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ON THE EXISTENCE OF BUBBLE-TYPE SOLUTIONS OF NONLINEAR SINGULAR PROBLEMS *

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Abstract Considered in this paper is a class of singular boundary value problem, arising in hydrodynamics and nonlinear field theory, when centrally bubble-type solutions are sought:

$$(p(t)u')' = c(t)p(t)f(u), \qquad u'(0) = 0, \quad u(+\infty) = L > 0$$

in the half-line $[0, +\infty)$, where $p(0) = 0$. We are interested in strictly increasing solutions of this problem in $[0, \infty)$ having just one zero in $(0, +\infty)$ and finite limit at zero, which has great importance in applications or pure and applied mathematics. Sufficient conditions of the existence of such solutions are obtained by applying the critical point theory and by using shooting argument [1, 2] to better analysis the properties of certain solutions associated with the singular differential equation. To the authors' knowledge, for the first time, the above problem is dealt with when f satisfies non-Lipschitz condition. Recent results in the literature are generalized and significantly improved.

Keywords Singular boundary value problem, Monotone solution, Variational methods, Unbounded domain, Homoclinic solution.

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1. Introduction

The singular problem which we investigate in this paper appears when Cahn−Hillard theory has been developed to study the behavior of nonhomogeneous fluid (fluid−fluid, fluid−vapor, fluid−gas, etc., see, e.g., [3, 5] and references therein). If ρ is the density of the medium, $\mu(\rho)$ the chemical potential of a nonhomogeneous fluid and the motion of the fluid is absent, the state of the fluid in R^N is described by the equation

$$\gamma\Delta\rho = \mu(\rho) - \mu_0, \tag{1}$$

where γ and μ_0 are suitable constants. This equation can describe the formation of microscopical bubbles in a nonhomogeneous fluid, in particular, vapor inside one liquid. With this purpose, we add to Eq. (1) the boundary conditions for the

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bubbles. Follows from the central symmetry, it is necessary for the smoothness of solutions of (1) at the origin:

$$\rho'(0) = 0. \tag{2}$$

Since the bubble is surrounded by an external liquid with density ρ_l, the following condition holds at infinity:

$$\lim_{r\to\infty} \rho(r) = \rho_l > 0. \tag{3}$$

From (3) it follows that $\mu_0 = \mu(\rho_l)$. Whenever a strictly increasing solution to problem (1)−(3) exists, for some $\rho(0) = \rho_v$, with $0 < \rho_v < \rho_l$, then ρ_v is the density of the gas at the center of the bubble and the solution ρ determines an increasing mass density profile [12]. In the case of plane or spherical bubbles Eq. (1) takes the form

$$\gamma\left(\rho'' + \frac{N-1}{r}\rho'\right) = \mu(\rho) - \mu(\rho_l), \quad r \in (0, \infty), \tag{4}$$

where $N = 2$ or $N = 3$, respectively, and is known as the density profile equation [3, 6].

In the simplest models for nonhomogeneous fluid, the chemical potential μ is a third degree polynomial with three distinct real roots. After some substitution (see [12]), problem (4), (2) and (3) is reduced to the form

$$(r^{N-1}\rho'(r))' = 4r^{N-1}\lambda^2(\rho + 1)\rho(\rho - \xi), \quad 0 < r < \infty, \tag{5}$$

$$\rho'(0) = 0, \qquad \lim_{r\to\infty} \rho(r) = \xi > 0. \tag{6}$$

More general situation is that the constant coefficient $4\lambda^2$ depends on the variable r, r^{N-1} and the nonlinear term are generalized by $p(r)$ and $f(\rho)$, respectively. Noting that the nonlinear boundary value problem (5), (6) has at least the solution $\rho(r) \equiv \xi > 0$. We are interested in solutions which are strictly increasing and have just exactly one zero in $(0, \infty)$. If such solutions exist, many important physical properties of the bubbles depend on them (in particular, the gas density inside the bubble, the bubble radius and the surface tension). It is also interesting to remark that boundary value problems of same kind arise in nonlinear field theory [4]. Therefore, it becomes an interesting and challenging problem to study the more general system which we will discuss in this paper.

We investigate in this paper a generalization of problem (5), (6), which refer to as the second order singular boundary value problem (BVP for short) in the half-line:

$$(p(t)u')' = c(t)p(t)f(u), \quad 0 \le t < \infty, \tag{7}$$

$$u(0) = 0, \quad u(+\infty) = L > 0, \tag{8}$$

where p, c and f are given continuous functions satisfying some assumptions and $p(0) = 0$. We consider the existence of a strictly increasing solution of problem (7), (8) having just one zero in $(0, \infty)$ and belonging to $C^1([0, \infty)) \bigcap C^2((0, \infty))$.

When $c(t) \equiv 1$, the singular BVP (7), (8) has been investigated in [15, 16] and [17] by means of differential and integral inequalities, upper and lower functions approach, respectively. We mention here that if $c(t) \not\equiv 1$, some arguments in [15]-[17] are unavailable. Problem (7), (8) can be transformed into a problem about the existence of a strictly decreasing and positive solution in the positive half-line which is of significant importance in many disciplines of science such as engineering or pure and applied mathematics. For $p(t) = t^k$, $k \in N$ or $k \in (1, \infty)$, such a problem was solved by shooting argument combined with variational methods in [1] and [2], respectively. It is worth pointing out that in this paper, if $p(t)$ reduces to t^k, we can extend $k \in (0, \infty)$. As for BVP (5) and (6), analytical−numerical investigation and numerical simulation of the problem can be found in [12] and [3, 8], respectively. We emphasize, if BVP (7), (8) reduces to BVP (5), (6), some sufficient conditions obtained in this paper are also necessary. It should be mentioned here that the critical point theory is a powerful tool to deal with the boundary value problems of differential equations on the bounded and unbounded domain, see for instance [11, 13, 14]. In particular, the existence of homoclinic solutions of differential equations has been extensive and intensive studied, see [7, 9, 18, 19] and the references listed therein for information on this subject. Note that the strictly increasing solutions of BVP (7), (8) having just one zero in $(0, \infty)$ and finite limit at zero can also be called homoclinic solutions [15]-[17].

