\dot{e}_{ij}\dot{e}_{ij} + \dot{\varphi}_{,j}\dot{\varphi}_{,j} + |\dot{\varphi}|^2 + \theta_{,i}\theta_{,i})$$

is satisfied for a positive constant  $C_2$ .

In the second part of this paper we will prove the analyticity of the solutions for the problem determined by our system in case that the internal energy and the dissipation are strictly positive. To make easier the read we will restrict this study to the one-dimensional and homogeneous case. The general study would need of very cumbersome expressions, but from the mathematical point of view the analysis would agree. Thus, we believe that it is much better to do that to simplify expressions. The only point to pay attention is that for the three-dimensional case we would need to use of the Korn inequality which is not needed for dimension one. If we denote  $J = \rho\kappa$  and consider  $\Omega = (0, \pi)$ , our system reduces to

$$\rho\ddot{u} = \mu u_{xx} + b\varphi_x + D\varphi_{xx} - \beta\theta_x + \gamma\dot{u}_{xx} + b^*\dot{\varphi}_x + D^*\dot{\varphi}_{xx} + M\theta_{xx} \quad (2.7)$$

$$J\ddot{\varphi} = A\varphi_{xx} + Du_{xx} - (a + R)\theta_x - bu_x - \xi\varphi + m\theta + A^*\dot{\varphi}_{xx} + G^*\dot{u}_{xx} + d^*\dot{\varphi}_x + P\theta_{xx} - F^*\dot{u}_x - \xi^*\dot{\varphi} \quad (2.8)$$

$$c\dot{\theta} = k\theta_{xx} + f\dot{u}_{xx} + (b^{**} - a)\dot{\varphi}_x - \beta\dot{u}_x - m\dot{\varphi} + a^*\dot{\varphi}_{xx} \quad (2.9)$$

the boundary conditions will be

$$u(0, t) = u(\pi, t) = \varphi(0, t) = \varphi(\pi, t) = \theta(0, t) = \theta(\pi, t) = 0, \quad (2.10)$$

and the initial conditions

$$u(x, 0) = u_0(x), \dot{u}(x, 0) = u_1(x), \varphi(x, 0) = \varphi_0(x), \dot{\varphi}(x, 0) = \varphi_1(x), \theta(x, 0) = \theta_0(x). \quad (2.11)$$

The internal energy and the dissipation of the system will have the form\*

$$2U = \mu u_x^2 + A\varphi_x^2 + \xi\varphi^2 + 2b\varphi u_x + 2Du_x\varphi_x,$$

and

$$\begin{aligned} \Pi = & \gamma|\dot{u}_x|^2 + A^*|\dot{\varphi}_x|^2 + \xi^*|\dot{\varphi}|^2 + k\theta_x^2 + (b^* + F^*)\dot{u}_x\dot{\varphi} + (D^* + G^*)\dot{u}_x\dot{\varphi}_x \\ & + (M + f)\dot{u}_x\theta_x + d^*\dot{\varphi}\dot{\varphi}_x + (R + b^{**})\dot{\varphi}\theta_x + (P + a^*)\dot{\varphi}_x\theta_x. \end{aligned}$$

### 3. Uniqueness

The aim of this section is to obtain a uniqueness result for the solutions of the problem determined by the thermoviscoelasticity with voids. We recall that a uniqueness

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\*Note that for homogeneous materials  $(d_i\varphi)_{,i} - d_i\varphi_{,i} = 0$  and for this reason we do not consider the counterpart of the constitutive tensor  $d_i$  in the system (2.8)-(2.10) neither in the internal energy.

result was obtained by Ieşan [6], but under the assumption that the internal energy and the dissipation are always greater or equal than zero. We here do not impose such restrictive assumptions. We do not assume any sign for the internal energy, but we need to assume the strictly positivity of the dissipation function  $\Pi$ .

As we want to prove the uniqueness of solutions, it is enough to see that the only solution for the problem determined by the null initial conditions

$$u_i(\mathbf{x}, 0) = 0, \dot{u}_i(\mathbf{x}, 0) = 0, \varphi(\mathbf{x}, 0) = 0, \dot{\varphi}(\mathbf{x}, 0) = 0, \theta(\mathbf{x}, 0) = 0, \mathbf{x} \in \Omega, \quad (3.1)$$

is the null solution.

For the initial conditions the conservation of the energy reads

$$\frac{1}{2} \int_{\Omega} (\rho \dot{u}_i \dot{u}_i + \rho \kappa |\dot{\varphi}|^2 + a \theta^2 + 2U) dv + \int_0^t \int_{\Omega} \Pi dv ds = 0. \quad (3.2)$$

The function

$$F(t) = \frac{1}{2} \int_{\Omega} (\rho \dot{u}_i \dot{u}_i + \rho \kappa |\dot{\varphi}|^2 + a \theta^2) dv + \int_0^t \int_{\Omega} \Pi dv ds,$$

satisfies

$$F(t) = -\frac{1}{2} \int_{\Omega} 2U dv.$$

In view of the null initial conditions, we have

$$F(t) = - \int_0^t \int_{\Omega} \Upsilon dv ds,$$

where

$$\begin{aligned} \Upsilon = & C_{ijrs} e_{ij} \dot{e}_{rs} + A_{ij} \dot{\varphi}_{,i} \varphi_{,j} + \xi \varphi \dot{\varphi} + B_{ij} (\varphi \dot{e}_{ij} + \dot{\varphi} e_{ij}) \\ & + D_{ijk} (\dot{e}_{ij} \varphi_{,k} + e_{ij} \dot{\varphi}_{,k}) + d_i (\dot{\varphi} \varphi_{,i} + \varphi \dot{\varphi}_{,i}). \end{aligned}$$

