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EXPLICIT SOLITON SOLUTIONS OF THE KAUP-KUPERSHMIDT EQUATION THROUGH THE DYNAMICAL SYSTEM APPROACH*

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Abstract In this paper, we study the traveling wave solutions of the Kaup-Kupershmidt (KK) equation through using the dynamical system approach, which is an integrable fifth-order wave equation. Based on Cosgrove's work [3] and the phase analysis method of dynamical systems, infinitely many soliton solutions are presented in an explicit form. To guarantee the existence of soliton solutions, we discuss the parameters range as well as geometrical explanation of soliton solutions.

Keywords Soliton solution, quasi-periodic solution, periodic solution, Kaup-Kupershmidt equation, homoclinic manifold, center manifold.

MSC(2000) 34C23, 34C37, 35Q55, 37K45, 58Z05, 74J30..

1. Introduction

Recently, we studied the bifurcation and traveling wave solutions [10] of the KdV6 equation

$$u_{xxxxxx} + au_xu_{xxxx} + bu_{xx}u_{xxx} + cu_x^2u_{xx} + du_{tt} + eu_{xxxt} + fu_xu_{xt} + gu_tu_{xx} = 0, \quad (1)$$

where a, b, c, d, e, f and g are real constants.This equation was derived by Karasu-Kalkani and his coworkers [7] and has Lax pair and soliton solutions. As we did in [10], letting $u(x,t) = u(x - vt) = u(\xi)$ and $\phi = u_\xi$ and integrating equation (1) with respect to ξ once, then we have

$$\phi^{(iv)} + a\phi\phi'' + \frac{(b-a)}{2}(\phi')^2 - ev\phi'' + \frac{c}{3}\phi^3 - \frac{v(f+g)}{2}\phi^2 + dv^2\phi + \beta_0 = 0, \quad (2)$$

where β_0 is an integral constant and "$'$" stands for the derivative with respect to ξ. The transformation $y = -(\phi - \frac{ev}{a})$ sends (2) to the following fourth-order ordinary

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differential equation:

$$y^{(iv)} = ayy'' + \frac{(b-a)}{2}(y')^2 - \frac{c}{3}y^3 - \gamma y^2 + \alpha y + \beta, \tag{3}$$

where $\gamma = -\frac{cev}{a} + \frac{v(f+g)}{2}$, $\alpha = -(\frac{ce^2v^2}{a^2} - \frac{ev^2(f+g)}{a} + dv^2)$, and $\beta = \frac{cv^3e^3}{3a^3} - \frac{e^2v^3(f+g)}{2a^2} + \frac{dev^3}{a} + \beta_0$.

In fact, there are a number of nonlinear wave equations with physical background, such as Sawada-Kotera-Caudrey-Dodd-Gibbon equation [13, 2], Kaup-Kupershmidt equation [8, 9, 6] and other higher order integrable equations [5], whose traveling wave equations are able to be the special forms of equation (3). In this paper, we study the traveling wave solutions of the Kaup-Kupershmidt (KK) equation [8, 9]:

$$u_t + \frac{\partial}{\partial x}\left(u_{xxxx} + \frac{45}{2}u_x^2 + 30uu_{xx} + 60u^3\right) = 0, \tag{4}$$

through using the bifurcation approach.

Letting $u(x,t) = u(x-vt) = \phi(\xi)$, integrating once with respect to ξ, and taking $\phi = -\frac{y}{2}$, we obtain the following four order ordinary differential equation

$$y^{(iv)} = 15yy'' + \frac{45}{4}(y')^2 - 15y^3 + vy + \beta, \tag{5}$$

which is apparently a special form of equation (3). The first integrals and some solution formulas of equations (5) have been studied by Cosgrove [3], where (5) corresponds to the F-III form of the higher-order Painleve equations in Cosgrove's paper [3].