When f satisfies non-Lipschitz condition, as far as authors know, there is no research about the existence of monotone solutions of BVP (7), (8) or some other similar problems. In the present paper, we are interested in the case that f satisfies non-Lipschitz condition, and motivated mainly by the papers [2, 15, 16], we consider the more general problem (7), (8) by applying the critical point theory and by using shooting argument [1, 2] to better analysis the properties of certain solutions associated with the singular differential equation. Recent results in the literature are generalized and significantly improved.

Now we present the basic assumptions in order to obtain the main results in this paper:
(H_1) $c(t)$ is a continuous function in $[0, \infty)$ and there exist real numbers c_1 and c_2 such that $0 < c_1 \le c(t) \le c_2$, $\forall t \in [0, \infty)$;
(H_2) let $L_0 < 0$, f is a continuous function on $[L_0, L]$, thus there exists a constant $K > 0$ such that $\forall x \in [L_0, L]$, $|f(x)| \le K$;
(H_3) $f(L_0) = f(0) = f(L) = 0$, $xf(x) < 0$ for $x \in (L_0, L)\backslash\{0\}$, moreover, there exists $\bar{B} \in (L_0, 0)$ such that $F(\bar{B}) = F(L)$, where

$$F(x) = -\int_0^x f(z)dz, \quad x \in [L_0, L]; \tag{9}$$

(H_4) $p \in C([0, \infty)) \bigcap C^1((0, \infty))$, $p(0) = 0$, $p' > 0$ in $(0, \infty)$.

In addition, we need the following hypothesis on the function p to announce the first result in this paper:
(H_5) there exists $\alpha \in (0, 1)$ such that $p^\alpha(t)/p'(t)$ is bounded as $t \to 0$;
(H_6) $p'(t)/p(t)$ is bounded as $t \to \infty$, and there exists a constant $\bar{b} > 0$, such that

$$\int_0^{\bar{b}} p(t)dt > \frac{1 + 2c_2 F(L)}{2c_1(F(L_0) - F(L))} \int_{\bar{b}}^{\bar{b}-L_0+L} p(t)dt = F_0 \int_{\bar{b}}^{\bar{b}-L_0+L} p(t)dt, \tag{10}$$

where c_2 and F are given by (H_1) and (9), respectively.

Remark 1.1. We will show in Appendix that under the condition (H_4), the explicit condition

$$\lim_{t\to\infty} \frac{p(t)}{p'(t)} = +\infty \tag{11}$$

is a sufficient condition for (H_6). There are many functions satisfy $(H_4)-(H_6)$, for example,

$$p(t)=t^k,\ p(t)=t^k\ln(1+t),\ \text{for}\ k>0,\qquad p(t)=t+\beta\sin t,\ \text{for}\ \beta\in(-1,1),$$
$$p(t)=\frac{t^k}{1+t^l},\ \text{for}\ k\ge l>0,\qquad p(t)=e^t-1,\ \text{for}\ e^{L-L_0}<1/F_0+1.$$

Up until now, we can state our first main result.

Theorem 1.1. *Assume that* $(H_1)-(H_6)$ *are satisfied. If* $c_2/c_1<1+F(L)/F(L_0)$, *then BVP (7), (8) possesses at least one strictly increasing solution* u *with just one zero and* $u(0)\in[L_0,0)$.

Remark 1.2. In the particular case that $c(t)\equiv 1$, assume that $f\in \mathrm{Lip}([L_0,L])$, $(H_3),(H_4)$ and (11) hold. In [16], Rachůnková et al. obtained the existence of escape solutions of BVP (7), (8) which can be used to find the strictly increasing solution having just one zero in $(0,\infty)$ (also called homoclinic solution) for BVP (7), (8). Note that, the homoclinic solutions were obtained under similar conditions in [15] by means of differential and integral inequalities and also obtained under stronger conditions in [17] by using upper and lower functions approach. However, in Theorem 1.1, we do not require f satisfies Lipschitz condition. Moreover, as we will show in Appendix, if function p satisfies (11), then (H_6) holds, but the reverse is not true. In fact, if $p(t)=e^t-1$ for $e^{L-L_0}<1/F_0+1$, then it is easy to check that $(H_4)-(H_6)$ hold but (11) does not. Therefore, we generalize and improve the results in [15]-[17] in some sense.

In addition, if we substitute $c_2/c_1<1+F(L)/F(L_0)$ by $f'(0)<0$, then we have the following theorem.

Theorem 1.2. *Assume that* $(H_1)-(H_5)$ *and* (11) *are satisfied. If* $f'(0)<0$, *then BVP (7), (8) possesses at least one strictly increasing solution* u *with just one zero and* $u(0)\in[L_0,0)$.

Remark 1.3. Problem (7), (8) can be transformed into problems concerning the existence of a strictly decreasing and positive solution which have been considered in [1] and [2] with $p(t)=t^k$ for $k\in N$ and $k\in(1,\infty)$, respectively. However, in this paper, if $p(t)$ reduces to t^k, we do not need any requirement on k expect that $k\in(0,\infty)$ by using an original decomposition technique to better estimate functions in a new function space which we construct in Section 2. In this point of view, we improve and generalize the results in [1, 2].