In view of the Holder inequality and the positivity of the dissipation, we see the existence of a positive constant  $C_4$  such that

$$F(t) \leq C_4 \left( \int_0^t \int_{\Omega} (e_{ij} e_{ij} + \varphi_{,i} \varphi_{,i} + |\varphi|^2) dv ds \right)^{1/2} \left( \int_0^t \int_{\Omega} \Pi dv ds \right)^{1/2}, \quad (3.3)$$

In view of the initial conditions and the Poincaré inequality we know that

$$\int_0^t \int_{\Omega} (e_{ij} e_{ij} + \varphi_{,i} \varphi_{,i} + |\varphi|^2) dv ds \leq \frac{4t^2}{\pi^2} \int_0^t \int_{\Omega} (\dot{e}_{ij} \dot{e}_{ij} + \dot{\varphi}_{,i} \dot{\varphi}_{,i} + |\dot{\varphi}|^2) dv ds. \quad (3.4)$$

Using again the positivity of the dissipation, we see the existence of a positive constant  $C_5$  such that

$$F(t) \leq C_5 t \int_0^t \int_{\Omega} \Pi dv ds \leq C_5 t F(t), \quad (3.5)$$

where the last inequality follows from the definition of  $F(t)$ .

It then follows that

$$(1 - C_5 t) F(t) \leq 0. \quad (3.6)$$

If we take  $t_0 = C_5^{-1}$ , we obtain that  $F(t)$  vanishes in the interval  $(0, t_0)$ . If we take into account the definition of  $F(t)$ , it follows that  $\theta \equiv 0$ ,  $\dot{\varphi} \equiv 0$  and  $\dot{u}_i \equiv 0$  for every  $t \leq t_0$ . Thus, we have proved that the problem determined by our system with the homogeneous boundary conditions and the null initial condition has only the null solution in the interval  $[0, t_0]$ . If we apply the same argument to the problem determined by our system, the same boundary conditions and initial conditions

$$u_i(\mathbf{x}, t_0) = \dot{u}_i(\mathbf{x}, t_0) = \varphi(\mathbf{x}, t_0) = \dot{\varphi}(\mathbf{x}, t_0) = \theta(\mathbf{x}, t_0) = 0, \quad \mathbf{x} \in B \quad (3.7)$$

we can conclude  $\theta \equiv 0$ ,  $\phi \equiv 0$  and  $u_i \equiv 0$  for every  $t \leq 2t_0$ .

After a recurrent argument we obtain the following result.

**Theorem 3.1.** *Let us assume that the mass density and the specific heat are strictly positive and that the dissipation function is strictly positive. Then the boundary-initial-value problem has at most one solution.*

### 4. Well posed problem

In this section we prove the well-posedness of the problem determined by (2.8)-(2.12) when the internal energy and the dissipation are strictly positive. To write the proof in the less cumbersome case we assume that the material is homogeneous and one dimensional.

We consider the Hilbert space

$$\mathcal{H} = H_0^1(0, \pi) \times L^2(0, \pi) \times H_0^1(0, \pi) \times L^2(0, \pi) \times L^2(0, \pi)$$

where  $H_0^i, H^i$  and  $L^2$  are the well known Hilbert spaces. If  $U = (u, v, \varphi, \phi, \theta)$  and  $U^* = (u^*, v^*, \varphi^*, \phi^*, \theta^*)$  we define the inner product

$$\begin{aligned} \langle U, U^* \rangle_{\mathcal{H}} &= \int_0^\pi [\rho v \bar{v}^* + \mu u_x \bar{u}_x^* + J \phi \bar{\phi}^* + A \varphi_x \bar{\varphi}_x^* + \xi \varphi \bar{\varphi}^* + c \theta \bar{\theta}^* \\ &\quad + b(u_x \bar{\varphi}^* + \bar{u}_x^* \varphi) + D(u_x \bar{\varphi}_x^* + \bar{u}_x^* \varphi_x)] dx, \end{aligned}$$

where the bar denotes the conjugate complex number, and the corresponding norm

$$\|U\|_{\mathcal{H}} = \int_0^\pi [\rho |v|^2 + \mu |u_x|^2 + J |\phi|^2 + A |\varphi_x|^2 + \xi |\varphi|^2 + c |\theta|^2 + 2b \operatorname{Re} u_x \bar{\varphi} + 2D \operatorname{Re} u_x \bar{\varphi}_x] dx.$$

Let us introduce the operator

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 & 0 & 0 \\ \mathcal{A}_{21} & \mathcal{A}_{22} & \mathcal{A}_{23} & \mathcal{A}_{24} & \mathcal{A}_{25} \\ 0 & 0 & 0 & I & 0 \\ \mathcal{A}_{41} & \mathcal{A}_{42} & \mathcal{A}_{43} & \mathcal{A}_{44} & \mathcal{A}_{45} \\ 0 & \mathcal{A}_{52} & 0 & \mathcal{A}_{54} & \mathcal{A}_{55} \end{pmatrix} \quad (4.1)$$

where  $I$  is the identity operator  $\mathcal{A}_{21} = \rho^{-1} \mu \partial^2$ ,  $\mathcal{A}_{22} = \rho^{-1} \gamma \partial^2$ ,  $\mathcal{A}_{23} = \rho^{-1} (b \partial + D \partial^2)$ ,  $\mathcal{A}_{24} = \rho^{-1} (b^* \partial + D^* \partial^2)$ ,  $\mathcal{A}_{25} = -\rho^{-1} (\beta \partial - M \partial^2)$ ,  $\mathcal{A}_{41} = -J^{-1} (b \partial - D \partial^2)$ ,  $\mathcal{A}_{42} = -J^{-1} (F^* \partial - G^* \partial^2)$ ,  $\mathcal{A}_{43} = J^{-1} (A \partial^2 - \xi I)$ ,  $\mathcal{A}_{44} = J^{-1} (A^* \partial^2 + d^* \partial - \xi^* I)$ ,  $\mathcal{A}_{45} = J^{-1} (mI + P \partial^2 - (a + R) \partial)$ ,  $\mathcal{A}_{52} = -c^{-1} (\beta \partial - f \partial^2)$ ,  $\mathcal{A}_{54} = c^{-1} (a^* \partial^2 + (b^{**} - a) \partial - mI)$ ,  $\mathcal{A}_{55} = c^{-1} k \partial^2$  and  $\partial^i = \frac{d^i}{dx^i}$ .