Let $x_1 = y,\ x_2 = x_1' = y',\ x_3 = x_2' = y'',\ x_4 = x_3' = y'''$. Then, equation (5) is equivalent to the following four dimensional system

$$x_1' = x_2, \quad x_2' = x_3, \quad x_3' = x_4, \quad x_4' = 15x_1x_3 + \frac{45}{4}x_2^2 - 15x_1^3 + vx_1 + \beta, \tag{6}$$

which has the following conservation laws [3]:

$$\begin{aligned}\Phi_1(x_1,x_2,x_3,x_4) = \ & (x_4 - 12x_1x_2)^2 - 3x_1x_3^2 + (\tfrac{3}{2}x_2^2 + 30x_1^3)x_3 - 9x_1^2x_2^2 - 72x_1^5 \\ & -v(2x_1x_3 - x_2^2 - 8x_1^3) - 2\beta(x_3 - 6x_2^2) - \tfrac{4}{3}v\beta = K_1\end{aligned} \tag{7}$$

and

$$\begin{aligned}\Phi_2(x_1,x_2,x_3,x_4) = \ & x_1x_4^2 - (x_3 + 18x_1^2)x_2x_4 + \tfrac{1}{3}x_3^2 - 6x_1^2x_3^2 + (\tfrac{27}{2}x_1x_2^2 + 30x_1^4)x_3 \\ & -\tfrac{9}{16}x_2^4 + \tfrac{135}{2}x_1^3x_2^2 - 45x_1^6 - v(\tfrac{2}{3}x_2x_4 - \tfrac{1}{3}x_3^2 + 2x_1^2x_3 \\ & -\tfrac{15}{2}x_1x_2^2 - 2x_1^4) - \beta(2x_1x_3 - \tfrac{3}{2}x_2^2 - 6x_1^3) \\ & +\tfrac{1}{3}v^2x_1^2 + \tfrac{2}{3}v\beta x_1 - \tfrac{4}{81}v^3 - \beta^2 = K_2,\end{aligned} \tag{8}$$

where K_1 and K_2 are real constants.

We shall study dynamical behaviors and exact solutions of (6) in the four dimensional phase space $R^4 = (x_1, x_2, x_3, x_4)$ for the case $\beta = 0$.

Let $f(x_1) = 15x_1^3 - vx_1 - \beta$. Then $\frac{df(x_1)}{dx_1} = 45x_1^2 - v$. For $\beta = 0$, and $v \geq 0$, (6) has the following three equilibrium points $E_1\left(-\frac{\sqrt{15v}}{15}, 0, 0, 0\right), E_2(0,0,0,0)$

and $E_3\left(\frac{\sqrt{15v}}{15},0,0,0\right)$. Let $M(x_{1_j},0,0,0)$ be the coefficient matrix of the linearized system of (6) at the equilibrium point $E_j (j=1,2,3)$. Then we have

$$M(x_{1_j},0,0,0)=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ v-45x_{1_j}^2 & 0 & 15x_{1_j} & 0 \end{pmatrix}. \tag{9}$$

At these three equilibrium points, one can easily calculate the eigenvalues of $M(x_{1_j},0,0,0)$, $j=1,2,3$

$$\pm\lambda_1 i,\ \pm\lambda_2 i; \quad \pm v^{\frac{1}{4}},\ \pm v^{\frac{1}{4}}i; \quad \pm\lambda_1,\pm\lambda_2, \tag{10}$$

where $\lambda_1=\frac{\sqrt{2}v^{\frac{1}{4}}}{2}(\sqrt{15}+\sqrt{7})^{\frac{1}{2}}$, $\lambda_2=\frac{\sqrt{2}v^{\frac{1}{4}}}{2}(\sqrt{15}-\sqrt{7})^{\frac{1}{2}}$.

Therefore, equilibrium point $E_j, j=1,2,3$ is a center-center, center-saddle and saddle-saddle, respectively.

In our earlier paper [10], one of our results showed the dynamics of (6) at the equilibrium point E_2 and some explicit solutions. In the current paper, we are dealing with the dynamics of (6) at the equilibrium point E_1 and E_3 and discuss the orbits that are homoclinic to the equilibrium points E_3 and E_1. We will give the parametric representations of these orbits. Furthermore, we find infinitely many soliton solutions of the KK equation, which are different from the typical solitons of the original KdV equation. Those solitons lie in a two dimensional global homoclinic manifold assigned by the equilibrium points E_3 and E_1 in four dimensional phase space through the two conservation laws Φ_1, Φ_2.