Remark 1.4. When BVP (7), (8) reduces to the problem (5), (6), after a simple calculation, we get that the conditions of Theorem 1.1 and 1.2 are satisfied if and only if $0<\xi<1$. On the other hand, according to [12](Proposition 4), $0<\xi<1$ is also a necessary condition for the existence of at least a strictly increasing solution having exactly one zero in $(0,\infty)$ for problem (5), (6).

Now we give the main idea of this paper. Similar to [15, 16], consider an auxiliary equation

$$(p(t)u')'=c(t)p(t)\tilde f(u),\quad 0\le t<\infty, \tag{12}$$

where

$$\tilde f(x)=\begin{cases} f(x), & x\in[L_0,L],\\ 0, & x\in R\backslash[L_0,L]. \end{cases} \tag{13}$$

It is obvious that if BVP (12) and (8) has a strictly increasing solution u having just one zero with $u(0) \in [L_0, 0)$, then u is also a solution of BVP (7), (8) with required properties. Therefore, we only need to consider the problem (12) and (8) in the rest of the paper.

Firstly, we consider the case that f satisfies Lipschitz condition. Motivated mainly by the papers [15, 16], we discuss the initial value problem (IVP for short) (12) with initial value condition

$$u(0) = B \leq 0, \quad u'(0) = 0, \tag{14}$$

by means of contraction mapping theorem, differential and integral inequalities. On the other hand, motivated by [2], by using variational method and estimating the values of the variational functional (24) at critical points, we carry out a study of the existence and properties of solutions to BVP (12) with boundary value condition

$$u'(0) = 0, \quad u(T) = L, \quad T > 0. \tag{15}$$

It is suffices then to invoke the shooting argument of [1] to obtain the existence results for BVP (12), (8) and thus for BVP (7), (8) in the case that f satisfies Lipschitz condition.

Let us describe the main idea of the proof. Set $I = (L_0, 0)$ and let $I_i (i = 1, 2)$ be the subset of I, consisting of all B such that the solution of IVP (12), (14) corresponding to B is type $(i), i = 1, 2$, see Proposition 4.1 for the precise definitions of type $(i), i = 1, 2, 3$. We then prove that $I_i (i = 1, 2)$ are disjoint, nonempty open sets, from which we can conclude that there exist elements $B \in I$ which belong neither to I_1 nor to I_2. We conclude the proof by showing that such an element B yields a solution of BVP (12) and (8) with required properties.

Finally, we study BVP (12), (8) under non-Lipschitz condition. Motivated by [10], we first construct a sequence of Lipschitz functions $\tilde{f}_n$ which gives a nice approximation of continuous function $\tilde{f}$ in $(-\infty, +\infty)$ as $n \to \infty$. Then we consider the problem (12) and (8) with $\tilde{f}$ replaced by $\tilde{f}_n$. By using the results obtained in Section 4, we have a certain sequence of strictly increasing functions u_n in $(0, \infty)$ with just one zero and $u_n(0) \in (L_0, 0)$. We prove that, as $n \to \infty$ the limit of u_n exists and is the solution of BVP (12), (8) with required properties.

Part of the difficulty in treating the non-Lipschitz case is caused by the fact that in order to use the results obtained in Section 4, we need that the properties of $\tilde{f}_n$ are similar to $\tilde{f}$ in $(-\infty, +\infty)$. Moreover, the limit of the functions u_n should also be a solution of BVP (12), (8) which satisfies required properties.

The remaining of this paper is organized as follows. In Section 2, we develop a Hilbert space and exhibit a variational functional for BVP (12), (15). Some inequalities and properties are proven which yield the basis for the subsequent use of critical point theory in what follows. In Section 3, in the case that $f \in \mathrm{Lip}([L_0, L])$, some basic properties of solutions of IVP (12), (14) and BVP (12), (15) are discussed, and also in Section 4, several existence criteria of strictly increasing solutions of BVP (12), (8) having just one zero with initial values belong to $(L_0, 0)$ are obtained under the Lipschitz condition. In Section 5, the case that f satisfies non-Lipschitz condition is discussed and the proofs of Theorem 1.1 and 1.2 are given.

Throughout of this paper, we denote by C some positive constant that may change from line to line.

2. Variational structure for BVP (12) and (15)

We shall make use of a variational setting, where solutions of BVP (12), (15) are in correspondence with critical points of a functional. The main idea of this section comes from [2], in which the Sobolev space with weight $t^k, k > 1$ has been considered. As we pointed out in the Introduction, if $p(t)$ reduces to t^k, we can extend $k \in (0, \infty)$.

Given $T \in (0, \infty)$. We introduce a function space $H(0, T)$ consisting of u absolutely continuous in $[0, T]$ such that

$$\|u\|_T^2 := \int_0^T p(t)u'^2(t)dt \tag{16}$$

is finite and $u(T) = 0$. The right-hand side of (16) defines the square of a norm in this space.

It is easy to verify that $H(0, T)$ is a reflexive and separable Banach space. In fact, Let $\{u_n\}$ be a Cauchy sequence in $H(0, T)$. Then the completeness of $L^2([0, T])$ implies that there exists $v \in L^2([0, T])$ such that $\sqrt{p}u_n' \to v$ in $L^2([0, T])$ as $n \to +\infty$. Consider $u(t) = \int_T^t v(s)/\sqrt{p(s)}ds, t \in [0, T]$, we have $u(T) = 0$ and $\sqrt{p}u' = v \in L^2([0, T])$. Therefore, $u \in H(0, T)$ and $\sqrt{p}u_n' \to \sqrt{p}u'$ in $L^2([0, T])$, which means that $u_n \to u$ in $H(0, T)$ as $n \to +\infty$. The reflexivity and separability of $H(0, T)$ can be verified by standard theories.

Here we define the inner product over $H(0, T)$ by

$$< u, v >= \int_0^T p(t)u'(t)v'(t)dt, \qquad u, v \in H(0, T),$$

and $H(0, T)$ is a Hilbert space respect to the inner product.