Our initial-boundary value problem is equivalent to the problem

$$U_t = \mathcal{A}U, \quad U(0) = U_0 \in \mathcal{D}(\mathcal{A}), \tag{4.2}$$

where  $U_0 = (u_0, u_1, \varphi_0, \varphi_1, \theta_0)$  and  $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ . The domain of  $\mathcal{A}$  can be easily calculated, but we note that it contains  $(H^2 \cap H_0^1)^5$  which is dense in our Hilbert space. We note that for every  $U \in \mathcal{D}(\mathcal{A})$

$$\operatorname{Re} \langle \mathcal{A}U, U \rangle_{\mathcal{H}} = -\frac{1}{2} \int_0^\pi \operatorname{Re} \Gamma dx$$

where

$$\begin{aligned} \Gamma = & \gamma|v_x|^2 + \xi^*|\phi|^2 + k|\theta_x|^2 + A^*\phi_x^2 + (b^*\phi\bar{v}_x + F^*v_x\bar{\phi}) \\ & + (P\theta_x\bar{\phi}_x + a^*\bar{\theta}_x\phi_x) + (G^*v_x\bar{\phi}_x + D^*\bar{v}_x\phi_x) + (M\bar{v}_x\theta_x + f v_x\bar{\theta}_x) + (R\theta_x\bar{\phi} + b^{**}\phi\bar{\theta}_x). \end{aligned}$$

In view of the positivity of the dissipation we see the existence of a positive constant such that

$$\operatorname{Re} \langle \mathcal{A}U, U \rangle_{\mathcal{H}} \leq -\frac{1}{2}M_1 \int_0^\pi (\gamma|v_x|^2 + \xi|\phi|^2 + k|\theta_x|^2 + A|\phi_x|^2)dx \leq 0,$$

where  $M_1$  is a positive calculable constant. Then  $\mathcal{A}$  is dissipative.

**Lemma 4.1.** *Under the above notations we have that  $0 \in \varrho(\mathcal{A})$ , where  $\varrho(\mathcal{A})$  is a set resolvent of  $\mathcal{A}$ .*

**Proof.** For any  $F = (f_1, f_2, f_3, f_4, f_5) \in \mathcal{H}$ , we want to find  $U = (u, v, \varphi, \phi, \theta) \in \mathcal{D}(\mathcal{A})$  such that

$$\mathcal{A}U = F. \tag{4.3}$$

In terms of the components we get

$$v = f_1 \tag{4.4}$$

$$\mu u_{xx} + b\varphi_x + \gamma v_{xx} - \beta\theta_x + b^*\phi_x + D\varphi_{xx} + D^*\phi_{xx} + M\theta_{xx} = \rho f_2$$

$$\phi = f_3 \tag{4.5}$$

$$\begin{aligned} A\varphi_{xx} - bu_x - \xi\varphi + m\theta + A^*\phi_{xx} + P\theta_{xx} - F^*v_x - \xi^*\phi + d^*\phi_x \\ + Du_{xx} + G^*v_{xx} - (a + R)\theta_x = Jf_4 \end{aligned} \tag{4.6}$$

$$k\theta_{xx} - \beta v_x + f v_{xx} - m\phi + a^*\phi_{xx} + (b^{**} - a)\phi_x = cf_5. \tag{4.7}$$

We have

$$v, \phi \in H_0^1(0, \pi). \tag{4.8}$$

We can write

$$k\theta_{xx} = \beta(f_1)_x - f(f_1)_{xx} + mf_3 - a^*(f_3)_{xx} - (b^{**} - a)(f_3)_x + cf_5 \in H^{-1}(0, \pi). \tag{4.9}$$

We conclude that there exists a unique function  $\theta \in H^1(0, \pi)$  satisfying (4.9).

Then, the remanning point is to prove that there exist  $u$  and  $\varphi$  satisfying

$$\mu u_{xx} + b\varphi_x + D\varphi_{xx} = F_1 \tag{4.10}$$

$$A\varphi_{xx} - bu_x - \xi\varphi + Du_{xx} = G_1 \tag{4.11}$$



where

$$F_1 := -\gamma(f_1)_{xx} + \beta\theta_x + \rho f_2 - b^*(f_3)_x - D^*(f_3)_{xx} - M\theta_{xx} \in H^{-1}(0, \pi),$$

and

$$G_1 := -m\theta + Jf_4 + F^*(f_1)_x + \xi^* f_3 - A^*(f_3)_{xx} - P\theta_{xx} - d^*(f_3)_x - G^*(f_1)_{xx} + (a + R)\theta_x \in H^{-1}(0, \pi).$$

Introducing the space  $W = H_0^1(0, \pi) \times H_0^1(0, \pi)$ , and denoting the bilinear form

$$\begin{aligned} a(V, \tilde{V}) &= \mu \int_0^\pi u_x \tilde{u}_x dx + b \int_0^\pi (\varphi \tilde{u}_x + u_x \tilde{\varphi}) dx + D \int_0^\pi (\varphi_x \tilde{u}_x + u_x \tilde{\varphi}_x) dx \\ &\quad + A \int_0^\pi \varphi_x \tilde{\varphi}_x dx + \xi \int_0^\pi \varphi \tilde{\varphi} dx \end{aligned}$$

we conclude that  $a(\cdot, \cdot)$  is a coercive, continuous bilinear operator over the Hilbert space  $W$ . Therefore there exists a solution to the variational equation

$$a(U, V) = \langle (F_1, G_1), V \rangle$$

that is equivalent to system (4.10)–(4.11). □

Thus, we have proved

**Theorem 4.1.** *Under the above conditions we have that the operator  $\mathcal{A}$  is the infinitesimal generator of a  $C_0$ -semigroup  $\mathcal{T}(t)$  of contractions over the Hilbert space  $\mathcal{H}$ .*

## 5. Analyticity

To prove the main aim of this section we will use a result which can be found in the book by Liu and Zheng [11].