2. Explicit soliton solutions of (4)

Let K_1 and K_2 be two arbitrarily given constants. Then, the two level sets, defined by $\Phi_1(x_1,x_2,x_3,x_4)=K_1$ and $\Phi_2(x_1,x_2,x_3,x_4)=K_2$, determine two three dimensional invariant manifolds of system (6). Their intersection is a two dimensional manifold.

For our convenience in the following context, let us take $v=p^2, p>0$. Then, at the equilibrium point E_1 and E_3, the values of K_1 and K_2 in equations (7) and (8) are given by

$$K_{11}=\Phi_1(E_3)=-\frac{16\sqrt{15}}{1125}p^5, \quad K_{21}=\Phi_2(E_3)=-\frac{64}{2025}p^6.$$

$$K_{13}=\Phi_1(E_3)=\frac{16\sqrt{15}}{1125}p^5, \quad K_{23}=\Phi_2(E_3)=-\frac{64}{2025}p^6.$$

We investigate the solutions on the intersectional manifold of the two level sets $\Phi_1(x_1,x_2,x_3,x_4)=K_{1j}$ and $\Phi_2(x_1,x_2,x_3,x_4)=K_{2j}$, which pass through the equilibrium point $E_j\left(\pm\frac{\sqrt{15}p}{15},0,0,0\right)$, $j=1,3$. In this section, we first consider the equilibrium point E_3.

By Cosgrove's results [3], we have

$$y=x_1(\xi)=\frac{(U'(\xi)+V'(\xi))^2}{(U(\xi)+V(\xi))^2}-U^2(\xi)-V^2(\xi)-2A_3, \tag{11}$$

where U and V are defined through inversions of the following two hyper-elliptic integrals

$$I_1 \equiv \int_\infty^U \frac{d\tau}{\sqrt{P_3(\tau)}}+\int_\infty^V \frac{d\tau}{\sqrt{P_3(\tau)}} = C_1, \quad I_2 \equiv \int_\infty^U \frac{\tau d\tau}{\sqrt{P_3(\tau)}}+\int_\infty^V \frac{\tau d\tau}{\sqrt{P_3(\tau)}} = C_2+\xi, \tag{12}$$

$$P_3(t) = (t^2+A_3)^3 - \frac{1}{3}p^2(t^2+A_3) + B_3 t + \frac{1}{3}\beta \tag{13}$$

with $B_3^2 = \frac{1}{9}K_{13}, A_3 = \frac{K_{23}}{9B_3^2} = -\frac{4\sqrt{15}}{27}p$, and C_1 and C_2 are integral constants.

Under the assumption of $v = p^2, \beta = 0$, $P_3(t)$ can be cast into the following form

$$\begin{aligned} P_3(t) = \; & t^6 - \tfrac{4\sqrt{15}}{9}pt^4 + \tfrac{53}{81}p^2t^2 + \tfrac{4(5\sqrt{15})^{\frac{1}{2}}}{225}p^{\frac{5}{2}}t + \tfrac{4\sqrt{15}}{6561}p^3 \\ & = \tfrac{1}{19683}\left(27t^2 + \tfrac{18(45\sqrt{15}p)^{\frac{1}{2}}}{5}t + 5\sqrt{15}p\right)\left(27t^2 - \tfrac{9(45\sqrt{15}p)^{\frac{1}{2}}}{5}t - \tfrac{2\sqrt{15}}{5}p\right)^2 \\ & = (t+r_3)(t+r_4)[(t-r_1)(t-r_2)]^2, \end{aligned} \tag{14}$$

where $r_1 = \frac{(15)^{\frac{1}{4}}}{6}\left(\frac{1}{5}\sqrt{45}+\frac{1}{3}\sqrt{21}\right)\sqrt{p},\ r_2 = \frac{(15)^{\frac{1}{4}}}{6}\left(\frac{1}{5}\sqrt{45}-\frac{1}{3}\sqrt{21}\right)\sqrt{p}$,
$r_3 = \frac{(15)^{\frac{1}{4}}}{15}\left(\sqrt{45}+\frac{1}{3}\sqrt{30}\right)\sqrt{p},\ r_4 = \frac{(15)^{\frac{1}{4}}}{15}\left(\sqrt{45}-\frac{1}{3}\sqrt{30}\right)\sqrt{p}$. One can easily check the following relationships $r_1^2+r_1(r_4+r_3)+r_3r_4 = \lambda_1^2,\ r_2^2+r_2(r_4+r_3)+r_3r_4 = \lambda_2^2$.