We can now give some useful estimates.

Proposition 2.1. For $T > 0$ and $u \in H(0, T)$, if (H_4) and (H_5) are satisfied, then inequality

$$\left(\int_0^T p'(t)(p(t))^{\alpha-1}u^2(t)dt\right)^{1/2} \le C\|u\|_T \tag{17}$$

holds, where α is given in (H_5) and $C > 0$ is a constant depends on T.

Proof. For any $u \in H(0, T)$, noting that $u(T) = p(0) = 0$ and $0 < \alpha < 1$, we have

$$\begin{aligned}
\int_0^T p'(t)(p(t))^{\alpha-1}u^2(t)dt &= \int_0^T (p(t))^{\alpha-1}u^2(t)dp(t) \\
&= -\int_0^T p(t)((p(t))^{\alpha-1}u^2(t))'dt \\
&= -2\int_0^T p^\alpha(t)u(t)u'(t)dt \\
&\quad +(1-\alpha)\int_0^T p'(t)(p(t))^{\alpha-1}u^2(t)dt.
\end{aligned}$$

Therefore, by using Cauchy−Schwartz inequality we obtain

$$\begin{aligned}\alpha\int_0^T p'(t)(p(t))^{\alpha-1}u^2(t)dt &= -2\int_0^T (p(t))^{\alpha-1/2}u(t)(p(t))^{1/2}u'(t)dt \\ &\leq 2\bigg(\int_0^T (p(t))^{2\alpha-1}u^2(t)dt\bigg)^{1/2}\|u\|_T. \qquad (18)\end{aligned}$$

According to (H_4) an (H_5), we know that there exists a constant C such that

$$\frac{p^\alpha(t)}{p'(t)} \leq C \qquad \text{for } t\in[0,T],$$

which means that

$$\int_0^T (p(t))^{2\alpha-1}u^2(t)dt \leq C\int_0^T p'(t)(p(t))^{\alpha-1}u^2(t)dt.$$

Combining this with (18), we obtain (17) and the proof is complete. □

Let us recall that for $v\in C([0,T])$, $\|v\|_\infty = \max_{t\in[0,T]}|v(t)|$.

Proposition 2.2. For $T>0$ and $u\in H(0,T)$, if (H_4) and (H_5) are satisfied, then there exists a constant $C>0$ depends on T such that

$$\|pu\|_\infty \leq C\|u\|_T. \qquad (19)$$

Proof. For any $u\in H(0,T)$ and $t\in[0,T]$, we compute by applying Cauchy− Schwartz inequality and (17)

$$\begin{aligned} p(t)u(t) &= \int_0^t (p(s)u(s))'ds \\ &= \int_0^t p'(s)u(s)ds + \int_0^t p(s)u'(s)ds \\ &= \int_0^t \sqrt{p'(s)}(p(s))^{(1-\alpha)/2}\sqrt{p'(s)}(p(s))^{(\alpha-1)/2}u(s)ds \\ &\quad + \int_0^t \sqrt{p(s)}\sqrt{p(s)}u'(s)ds \\ &\leq \bigg(\int_0^T p'(s)(p(s))^{(1-\alpha)}ds\bigg)^{1/2}\bigg(\int_0^T p'(t)(p(t))^{\alpha-1}u^2(t)dt\bigg)^{1/2} \\ &\quad + \bigg(\int_0^T p(s)ds\bigg)^{1/2}\|u\|_T \\ &\leq C(p(T))^{1-\alpha/2}\|u\|_T + \bigg(\int_0^T p(s)ds\bigg)^{1/2}\|u\|_T \\ &\leq C\|u\|_T. \qquad (20)\end{aligned}$$

The proof is complete. □

Proposition 2.3. Let $T > 0$ and $u \in H(0,T)$, if (H_4) and (H_5) are satisfied, then there exists a constant $C > 0$ depends on T such that

$$\int_0^T p(t)u^2(t)dt \leq C\|u\|_T^2. \tag{21}$$

Proof. For any $u \in H(0,T)$ and $t \in [0,T]$, since (H_4) and (H_5), we may write

$$p(t)u^2(t) = p'(t)(p(t))^{\alpha-1}u^2(t)\left(\frac{(p(t))^{2-\alpha}}{p'(t)}\right)$$

as product of two functions. Notice that the second function of the right-hand side is bounded for $t \in [0,T]$, we then apply (17) to conclude. The proof is complete. $\square$

These properties have as an immediate consequence the following proposition.

Proposition 2.4. Let $T > 0$ and $u \in H(0,T)$. Assume that (H_4) and (H_5) are satisfied and the sequence $\{u_k\}$ converges weakly to u in $H(0,T)$ i.e. $u_k \rightharpoonup u$. Then $pu_k \to pu$ in $C([0,T])$, i.e. $\|pu - pu_k\|_\infty = 0$, as $k \to \infty$.

Proof. Consider the function set

$$C_p(0,T) = \{u : (0,T] \to R \text{ is continuous} \mid pu \in C([0,T])\},$$

with the norm $\|\cdot\|_p$ defined by

$$\|u\|_p = \|pu\|_\infty = \max_{t\in[0,T]} |p(t)u(t)|.$$

Then $(C_p(0,T), \|\cdot\|_p)$ is a Banach space by the similar arguments we used to discuss $H(0,T)$.