**Theorem 5.1.** *Let us consider  $S(t) = e^{\mathcal{A}t}$  a  $C_0$ -semigroup of contractions generated for operator  $\mathcal{A}$  in Hilbert space  $\mathcal{H}$ . If  $i\mathbb{R} \subseteq \rho(\mathcal{A})$ , then  $S(t)$  is analytic if and only if*

$$\overline{\lim}_{|\beta| \rightarrow \infty} \|\beta(i\beta I - \mathcal{A})^{-1}\| < \infty, \quad \beta \in \mathbb{R}.$$

To apply this theorem to our situation we need to consider the resolvent equation which is given by

$$\lambda U - \mathcal{A}U = F \tag{5.1}$$

where

$$U = \begin{pmatrix} u \\ v \\ \varphi \\ \phi \\ \theta \end{pmatrix}, \quad F = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{pmatrix} \quad \text{and } \lambda \in \mathbb{C}.$$

To show the analyticity we shall take  $\lambda = i\alpha, \alpha \in \mathbb{R}$ . Written the equation (5.1) with  $\lambda = i\alpha, \alpha \in \mathbb{R}$  we have

$$i\alpha u - v = f_1 \quad (5.2)$$

$$\begin{aligned} i\alpha \rho v - (\mu u_{xx} + b\varphi_x + \gamma v_{xx} - \beta\theta_x + b^*\phi_x \\ + D\varphi_{xx} + D^*\phi_{xx} + M\theta_{xx}) = \rho f_2 \end{aligned} \quad (5.3)$$

$$i\alpha\varphi - \phi = f_3 \quad (5.4)$$

$$\begin{aligned} i\alpha J\phi - (A\varphi_{xx} - bu_x - \xi\varphi + m\theta + A^*\phi_{xx} + P\theta_{xx} - F^*v_x \\ - \xi^*\phi + d^*\phi_x + Du_{xx} + G^*v_{xx} - (a+R)\theta_x) = Jf_4 \end{aligned} \quad (5.5)$$

$$i\alpha c\theta - (k\theta_{xx} - \beta v_x + f v_{xx} - m\phi + a^*\phi_{xx} + (b^{**} - a)\phi_x) = cf_5. \quad (5.6)$$

We need of the following lemmas.

**Lemma 5.1.** *For any  $F \in \mathcal{H}$ , there exists a positive constant  $c_1$  such that*

$$\int_0^\pi (\gamma|v_x|^2 + \xi^*|\phi|^2 + k|\theta_x|^2 + A^*|\phi_x|^2)dx \leq c_1\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \quad (5.7)$$

**Proof.** Multiplying the equations (5.2)-(5.6), respectively, for  $-\mu\bar{u}_{xx}, \bar{v}, -A\bar{\varphi}_{xx}$  and  $\xi\bar{\varphi}, \bar{\phi}, \bar{\theta}$ , integration from 0 to  $\pi$  and summing the equations, we find that

$$\begin{aligned} i\alpha \int_0^\pi [\rho|v|^2 + \mu|u_x|^2 + A|\varphi_x|^2 + J|\phi|^2 + \xi|\varphi|^2 + c|\theta|^2]dx \\ + \mu \int_0^\pi (u_x\bar{v}_x - \bar{u}_x v_x)dx + b \int_0^\pi (\varphi\bar{v}_x + u_x\bar{\phi})dx + m \int_0^\pi (\phi\bar{\theta} - \theta\bar{\phi})dx \\ + A \int_0^\pi (\varphi_x\bar{\phi}_x - \bar{\varphi}_x\phi_x)dx + \beta \int_0^\pi (\theta_x\bar{v} - \bar{\theta}_x v)dx + \xi \int_0^\pi (\varphi\bar{\phi} - \bar{\varphi}\phi)dx \\ + D \int_0^\pi (\varphi_x\bar{v}_x + u_x\bar{\phi}_x)dx + \int_0^\pi (D^*\phi_x\bar{v}_x + G^*v_x\bar{\phi}_x)dx \\ + \int_0^\pi (M\theta_x\bar{v}_x + f v_x\bar{\theta}_x)dx + d^* \int_0^\pi \phi\bar{\phi}_x dx + \int_0^\pi (R\theta_x\bar{\phi} + b^{**}\phi\bar{\theta}_x)dx \\ + a \int_0^\pi (\theta_x\bar{\phi} - \phi\bar{\theta}_x)dx + \int_0^\pi (P\theta_x\bar{\phi}_x + a^*\phi_x\bar{\theta}_x)dx + \int_0^\pi (b^*\phi\bar{v}_x + F^*v_x\bar{\phi})dx \\ + \int_0^\pi (\gamma|v_x|^2 + A^*|\phi_x|^2 + \xi^*|\phi|^2 + k|\theta_x|^2)dx = R_1 \end{aligned} \quad (5.8)$$

where  $|R_1| \leq C\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}$  for a positive constant  $C$ . We note that

$$\varphi\bar{v}_x + u_x\bar{\phi} = \overline{\varphi(i\alpha u_x - (f_1)_x)} + u_x \overline{(i\alpha\varphi - f_3)} = -i\alpha(\varphi\bar{u}_x + u_x\bar{\varphi}) - \varphi(\bar{f}_1)_x - u_x\bar{f}_3,$$

and

$$\begin{aligned} \varphi_x\bar{v}_x + u_x\bar{\phi}_x &= \overline{\varphi(i\alpha u_x - (f_1)_x)} + u_x \overline{(i\alpha\varphi_x - (f_3)_x)} \\ &= -i\alpha(\varphi_x\bar{u}_x + u_x\bar{\varphi}_x) - \varphi_x(\bar{f}_1)_x - u_x(\bar{f}_3)_x. \end{aligned}$$