Let $e_1 = \frac{1}{2}(r_4+r_3+2r_1) = \frac{(15)^{\frac{1}{4}}}{2}\left(\frac{1}{5}\sqrt{45}+\frac{1}{9}\sqrt{21}\right)\sqrt{p}$ and $e_2 = \frac{1}{2}(r_4+r_3+2r_2) = \frac{(15)^{\frac{1}{4}}}{2}\left(\frac{1}{5}\sqrt{45}-\frac{1}{9}\sqrt{21}\right)\sqrt{p}$. Then, by (12) and (14), we obtain

$$\frac{\left(\lambda_1^2+e_1U_1+\lambda_1\sqrt{U_1^2+2e_1U_1+\lambda_1^2}\right)\left(\lambda_1^2+e_1V_1+\lambda_1\sqrt{V_1^2+2e_1V_1+\lambda_1^2}\right)}{U_1V_1} = C_1e^{-\lambda_1\xi}, \tag{15}$$

and

$$\frac{\left(\lambda_2^2+e_2U_2+\lambda_2\sqrt{U_2^2+2e_2U_2+\lambda_2^2}\right)\left(\lambda_2^2+e_2V_2+\lambda_2\sqrt{V_2^2+2e_2V_2+\lambda_2^2}\right)}{U_2V_2} = C_2e^{-\lambda_2\xi}, \tag{16}$$

where $U_j = U(\xi) - r_j,\ V_j = V(\xi) - r_j,\ j = 1,2$.

To solve for $U(\xi)$ and $V(\xi)$ from equations (15) and (16), let

$$\lambda_1^2 + e_1V_1 + \lambda_1\sqrt{V_1^2+2e_1V_1+\lambda_1^2} = a_1, \quad \lambda_2^2 + e_2U_2 + \lambda_2\sqrt{U_2^2+2e_2U_2+\lambda_2^2} = a_2,$$

where a_1 and a_2 are two constants, which are given through equations (24) and (25) below. Therefore, equations (15) and (16) admit the following solutions

$$U(\xi) = U_a(\xi) = r_1 + \frac{C_1\lambda_1}{-2e_1C_1 + C_1e^{-\lambda_1\xi} + (e_1^2-\lambda_1^2)e^{\lambda_1\xi}}, \quad V(\xi) = V_a \equiv r_1 + V_{10}, \tag{17}$$

where $V_{10} = \frac{a_1e_1 - \sqrt{a_1^2e_1^2-(a_1^2-2\lambda_1^2a_1)(e_1^2-\lambda_1^2)}}{e_1^2-\lambda_1^2}$; and

$$V(\xi) = V_b(\xi) = r_2 + \frac{C_2\lambda_2}{-2e_2C_2 + C_2e^{-\lambda_2\xi} + (e_2^2-\lambda_2^2)e^{\lambda_2\xi}}, \quad U(\xi) = U_b \equiv r_2 + U_{20}, \tag{18}$$

where $U_{20} = \frac{a_2e_2 - \sqrt{a_2^2e_2^2 - (a_2^2 - 2\lambda_2^2 a_2)(e_2^2 - \lambda_2^2)}}{e_2^2 - \lambda_2^2}$. Apparently, one can easily calculate the derivatives of $U_a(\xi)$ and $V_b(\xi)$

$$U_a'(\xi) = \frac{C_1\lambda_1^2[C_1e^{-\lambda_1\xi} - (e_1^2 - \lambda_1^2)e^{\lambda_1\xi}]}{[-2e_1C_1 + C_1e^{-\lambda_1\xi} + (e_1^2 - \lambda_1^2)e^{\lambda_1\xi}]^2} \tag{19}$$

and

$$V_b'(\xi) = \frac{C_2\lambda_2^2[C_2e^{-\lambda_2\xi} - (e_2^2 - \lambda_2^2)e^{\lambda_2\xi}]}{[-2e_2C_2 + C_2e^{-\lambda_2\xi} + (e_2^2 - \lambda_2^2)e^{\lambda_2\xi}]^2}. \tag{20}$$