According to (19), the injection of $H(0,T)$ into $C_p(0,T)$ is continuous, i.e. if $u_k \to u$ in $H(0,T)$, then $u_k \to u$ in $C_p(0,T)$. Since $u_k \rightharpoonup u$ in $H(0,T)$, it follows that $u_k \rightharpoonup u$ in $C_p(0,T)$. By the Banach$-$Steinhaus theorem, $\{u_k\}$ is bounded in $H(0,T)$ and, hence, in $C_p(0,T)$. Moreover, the sequence $\{pu_k\}$ is equi-uniformly continuous since, for $0 \leq t_1 < t_2 \leq T$, by applying (17) and in view of (20), we have

$$\begin{aligned}
& |p(t_2)u_k(t_2) - p(t_1)u_k(t_1)| \\
= & \left|\int_{t_1}^{t_2} (p(t)u_k(t))'dt\right| \\
\leq & \left|\int_{t_1}^{t_2} p'(s)u_k(s)ds\right| + \left|\int_{t_1}^{t_2} p(s)u_k'(s)ds\right| \\
\leq & \left(C\left(\int_{t_1}^{t_2} p'(s)(p(s))^{(1-\alpha)}ds\right)^{1/2} + \left(\int_{t_1}^{t_2} p(s)ds\right)^{1/2}\right)\|u_k\|_T \\
\to & \ 0, \quad \text{as } t_1 \to t_2.
\end{aligned}$$

By the Ascoli$-$Arzela theorem, $\{pu_k\}$ is relatively compact in $C([0,T])$, and thus going to a subsequence if necessary, we may assume that $pu_k \to u^*$ in $C([0,T])$. Hence, $u_k \to u^*/p$ in $C_p(0,T)$. By the uniqueness of the weak limit in $C_p(0,T)$, every uniformly convergent subsequence of $\{u_k\}$ converges uniformly on $[0,T]$ to

u in $C_p(0,T)$, which means that $pu_k \to pu$ in $C([0,T])$ and this completes the proof. $\square$

We are now in a position to establish a variational structure which enables us to reduce the existence of solutions of BVP (12), (15) to the one of finding critical points of corresponding functional defined on the space $H(0,T)$.

First of all, Consider BVP

$$\begin{cases} (p(t)u')' = p(t)\tilde{f}(u+L), & t \in [0,T], \\ u'(0) = 0, \quad u(T) = 0. \end{cases} \tag{22}$$

If $u \in C^1([0,T]) \bigcap C^2((0,T])$, it is obvious that u is a solution of BVP (22) if and only if $u+L$ is a solution of BVP (12), (15). Therefore, we seek a solution of BVP (12), (15) which , of course, corresponds to the solution of BVP (22).

Our task is now to establish a variational functional on $H(0,T)$ such that the critical points of this functional are indeed solutions of BVP (22), and therefore, are solutions of BVP (12), (15).

Proposition 2.5. Assume that $\tilde{f} : R \to R$ is continuous and bounded. Set

$$\tilde{F}(u) = \int_0^u \tilde{f}(x+L)dx. \tag{23}$$

If (H_1), (H_4) and (H_5) are satisfied, then
(i) the functional $\varphi : H(0,T) \to R$ defined as

$$\varphi(u) = \int_0^T p(t)\left[\frac{u'^2(t)}{2} + c(t)\tilde{F}(u(t))\right]dt \tag{24}$$

is continuously differentiable on $H(0,T)$ and for any $u, v \in H(0,T)$, we have

$$< \varphi'(u), v >= \int_0^T [p(t)u'(t)v'(t) + c(t)p(t)\tilde{f}(u(t)+L)v(t)]dt; \tag{25}$$

(ii) φ is weakly lower semicontinuous and coercive on $H(0,T)$;
(iii) a critical point u of φ in $H(0,T)$ with $u(0) \neq 0$ belongs to $C^1([0,T]) \bigcap C^2((0,T])$ and is a solution of BVP (22).

Proof. The Proof of (i) follows from the standard line (see, e.g., [13](Theorem 1.4)), using Proposition 2.2−2.4, so we omit it. For the proof of (ii), φ is weakly lower semi-continuous functional on $H(0,T)$ as the sum of a convex continuous function [13](Theorem 1.2) and of a weakly continuous one [13](Proposition 1.2).

In fact, according to Proposition 2.4, if $u_k \rightharpoonup u$ in $H(0,T)$, then $pu_k \to pu$ in $C([0,T])$. Therefore, $p(t)c(t)\tilde{F}(u_k(t)) \to p(t)c(t)\tilde{F}(u(t))$ for $t \in [0,T]$. By the Lebesgue dominated convergence theorem, we have $\int_0^T p(t)c(t)\tilde{F}(u_k(t))dt \to \int_0^T p(t)c(t)\tilde{F}(u(t))dt$, which means that the functional $u \to \int_0^T p(t)c(t)\tilde{F}(u(t))dt$ is weakly continuous on $H(0,T)$.

On the other hand, it follows from Proposition 2.2 that

$$\begin{aligned}\varphi(u) &= \int_0^T p(t)\left[\frac{u'^2(t)}{2} + c(t)\tilde{F}(u(t))\right]dt \\ &\geq \frac{1}{2}\int_0^T p(t)u'^2(t)dt - C\int_0^T p(t)u(t)dt \\ &\geq \frac{1}{2}\|u\|_T^2 - C\|u\|_T \to +\infty, \quad \text{as } \|u\|_T \to \infty.\end{aligned}$$

Hence, φ is coercive.

For the proof of (iii), we first note that a critical point u of φ satisfies $u(T) = 0$ and $< \varphi'(u), v >= 0$ for any $v \in H(0,T)$, and of course for $v \in C_0^\infty([0,T])$, where $C_0^\infty([0,T])$ is the set of all functions $v \in C^\infty([0,T])$ with $v(0) = v(T) = 0$. By (25), we have that $p(t)c(t)\tilde{f}(u(t) + L)$ is the weak derivative of $p(t)u'(t)$. Since $p(t)c(t)\tilde{f}(u(t) + L)$ is continuous in $[0,T]$, we see that $p(t)c(t)\tilde{f}(u(t) + L)$ is the classical derivative of $p(t)u'(t)$, which means that

$$(p(t)u'(t))' = c(t)p(t)\tilde{f}(u(t) + L), \quad t \in [0,T]. \tag{26}$$

Integrating (26) between $t_1 > 0$ and $t_2 > t_1$, using the boundness of functions c and $\tilde{f}$, we conclude that pu' satisfies the Cauchy condition at $t = 0$ and $t = T$, so that $p(t)u'(t)$ has a finite limit as $t \to 0^+$ or $t \to T^-$. We shall show that $p(t)u'(t) \to 0$ as $t \to 0^+$.