Taking real part in the equation (5.8), using the positivity conditions for the dissipation and the definition of norm in  $\mathcal{H}$  we obtain the estimate.  $\square$

**Lemma 5.2.** For any  $F \in \mathcal{H}$ , there exists  $C > 0$  such that

$$|\alpha| \|U\|_{\mathcal{H}} \leq C \|F\|_{\mathcal{H}}, \quad \forall \alpha \in \mathbb{R},$$

where  $U$  is the solution for (5.1) with  $\lambda = i\alpha$ .

**Proof.** Multiplying the equations (5.2)-(5.6), respectively, for  $i\mu\bar{u}_{xx}$ ,  $-i\bar{v}$ ,  $iA\bar{\varphi}_{xx}$  and  $-i\xi\bar{\varphi}$ ,  $-i\bar{\phi}$ ,  $-i\bar{\theta}$ , integration from 0 to  $\pi$  and summing the equations, we find that

$$\begin{aligned} & \alpha \int_0^\pi [\rho|v|^2 + \mu|u_x|^2 + A|\varphi_x|^2 + J|\phi|^2 + \xi|\varphi|^2 + c|\theta|^2] dx \\ & - ib \int_0^\pi (\varphi\bar{v}_x + u_x\bar{\phi}) dx + i\mu \int_0^\pi (v_x\bar{u}_x - \bar{v}_x u_x) dx \\ & + i\xi \int_0^\pi (\phi\bar{\varphi} - \varphi\bar{\phi}) dx + im \int_0^\pi (\theta\bar{\phi} - \phi\bar{\theta}) dx + i\beta \int_0^\pi (v\bar{\theta}_x - \bar{v}\theta_x) dx \\ & + iA \int_0^\pi (\phi_x\bar{\varphi}_x - \varphi_x\bar{\phi}_x) dx - iD \int_0^\pi (\varphi_x\bar{v}_x + u_x\bar{\phi}_x) dx \\ & - i \int_0^\pi (D^* \phi_x\bar{v}_x + G^* v_x\bar{\phi}_x) dx - i \int_0^\pi (M\theta_x\bar{v}_x + f v_x\bar{\theta}_x) dx \\ & - id^* \int_0^\pi \phi\bar{\phi}_x dx - i \int_0^\pi (R\theta_x\bar{\phi} + b^{**}\phi\bar{\theta}_x) dx - ia \int_0^\pi (\theta_x\bar{\phi} - \phi\bar{\theta}_x) dx \\ & - i \int_0^\pi (P\theta_x\bar{\phi}_x + a^* \phi_x\bar{\theta}_x) dx - i \int_0^\pi (b^* \phi\bar{v}_x + F^* v_x\bar{\phi}) dx \\ & - i \int_0^\pi (\gamma|v_x|^2 + A^*|\phi_x|^2 + \xi^*|\phi|^2 + k|\theta_x|^2) dx = \tilde{R} \end{aligned} \quad (5.9)$$

where  $|\tilde{R}| \leq C \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}}$  for a calculable positive constant  $C$ .

Taking the real part in the former equality we have

$$\begin{aligned} & \alpha \int_0^\pi [\rho|v|^2 + \mu|u_x|^2 + A|\varphi_x|^2 + J|\phi|^2 + \xi|\varphi|^2 + c|\theta|^2] dx \\ & \leq \operatorname{Re} \left\{ ib \int_0^\pi (\varphi\bar{v}_x + u_x\bar{\phi}) dx + iD \int_0^\pi (\varphi_x\bar{v}_x + u_x\bar{\phi}_x) dx \right. \\ & + i \int_0^\pi (D^* \phi_x\bar{v}_x + G^* v_x\bar{\phi}_x) dx + i \int_0^\pi (M\theta_x\bar{v}_x + f v_x\bar{\theta}_x) dx \\ & + id^* \int_0^\pi \phi\bar{\phi}_x dx + i \int_0^\pi (R\theta_x\bar{\phi} + b^{**}\phi\bar{\theta}_x) dx - i\xi \int_0^\pi (\phi\bar{\varphi} - \varphi\bar{\phi}) dx \\ & - im \int_0^\pi (\theta\bar{\phi} - \phi\bar{\theta}) dx - i\beta \int_0^\pi (v\bar{\theta}_x - \bar{v}\theta_x) dx \\ & - iA \int_0^\pi (\phi_x\bar{\varphi}_x - \varphi_x\bar{\phi}_x) dx + ia \int_0^\pi (\theta_x\bar{\phi} - \phi\bar{\theta}_x) dx \\ & - i\mu \int_0^\pi (v_x\bar{u}_x - \bar{v}_x u_x) dx + i \int_0^\pi (P\theta_x\bar{\phi}_x + a^* \phi_x\bar{\theta}_x) dx \\ & \left. + i \int_0^\pi (b^* \phi\bar{v}_x + F^* v_x\bar{\phi}) dx \right\} + C \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}}. \end{aligned} \quad (5.10)$$