So, by equation (11), system (6) has the following three classes of explicit solutions:

$$x_1(\xi) = y(\xi) = x_{11}(\xi) = \frac{(U_a'(\xi))^2}{(U_a(\xi) + V_a)^2} - (U_a(\xi))^2 - (V_a)^2 + \frac{8\sqrt{15}}{27}p, \tag{21}$$

$$x_1(\xi) = y(\xi) = x_{12}(\xi) = \frac{(V_b'(\xi))^2}{(U_b + V_b(\xi))^2} - (U_b)^2 - (V_b(\xi))^2 + \frac{8\sqrt{15}}{27}p, \tag{22}$$

$$x_1(\xi) = y(\xi) = x_{13}(\xi) = \frac{(U_a'(\xi) + V_b'(\xi))^2}{(U_a(\xi) + V_b(\xi))^2} - (U_a(\xi))^2 - (V_b(\xi))^2 + \frac{8\sqrt{15}}{27}p,. \tag{23}$$

It is not hard to see that the above three solutions (21), (22) and (23) have the following asymptotic behaviors as $\xi \to \infty$ or $\xi \to -\infty$

$$x_{11}(\xi) \to \left[-2r_1^2 - 2r_1V_{10} - V_{10}^2 + \frac{8\sqrt{15}}{27}p\right] \equiv x_{11}(\pm\infty),$$

$$x_{12}(\xi) \to \left[-2r_2^2 - 2r_2U_{20} - U_{20}^2 + \frac{8\sqrt{15}}{27}p\right] \equiv x_{12}(\pm\infty)$$

and

$$x_{13}(\xi) \to \left[-(r_1^2 + r_2^2) + \frac{8\sqrt{15}}{27}p\right] = \frac{\sqrt{15}p}{15}.$$

In order that $x_{11}(\pm\infty) = x_{12}(\pm\infty) = \frac{\sqrt{15}p}{15}$, solving the above first two equations for V_{10} and U_{20} leads to

$$V_{10} = -r_1 + \sqrt{\frac{31}{135}\sqrt{15}p - r_1^2},$$

$$U_{20} = -r_2 + \sqrt{\frac{31}{135}\sqrt{15}p - r_2^2}.$$

Therefore, the earlier two constants a_1 and a_2 should be given by

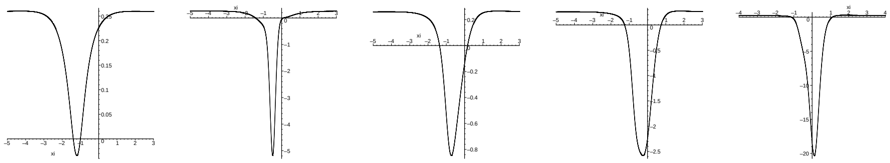
$$a_1 = \lambda_1^2 + e_1V_{10} + \lambda_1\sqrt{V_{10}^2 + 2e_1V_{10} + \lambda_1^2}, \tag{24}$$

$$a_2 = \lambda_2^2 + e_2U_{20} + \lambda_2\sqrt{U_{20}^2 + 2e_2U_{20} + \lambda_2^2}. \tag{25}$$

Obviously, three solutions $x_{11}(\xi)$, $x_{12}(\xi)$, and $x_{13}(\xi)$ of (6) with $\beta = 0$, given by (21), (22), and (23), respectively, depend on integral constants C_1 or C_2 or both. In general, these three solutions have singularities at some points where the denominators of $U_a(\xi)$ and $V_b(\xi)$ are equal to zeros. However, when we choose the two constants C_1 and C_2 satisfy the following conditions:

$$0 < C_1 < 1 - \frac{\lambda_1^2}{e_1^2} \approx 0.0173, 0 < C_2 < 1 - \frac{\lambda_2^2}{e_2^2} \approx 0.0855,$$

(21), (22) and (23) will yield the smooth soliton solutions. That is only the case where solitons happen. For example, for some fixed pairs (C_1, C_2), we have the graphs of $x_{13}(\xi)$, decaying to a non-zero constant $\frac{\sqrt{15}p}{15}$ (See Fig. 1 (1-1)-(1-5)).