Multiplying (26) by u and integrating between 0 and T, we get

$$p(T)u'(T)u(T) - p(0)u'(0)u(0) - \int_0^T p(t)u'^2(t)dt = \int_0^T p(t)c(t)\tilde{f}(u(t) + L)u(t)dt. \tag{27}$$

Since u is a critical point of φ in $H(0,T)$ with $u(0) \neq 0$ and $u(T) = 0$, we have by (25) and (27) that $p(0)u'(0) = 0$. This implies by integrating (26) again between 0 and t that

$$u'(t) = \frac{1}{p(t)}\int_0^t c(s)p(s)\tilde{f}(u(s) + L)ds, \quad t \in [0,T].$$

Noting that $p(t)$ is increasing in $(0,\infty)$ and the boundness of $c, \tilde{f}$, we have

$$|u'(t)| \leq \frac{1}{p(t)}\int_0^t c(s)p(s)|\tilde{f}(u(s) + L)|ds \leq Ct,$$

where $C > 0$ is a constant and the assertion follows. The proof is complete. □

Let us conclude Section 2 with some remarks. Consider the F and $\tilde{F}$ which we defined by (9) and (23), respectively. Assumptions (H_2) and (H_3) yield the following results.

Remark 2.1. F is continuous on $[L_0, L]$, decreasing in $[L_0, 0)$, increasing in $(0, L]$, $F(B) > F(L)$ for $B \in [L_0, \bar{B})$ and $F(B) < F(L)$ for $B \in (\bar{B}, L)$. furthermore, we have

$$-\int_0^x \tilde{f}(z)dz = \begin{cases} F(L_0), & x \in (-\infty, L_0], \\ F(x), & x \in [L_0, L], \\ F(L), & x \in [L, \infty). \end{cases} \tag{28}$$

Remark 2.2. $\tilde{F}$ is continuous in $(-\infty,+\infty)$ and $\tilde{F}(x) = F(L) - F(x+L),\ x \in [L_0 - L, 0]$. Therefore, $\tilde{F}$ is increasing on $[L_0 - L, -L]$, decreasing on $[-L, 0]$ and $\tilde{F}(-L) = F(L)$, $\tilde{F}(L_0 - L) = F(L) - F(L_0)$. Moreover, we have

$$\tilde{F}(x) \begin{cases} < 0, & x \in (-\infty, \bar{B} - L), \\ > 0, & x \in (\bar{B} - L, 0), \\ = 0, & x \in [0, +\infty) \bigcup \{\bar{B} - L\}, \\ = \tilde{F}(L_0 - L), & x \in (-\infty, L_0 - L]. \end{cases}$$

3. Some properties of solutions of IVP (12), (14) and BVP (12), (15)

We first consider IVP (12), (14). According to [15, 16], we have the following result.

Proposition 3.1. Let $\tilde{f} \in \mathrm{Lip}((-\infty,+\infty))$, $(H_1)-(H_4), (13)$ be satisfied. We have
(i) IVP (12), (14) with $B \in [L_0, 0]$ has a unique solution $u \in C^1([0,\infty)) \bigcap C^2((0,\infty))$. Moreover, if $B = L_0$ or $B = 0$, the solutions are constant functions $u = L_0$ or $u = 0$, respectively.
(ii) For each $b > 0, B_0 \in (L_0, 0)$ and each δ small enough such that $(B_0 - \delta, B_0 + \delta) \subseteq (L_0, 0)$, there exists $M = M(b, B_0, \delta) > 0$ such that

$$|u(t)| + |u'(t)| \le M, \quad t \in [0, b], \qquad \int_0^b \frac{p'(s)}{p(s)} |u'(s)| ds \le M \tag{29}$$

holds for each solution u of IVP (12), (14) with $B \in (B_0 - \delta, B_0 + \delta)$.
(iii) For each $b > 0$ and each $\epsilon > 0$, there exists $\delta > 0$ such that for any $B_1, B_2 \in [L_0, 0]$,

$$|B_1 - B_2| < \delta \Rightarrow |u_1(t) - u_2(t)| + |u_1'(t) - u_2'(t)| < \epsilon, \quad t \in [0, b]. \tag{30}$$

Here u_i is the unique solution of IVP (12), (14) with $B = B_i, i = 1, 2$.

Proof. Noting that $\tilde{f}$ is Lipschitz and bounded in $(-\infty,+\infty)$ and (H_1) implies the boundness of c. The proof of (i) is similar to that of [16](Lemma 4) and the method is contraction mapping theorem. The proof of (ii) follows the arguments of step 2 and step 3 in [15](Lemma 3). The proof of (iii) is similar to that of [16](Lemma 7) and the technical tool is Gronwall inequality. The proof is complete. □

Remark 3.1. Choose $a \ge 0$, and consider the initial conditions

$$u(a) = C \in (-\infty, +\infty), \quad u'(a) = 0. \tag{31}$$

We can prove as in the proof of (i) in Proposition 3.1 that IVP (12), (31) has a unique solution in $[a, +\infty)$. In particular, for $C = 0$, $C \ge L$ or $C \le L_0$, the unique solution of IVP (12), (31) is $u \equiv C$.