Using (5.7) we have that

$$\operatorname{Re}\left\{ib \int_0^\pi \varphi \bar{v}_x dx\right\} \leq K_1 \|F\|_{\mathcal{H}}^{1/2} \|U\|_{\mathcal{H}}^{3/2} \quad (5.11)$$

and

$$\operatorname{Re}\left\{iD^* \int_0^\pi \phi_x \bar{v}_x dx\right\} \leq K_2 \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}}, \quad (5.12)$$

where  $K_1, K_2$  are calculable constants. In a similar way we can estimate the other terms of the RHS of (5.10)

Therefore, we have that

$$\alpha \|U\|_{\mathcal{H}}^2 \leq C^* \|F\|_{\mathcal{H}}^{1/2} \|U\|_{\mathcal{H}}^{3/2} + C^{**} \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}},$$

for calculable positive constants  $C, C^*$  which are independent of  $\alpha$ . Then

$$|\alpha| \|U\|_{\mathcal{H}} \leq C \|F\|_{\mathcal{H}}, \quad (5.13)$$

where  $C > 0$  and when  $\alpha > 0$  is sufficiently greater. From where our conclusion follows.  $\square$

Now, we are in condition to show the main result of this section

**Theorem 5.2.** *The semigroup generated by operator  $\mathcal{A}$  given at (4.1) is analytic.*

**Proof.** Since  $\mathcal{A}$  is the infinitesimal generator of a strongly continuous semigroup,  $\mathbb{R}_+ \in \rho(\mathcal{A})$  and as  $0 \in \rho(\mathcal{A})$  we have  $i\mathbb{R} \subset \rho(\mathcal{A})$ . From Lemma 5.2 we have

$$\|\alpha(i\alpha I - \mathcal{A})^{-1} F\|_{\mathcal{H}} = |\alpha| \|U\|_{\mathcal{H}} \leq C \|F\|_{\mathcal{H}}.$$

Then

$$\overline{\lim}_{|\alpha| \rightarrow \infty} \|\alpha(i\alpha I - \mathcal{A})^{-1}\| < \infty.$$

Then conclusion following from 5.1.  $\square$

**Remark:** As consequence of the analyticity, the system (2.7)-(2.9) is exponentially stable. Moreover, the system have a regularity effect in the sense that the solution  $U = (u, u_t, \varphi, \varphi_t, \theta)$  satisfy

$$U \in C^\infty(0, T; \mathcal{D}(\mathcal{A}^\infty)).$$

However,  $\mathcal{D}(\mathcal{A})$  is not necessary a space regular, which in particular implies that the solution  $U$  is not in  $C^\infty(]0, T[ \times ]0, L[)$  when the initial data is not regular.

A consequence of the analyticity of solutions is the following result:

**Corollary 5.1.** Let  $(u, \varphi, \theta)$  be a solution of the problem determined by the system (2.7)-(2.9), the initial conditions (2.11) and the boundary conditions (2.10) such that  $u = \varphi = \theta \equiv 0$  after a finite time  $t_0 > 0$ . Then  $u = \varphi = \theta \equiv 0$  for every  $t \geq 0$ .

## 6. Case $A^* = 0$

When the parameters  $A^*$  vanishes the dissipation is not so strong. In order to guarantee the sign of the dissipation we also need to impose  $D^* + G^* = 0, d^* = 0$  and  $P + a^* = 0$ . The system of equations becomes

$$\rho \ddot{u} = \mu u_{xx} + b\varphi_x + D\varphi_{xx} - \beta\theta_x + \gamma \dot{u}_{xx} + b^* \dot{\varphi}_x + D^* \dot{\varphi}_{xx} + M\theta_{xx} \quad (6.1)$$

$$\begin{aligned} J\ddot{\varphi} &= A\varphi_{xx} + Du_{xx} - (a + R)\theta_x - bu_x - \xi\varphi + m\theta \\ &\quad - D^* \dot{u}_{xx} - a^* \theta_{xx} - F^* \dot{u}_x - \xi^* \dot{\varphi} \end{aligned} \quad (6.2)$$

$$c\dot{\theta} = k\theta_{xx} + f\dot{u}_{xx} + (b^{**} - a)\dot{\varphi}_x - \beta\dot{u}_x - m\dot{\varphi} + a^* \dot{\varphi}_{xx} \quad (6.3)$$

and the dissipation is

$$\Pi^* = \gamma|\dot{u}_x|^2 + \xi^*|\dot{\varphi}|^2 + k\theta_x^2 + (b^* + F^*)\dot{u}_x\dot{\varphi} + (M + f)\dot{u}_x\theta_x + (R + b^{**})\dot{\varphi}\theta_x.$$

In case that we assume that there exists a positive constant  $C$  such that the inequality

$$\Pi^* \geq C(|\dot{u}_x|^2 + |\dot{\varphi}|^2 + \theta_x^2),$$

holds we can obtain the exponential stability of the solutions for the problem determined by the system (6.1)-(6.3) and the boundary and initial conditions proposed previously. This is natural, because we also have sufficient dissipation mechanisms. Even more the aim of this section is to prove that in case that we have a strong coupling mechanism  $a^* \neq 0$ , then we also can guarantee the analyticity of solutions. We point out that our approach is inspired in the arguments used in the case of plates [10].