(1-1) (1-2) (1-3) (1-4) (1-5)

Fig.1 The graphs of the function $x_{13}(\xi)$ given by (23) for $p = 1$.

(1-1) $C_1 = 0.0008,\ C_2 = 0.008$. (1-2) $C_1 = 0.001,\ C_2 = 0.01$. (1-3) $C_1 = 0.003,\ C_2 = 0.03$. (1-4) $C_1 = 0.005,\ C_2 = 0.05$. (1-5) $C_1 = 0.007,\ C_2 = 0.07$.

The solution $x_{13}(\xi)$ of equation (6) depends on two integral constants C_1 and C_2, and generates a two dimensional homoclinic manifold pertain to the equilibrium point $E_3(\frac{\sqrt{15}p}{15}, 0, 0, 0)$ of (8). The manifold is a global flow, which is determined by the intersection of the two conservation laws $\Phi_1(x_1, x_2, x_3, x_4) = K_{13}$ and $\Phi_2(x_1, x_2, x_3, x_4) = K_{23}$.

3. Explicit quasi-periodic and periodic solutions of (4)

In this section, we discuss a dynamical flow, which is called the center manifold determined through the equilibrium point E_1. In this case, by equations (12) and (13) as well as Cosgrove's work [3], $P_1(t)$ should be

$$P_1(t) = t^6 + \frac{4\sqrt{15}}{9}pt^4 + \frac{53}{81}p^2t^2 + \frac{4(5\sqrt{15})^{\frac{1}{2}}}{225}(-p)^{\frac{5}{2}}t - \frac{4\sqrt{15}}{6561}p^3. \tag{26}$$

Clearly, by the parameter transformation $p \to -p$, the polynomial $P_1(t)$ becomes $P_3(t)$. Therefore, following the solutions (17) and (18) yields

$$U_{a1}(\xi) = r_1 + \frac{C_1 i\lambda_1}{-2e_1C_1 + C_1e^{-i\lambda_1\xi} + (e_1^2 + \lambda_1^2)e^{i\lambda_1\xi}}, \tag{27}$$

and

$$V_{b1}(\xi) = r_2 + \frac{C_2 i\lambda_2}{-2e_2C_2 + C_2e^{-i\lambda_2\xi} + (e_2^2 + \lambda_2^2)e^{i\lambda_2\xi}}. \tag{28}$$

Taking their real parts and imaginary parts, we obtain

$$U_{a1r}(\xi) = r_1 - \frac{C_1\lambda_1[-2e_1C_1 + (C_1 - e_1^2 - \lambda_1^2)\sin(\lambda_1\xi)]}{[-2e_1C_1 + (C_1 + e_1 + \lambda_1^2)\cos(\lambda_1\xi)]^2 + (C_1 - e_1^2 - \lambda_1)^2\sin^2(\lambda_1\xi)}, \tag{29}$$

$$U_{a1i}(\xi) = r_1 + \frac{C_1\lambda_1[-2e_1C_1 + (C_1 + e_1^2 + \lambda_1^2)\cos(\lambda_1\xi)]}{[-2e_1C_1 + (C_1 + e_1 + \lambda_1^2)\cos(\lambda_1\xi)]^2 + (C_1 - e_1^2 - \lambda_1)^2\sin^2(\lambda_1\xi)}; \tag{30}$$

and

$$V_{b1r}(\xi) = r_2 - \frac{C_2\lambda_1[-2e_2C_2 + (C_2 - e_2^2 - \lambda_2^2)\sin(\lambda_2\xi)]}{[-2e_2C_2 + (C_2 + e_2 + \lambda_2^2)\cos(\lambda_2\xi)]^2 + (C_2 - e_2^2 - \lambda_2)^2\sin^2(\lambda_2\xi)}, \tag{31}$$