The following result is similar to [16](Lemma 6), while the main idea of the proof borrowed from [2](Proposition 11) which is different from that of [16](Lemma 6).

Proposition 3.2. Assume that $(H_1) - (H_4)$ and (H_6) are satisfied. Let u be a solution of (12) such that $u \in C^1([0,\infty)) \bigcap C^2((0,\infty))$ is increasing in $(0, +\infty)$ and

$u(t) \in (L_0, L]$ for each $t \in [0, \infty)$. Then $\lim_{t\to\infty} u(t) \in \{0, L\}$ and in addition that (11) holds, $f'(0)$ exists and is nonzero, we have $\lim_{t\to\infty} u(t) = L$.

Proof. Since $u \in C^1([0, \infty)) \bigcap C^2((0, \infty))$, it follows from (12) that

$$u''(t) + \frac{p'(t)}{p(t)}u'(t) = c(t)\tilde{f}(u(t)), \quad t \in (0, \infty). \tag{32}$$

By virtue of (H_3) and (13) there exists $t_1 > 0$ such that $\tilde{f}(u(t))$ does not change sign for $t \geq t_1$. (H_4) and (H_6) imply that $\int_{t_1}^{t} p'(s)/p(s)u'(s)ds$ is bounded for $t \geq t_1$, note that $\lim_{t\to\infty} u'(t) = 0$ and (32) yields

$$u'(t) - u'(t_1) + \int_{t_1}^{t} \frac{p'(s)}{p(s)}u'(s)ds = \int_{t_1}^{t} c(s)\tilde{f}(u(s))ds, \quad t \geq t_1.$$

We infer that $\int_{t_1}^{+\infty} c(s)\tilde{f}(u(s))ds$ converges, which means that $c(t)\tilde{f}(u(t)) \to 0$ as $t \to +\infty$. By $(H_1), (H_3)$ and the assumptions on u, either $\lim_{t\to\infty} u(t) = 0$ or $\lim_{t\to\infty} u(t) = L$. The rest of the proof is similar to that of [2](Proposition 11) and so we omit it. The proof is complete. $\square$

Some ideas of the following result derive form [2](Proposition 12) and [15] (Theorem 13).

Proposition 3.3. Assume that $(H_1) - (H_4)$ and (H_6) are satisfied. If $B \in (L_0, 0)$ in (14) is sufficiently close to 0, then the solution $u \in C^1([0, \infty)) \bigcap C^2((0, \infty))$ of IVP (12), (14) either attains a local maximum belongs to $(0, L)$ at some point $\bar{t} > 0$ with u increasing in $(0, \bar{t})$, or u is increasing in $(0, \infty)$ with $\lim_{t\to\infty} u(t) = 0$.

Proof. Let $u \in C^1([0, \infty)) \bigcap C^2((0, \infty))$ be a solution of IVP (12), (14) with $B \in (L_0, 0)$ sufficiently close to 0. Integrating (12) over $[0, t]$, we have

$$u'(t) = \frac{1}{p(t)} \int_0^t c(s)p(s)\tilde{f}(u(s))ds, \quad t > 0. \tag{33}$$

Due to $(H_1), (H_3), (H_4)$ and (13), we see that u is strictly increasing for $t > 0$ as long as $u(t) \in (L_0, 0)$, which means that the values of local maximum are greater than 0.

Suppose on the contrary that u does not satisfy the conclusion of Proposition 3.3. Then according to Remark 3.1, Proposition 3.2, (12) and (H_3), there are two possibilities:

(i) there exists $b \in (0, \infty)$ such that $u(b) = L, u'(b) > 0$ and u is increasing in $(0, b]$;

(ii) $\lim_{t\to\infty} u(t) = L$ and u is increasing in $(0, \infty)$.

If u satisfies (i), then there exists $\theta > 0$ such that $u(\theta) = 0$. (32), (H_1), (H_3) and Remark 2.1 give by multiplying u' and integration

$$\begin{aligned} 0 < \frac{u'^2(b)}{2} + \int_0^b \frac{p'(s)}{p(s)}u'^2(s)ds &= \int_0^\theta c(s)\tilde{f}(u(s))u'(s)ds + \int_\theta^b c(s)\tilde{f}(u(s))u'(s)ds \\ &\leq c_2 \int_0^\theta \tilde{f}(u(s))u'(s)ds + c_1 \int_\theta^b \tilde{f}(u(s))u'(s)ds \\ &= c_2 F(B) - c_1 F(L) \leq 0 \end{aligned}$$

as B is close sufficiently to 0. A contradiction.

If u satisfies (ii), then u has a unique zero $\theta > 0$. Multiplying u' and integrating (32) over $[0, \theta]$, we get

$$\frac{u'^2(\theta)}{2} + \int_0^\theta \frac{p'(s)}{p(s)} u'^2(s)ds = \int_0^\theta c(s)\tilde{f}(u(s))u'(s)ds \le c_2 F(B),$$

and thus we have

$$u'^2(\theta) \le 2c_2 F(B). \tag{34}$$

On the other hand, integrating (32) over $[\theta, t]$, we obtain for $t > \theta$

$$\begin{aligned} \frac{u'^2(t)}{2} - \frac{u'^2(\theta)}{2} + \int_\theta^t \frac{p'(s)}{p(s)} u'^2(s)ds &= \int_\theta^t c(s)\tilde{f}(u(s))u'(s)ds \\ &\le c_1 \int_\theta^t \tilde{f}(u(s))u'(s)ds \\ &= -c_1 F(u(t)). \end{aligned} \tag{35}$$

Therefore, letting $t \to \infty$, we get $u'^2(\theta) \ge 2c_1 F(L)$ by (35). This together with (34) implies $c_1 F(L) \le c_2 F(B)$, which is a contradiction as B is sufficiently close to 0. The proof is complete. □

We are now in a position to consider BVP (12), (15). As we pointed out in Section 2, we only need consider BVP (22) and according to Proposition 2.5, we know that in order to find a solution of BVP (22), it suffices to obtain the critical point of functional φ given by (24).