We can consider the same Hilbert space and with the same inner product considered in the general case. We note that the semigroup of solutions is generated by the operator

$$\mathcal{A}_1 = \begin{pmatrix} 0 & I & 0 & 0 & 0 \\ \mathcal{A}_{21} & \mathcal{A}_{22} & \mathcal{A}_{23} & \mathcal{A}_{24} & \mathcal{A}_{25} \\ 0 & 0 & 0 & I & 0 \\ \mathcal{A}_{41} & \mathcal{A}_{42}^* & \mathcal{A}_{43} & \mathcal{A}_{44}^* & \mathcal{A}_{45}^* \\ 0 & \mathcal{A}_{52} & 0 & \mathcal{A}_{54} & \mathcal{A}_{55} \end{pmatrix} \quad (6.4)$$

where  $\mathcal{A}_{42}^* = -J^{-1}(F^*\partial + D^*\partial^2)$ ,  $\mathcal{A}_{44}^* = -J^{-1}\xi^*I$ ,  $\mathcal{A}_{45}^* = J^{-1}(mI - a^*\partial^2 - (a + R)\partial)$ , and the other operators were defined in the fourth section.

We can give an easy proof of the exponential stability by means of the energy arguments. If we consider the function

$$E(t) = \frac{1}{2} \int_0^\pi (\rho|\dot{u}|^2 + J|\dot{\varphi}|^2 + c|\theta|^2 + 2U)dx, \quad (6.5)$$

we obtain

$$\dot{E}(t) = - \int_0^\pi \Pi^* dx \leq -C \int_0^\pi (|\dot{u}_x|^2 + |\dot{\varphi}|^2 + \theta_x^2)dx. \quad (6.6)$$

The functions

$$R(t) = \int_0^\pi \rho \dot{u} u dx + \frac{1}{2} \int_0^\pi |u_x|^2 dx, \quad J(t) = \int_0^\pi J \dot{\varphi} \varphi dx, \quad (6.7)$$

satisfy

$$\begin{aligned} \dot{R}(t) &= \int_0^\pi \rho |\dot{u}|^2 dx - \mu \int_0^\pi |u_x|^2 dx - b \int_0^\pi u_x \varphi dx \\ &\quad - D \int_0^\pi u_x \varphi_x dx - \beta \int_0^\pi \theta_x u dx - b^* \int_0^\pi \dot{\varphi} u_x dx \\ &\quad - D^* \int_0^\pi u_x \dot{\varphi}_x dx - M \int_0^\pi u_x \theta_x dx, \end{aligned} \quad (6.8)$$

$$\begin{aligned}
\dot{J}(t) &= \int_0^\pi J|\dot{\varphi}|^2 dx - A \int_0^\pi |\varphi_x|^2 dx - D \int_0^\pi u_x \varphi_x dx - (a+R) \int_0^\pi \theta_x \varphi dx \\
&\quad - \xi \int_0^\pi |\varphi|^2 dx - b \int_0^\pi u_x \varphi dx + m \int_0^\pi \varphi \theta dx + D^* \int_0^\pi \dot{u}_x \varphi_x dx \\
&\quad + a^* \int_0^\pi \theta_x \varphi_x dx - F^* \int_0^\pi \dot{u}_x \varphi dx - \xi^* \int_0^\pi \dot{\varphi} \varphi dx.
\end{aligned} \tag{6.9}$$

Thus, the time derivative of the function  $S(t) = R(t) + J(t) + D^* \int_0^\pi u_x \varphi_x dx$  is

$$\begin{aligned}
\dot{S}(t) &= \int_0^\pi \rho |\dot{u}|^2 dx + \int_0^\pi J |\dot{\varphi}|^2 dx - \int_0^\pi 2U dx - \beta \int_0^\pi \theta_x u dx \\
&\quad - b^* \int_0^\pi \dot{\varphi} u_x dx - M \int_0^\pi u_x \theta_x dx - (a+R) \int_0^\pi \theta_x \varphi dx \\
&\quad + m \int_0^\pi \varphi \theta dx + 2D^* \int_0^\pi \dot{u}_x \varphi_x dx + a^* \int_0^\pi \theta_x \varphi_x dx \\
&\quad - F^* \int_0^\pi \dot{u}_x \varphi dx - \xi^* \int_0^\pi \dot{\varphi} \varphi dx.
\end{aligned} \tag{6.10}$$

We can see that the following estimate

$$\begin{aligned}
\dot{S}(t) &\leq \int_0^\pi \rho |\dot{u}|^2 dx + \int_0^\pi J |\dot{\varphi}|^2 dx - \int_0^\pi 2U dx + C^* \int_0^\pi (|\theta_x|^2 + |\dot{\varphi}|^2 + |\dot{u}_x|^2) dx \\
&\quad + \epsilon \int_0^\pi (|\varphi_x|^2 + |\varphi|^2 + |u_x|^2) dx.
\end{aligned} \tag{6.11}$$

holds, where  $\epsilon$  can be selected positive, but such small as we want and  $C^*$  also depends on  $\epsilon$ . But taking  $\epsilon$  small enough and recalling the positivity of the internal energy we see that

$$\begin{aligned}
\dot{S}(t) &\leq \int_0^\pi \rho |\dot{u}|^2 dx + \int_0^\pi J |\dot{\varphi}|^2 dx + C^* \int_0^\pi (|\theta_x|^2 + |\dot{\varphi}|^2 + |\dot{u}_x|^2) dx \\
&\quad - K \int_0^\pi (|\varphi_x|^2 + |\varphi|^2 + |u_x|^2) dx.
\end{aligned} \tag{6.12}$$

Here  $K$  is a strictly positive constant.