$$V_{b1i}(\xi) = r_2 + \frac{C_2\lambda_2[-2e_2C_2 + (C_2 + e_2^2 + \lambda_2^2)\cos(\lambda_2\xi)]}{[-2e_2C_2 + (C_2 + e_2 + \lambda_2^2)\cos(\lambda_2\xi)]^2 + (C_2 - e_2^2 - \lambda_2)^2\sin^2(\lambda_2\xi)}. \tag{32}$$

So, by the procedure similar to (23), system (6) has the following two real solutions (one is the real part $y_r(xi)$, and the other one the imaginary part $y_i(\xi)$):

$$x_1(\xi) = y_r(\xi) = \frac{(U'_{a1r}(\xi) + V'_{b1r}(\xi))^2}{(U_{a1r}(\xi) + V_{b1r}(\xi))^2} - (U_{a1r}(\xi))^2 - (V_{b1r}(\xi))^2 - \frac{8\sqrt{15}}{27}p \tag{33}$$

and

$$x_1(\xi) = y_i(\xi) = \frac{(U'_{a1i}(\xi) + V'_{b1i}(\xi))^2}{(U_{a1i}(\xi) + V_{b1i}(\xi))^2} - (U_{a1i}(\xi))^2 - (V_{b1i}(\xi))^2 - \frac{8\sqrt{15}}{27}p \tag{34}$$

Because $\frac{2\pi}{\lambda_1} \neq \frac{2\pi}{\lambda_2}$, (33) and (34) generally give two families of quasi-periodic solutions of (6) for any real number pair (C_1, C_2). However, in the special case of $C_1 = 0, C_2 \neq 0$ or $C_2 = 0, C_1 \neq 0$, we are able to obtain two families of periodic solutions of (6). All these solutions lie in the center manifold $M_1 = \{(x_1, x_2, x_3, x_4) \in R^4 | \Phi_1(x_1, x_2, x_3, x_4) = K_{11}, \Phi_2(x_1, x_2, x_3, x_4) = K_{21}\}$, which is relevant to the equilibrium E_1 of (6).

4. Conclusion and open problems

To conclude our results, we present the following theorem.

Theorem 4.1. *1. The traveling wave system (6) with $\beta = 0$ of the KK equation (4) is able to determine two two-dimensional manifolds: homoclinic one $M = \{(x_1, x_2, x_3, x_4) \in R^4 | \Phi_1(x_1, x_2, x_3, x_4) = K_{13}, \Phi_2(x_1, x_2, x_3, x_4) = K_{23}\}$, and center one $M_1 = \{(x_1, x_2, x_3, x_4) \in R^4 | \Phi_1(x_1, x_2, x_3, x_4) = K_{11}, \Phi_2(x_1, x_2, x_3, x_4) = K_{21}\}$, which are relevant to the two equilibrium points E_3 and E_1, respectively. The dynamical flows of (6) can be solved on homoclinic manifold M and center manifold M_1 by $(x_{13}(\xi), x'_{13}(\xi), x''_{13}(\xi), x'''_{13}(\xi))$, where $x_{13}(\xi)$ is given by (23) and (33) or (34) respectively.*

2. The KK equation (4) has infinitely many classical soliton solutions given by (21), (22),and (23) if two constants satisfy $0 < C_1 < 1 - \frac{\lambda_1^2}{e_1^2}$, $0 < C_2 < 1 - \frac{\lambda_2^2}{e_2^2}$, and also has infinitely many quasi-periodic solutions and periodic solutions given by (33) and (34).

In our paper we only deal with the case of $\beta = 0$ for the traveling wave system (4). What about the case $\beta \neq 0$? It can be discussed by the same method, if $P_j(t)$ has a double root. Any cuspon [14, 11], peakon [1, 4] or M/W-shape soliton [12] solutions appear for the system (4) like the CH, DP, and others? We do not know yet. Also, can (4) be written as a canonical Hamiltonian form?

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