We can always find a positive constant  $\epsilon_1$  such that for every  $0 < \epsilon \leq \epsilon_1$  the function  $E(t) + \epsilon S(t)$  is equivalent to  $E(t)$  and a positive constant  $\epsilon_2$  such that for  $0 < \epsilon \leq \epsilon_2$

$$\dot{E}(t) + \epsilon \dot{S}(t) \leq -K^*(\epsilon) \int_0^\pi (|\dot{u}|^2 + |\dot{\varphi}|^2 + |\varphi_x|^2 + |\varphi|^2 + |u_x|^2) dx. \tag{6.13}$$

By taking  $\epsilon \leq \min(\epsilon_1, \epsilon_2)$  we obtain that the function  $\Sigma(t) = E(t) + \epsilon S(t)$  satisfies

$$\dot{\Sigma}(t) + \kappa \Sigma(t) \leq 0, \tag{6.14}$$

for a positive  $\kappa$ . Thus the exponential decay of  $\Sigma(t)$  follows, and then the exponential decay of the energy function  $E(t)$ . That is, there exist two positive constants  $M^*, c^*$  such that

$$E(t) \leq M^* E(0) \exp(-c^* t),$$

for every solution and where  $M^*$  and  $c^*$  are uniform for every solution.

Our intention on the remain of this section is to prove that in case that  $a^*$  is different from zero the solutions are generated by an analytic semigroup. We note that the fact that the complex axis is contained in the resolvent of the operator is clear. Thus to prove our aim it is sufficient to prove the second condition of the Theorem 5.1.

Let us to assume that this condition does not hold. Then, there exists a sequence  $\alpha_n$  of positive numbers such that  $\alpha_n \rightarrow \infty$ ; and a sequence  $(u_n, v_n, \varphi_n, \phi_n, \theta_n)$  with

$$\|u_{n,x}\|^2 + \|v_n\|^2 + \|\varphi_{n,x}\|^2 + \|\phi_n\|^2 + \|\theta_n\|^2 = 1, \tag{6.15}$$

such that

$$\alpha_n^{-1}(i\alpha_n u_n - v_n) \rightarrow 0 \text{ in } H^1 \tag{6.16}$$

$$\begin{aligned} \alpha_n^{-1}(i\alpha_n \rho v_n - (\mu u_{n,xx} + b\varphi_{n,x} + \gamma v_{n,xx} - \beta\theta_{n,x} \\ + b^* \phi_{n,x} + D\varphi_{n,xx} + D^* \phi_{n,xx} + M\theta_{n,xx})) \rightarrow 0 \text{ in } L^2 \end{aligned} \tag{6.17}$$

$$\alpha_n^{-1}(i\alpha_n \varphi_n - \phi_n) \rightarrow 0 \text{ in } H^1 \tag{6.18}$$

$$\begin{aligned} \alpha_n^{-1}(i\alpha_n J\phi_n - (A\varphi_{n,xx} - bu_{n,x} - \xi\varphi_n + m\theta_n - a^*\theta_{n,xx} \\ - F^*v_{n,x} - \xi^*\phi_n + Du_{n,xx} - D^*v_{n,xx} - (a + R)\theta_{n,x})) \rightarrow 0 \text{ in } L^2 \end{aligned} \tag{6.19}$$

$$\begin{aligned} \alpha_n^{-1}(i\alpha_n c\theta_n - (k\theta_{n,xx} - \beta v_{n,x} + f v_{n,xx} - m\phi_n \\ + a^* \phi_{n,xx} + (b^{**} - a)\phi_{n,x})) \rightarrow 0 \text{ in } L^2. \end{aligned} \tag{6.20}$$

From the dissipation properties of the operator we have that

$$\alpha_n^{-1/2}(\|v_{n,x}\| + \|\phi_n\| + \|\theta_{n,x}\|) \rightarrow 0. \tag{6.21}$$

From (6.16) it follows that

$$\|u_{n,x}\| \rightarrow 0.$$

In view of (6.18), we see that  $\alpha_n^{-1}\phi_n$  is bounded in  $H^1$ . From (6.20), we note that  $\alpha_n^{-1}\|k\theta_n + f v_n + a^*\phi_n\|_{H^2}$  is uniformly bounded and  $\|k\theta_n + f v_n + a^*\phi_n\|_{L^2}$  is also bounded. Then, by Gagliardo-Nirenberg interpolation inequality  $\alpha_n^{-1/2}\|k\theta_n + f v_n + a^*\phi_n\|_{H^1}$  is also bounded. In view of (6.21)  $\alpha_n^{-1/2}\|\phi_n\|_{H^1}$  must be bounded. We now we multiply (6.20) by  $\theta_n$ . In view of (6.21) we also obtain

$$\|\theta_n\|^2 \rightarrow 0.$$

If we multiply (6.20) by  $\varphi_n$ , we obtain

$$\begin{aligned} \alpha_n^{-1}(k \langle \theta_{n,x}, \varphi_{n,x} \rangle + f \langle v_{n,x}, \varphi_{n,x} \rangle \\ + (a - b^{**}) \langle \phi_n, \varphi_{n,x} \rangle + m \langle \phi_n, \varphi_n \rangle + a^* \langle \phi_{n,x}, \varphi_{n,x} \rangle) \rightarrow 0. \end{aligned}$$

We see that the first fourth terms tend to zero and in view of (6.18), we may substitute  $\alpha_n^{-1}\phi_{n,x}$  by  $\varphi_{n,x}$ . Thus we see

$$a^* \langle \varphi_{n,x}, \varphi_{n,x} \rangle \rightarrow 0.$$

From (6.17) and (6.19), we obtain that  $v_n$  and  $\phi_n$  also tends to zero in the  $L^2$ . This contradicts (6.15). Thus, we have proved

**Theorem 6.1.** *The semigroup generated by operator  $\mathcal{A}_1$  given in (6.4) is analytic.*

**Corollary 6.1.** Let  $(u, \varphi, \theta)$  be a solution of the problem determined by the system (6.1)-(6.3), the initial conditions (2.11) and the boundary conditions (2.10) such that  $u = \varphi = \theta \equiv 0$  after a finite time  $t_0 > 0$ . Then  $u = \varphi = \theta \equiv 0$  for every  $t \geq 0$ .

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