In what follows we shall make use of a function $w : [0, \infty) \to R$ defined by

$$w(t) = \begin{cases} L_0 - L, & \text{if } t \in [0, \bar{b}], \\ t + L_0 - L - \bar{b}, & \text{if } t \in [\bar{b}, \bar{b} - L_0 + L], \\ 0, & \text{otherwise}, \end{cases} \tag{36}$$

where $\bar{b}$ is given by (H_6). According to Remark 2.2, $\tilde{F}(L_0 - L) < 0$.

Proposition 3.4. Assume that $(H_1) - (H_6)$ hold. Then for any $T \ge \bar{b} - L_0 + L$, BVP (22) has a nonzero solution $u \in C^1([0, T]) \bigcap C^2((0, T])$ which means that BVP (12), (15) has a nonconstant solution $u + L$ on $[0, T]$.

Proof. Since φ is coercive and weakly lower semicontinuous according to Proposition 2.5(ii), it follows from [13](Theorem 1.1) that φ attains its minimum at some point in $H(0, T)$, say u. Hence, by Proposition 2.5(iii) and Remark 3.1, it suffices to show that the critical point u is a nonzero function so that BVP (22) is solvable.

In fact, for any $T \ge \bar{b} - L_0 + L$, according to Remark 2.1 and 2.2 and noting that $\tilde{F}(L_0 - L) < 0$, we compute by (H_6)

$$\begin{aligned} &\varphi(w) \\ = \ & \int_0^{\bar{b}} c(s)p(s)\tilde{F}(L_0 - L)ds + \int_{\bar{b}}^{\bar{b}-L_0+L} \left(\frac{p(s)}{2} + c(s)p(s)\tilde{F}(s + L_0 - L - b)\right)ds \\ \le \ & -c_1(F(L_0) - F(L)) \int_0^{\bar{b}} p(s)ds + \left(\frac{1}{2} + c_2 F(L)\right) \int_{\bar{b}}^{\bar{b}-L_0+L} p(s)ds \\ < \ & 0, \end{aligned} \tag{37}$$

where w is given by (36). It is easy to see that (37) together with $\varphi(0) = 0$ implies that u is nonzero. The proof is complete. $\square$

Remark 3.2. Nontrivial solutions of BVP (22) must belong to $(L_0 - L, 0)$ in $[0, T)$, since if a solution of BVP (22) which does not belong to $(L_0 - L, 0)$ in some interval with some positive length cannot satisfy the boundary condition. Therefore, nonconstant solutions of BVP (12), (15) must belong to (L_0, L) in $[0, T)$.

Proposition 3.5. Assume that $(H_1) - (H_5)$ hold and let $T > 0$. If u is a nontrivial solution of BVP (22) that minimizes φ, then
(i) $u(0) \in (L_0 - L, \bar{B} - L)$;
(ii) u is strictly increasing in $(0, T]$.

Proof. The main idea of the proof is borrowed from [2](Proposition 10). First, observe that as $\varphi(0) = 0$, any nontrivial solution u of BVP (22) that minimizes φ satisfies $\varphi(u) \leq 0$. statement (i) is therefore an obvious consequence of Remark 3.2, the positivity of $\tilde{F}$ in $(\bar{B} - L, 0)$ and statement (ii) which we prove next. Since local extrema of u are isolated maxima or minima according to whether their values are greater or smaller than $-L$ by (22) and (H_3), in view of Remark 3.2, we must show that the assumptions

$$L_0 - L < u(t_1) < -L < u(t_0) < 0, \qquad 0 \leq t_0 < t_1 < T$$

lead to a contradiction, where t_0 is the first local maxima of u and t_1 is a minima of u.

Noticing that $\tilde{F}$ is decreasing in $[-L, 0]$, increasing in $[L_0 - L, -L]$ and $\tilde{F}(\bar{B} - L) = 0$ according to Remark 2.2, we let $b \in (\bar{B} - L, -L)$ be such that $\tilde{F}(u(t_0)) = \tilde{F}(b)$. Let $t_2 \in (t_0, T)$ be the smallest number in this interval with $u(t_2) = u(t_0)$. we can assume that $u(t_1) = \min_{t \in [t_0, t_2]} u(t)$.

If $u(t_1) \geq b$, then for the function defined on $[0, T]$ by

$$v(t) = \begin{cases} u(t_0), & \text{if } t \in [t_0, t_2], \\ u(t), & \text{otherwise}, \end{cases}$$

we have $\varphi(v) \leq \varphi(u)$, which is impossible.

If $u(t_1) < b$ and $t_0 = 0$, defining

$$v(t) = \begin{cases} u(t_1), & \text{if } t \in [0, t_1], \\ u(t), & \text{otherwise}, \end{cases}$$

again we would have $\varphi(v) < \varphi(u)$. Therefore if $u(t_1) < b$ we have $t_0 > 0$. Since t_0 is the first local maxima of u, we have $u(0) \in (L_0 - L, -L)$. If $\min_{t \in [0, t_0]} u(t) < u(t_1)$ let $t_3 \in (0, t_0)$ be the point nearest to t_0 such that $u(t_3) = u(t_1)$; otherwise set $t_3 = 0$. Defining

$$v(t) = \begin{cases} u(t_1), & \text{if } t \in [t_3, t_1], \\ u(t), & \text{otherwise}, \end{cases}$$

we conclude again $\varphi(v) < \varphi(u)$ which lead to a contradiction. The proof is complete. $\square$

4. Existence results for BVP (7), (8) with Lipschitz nonlinearities

In this section we are concerned with nonlinear terms $f \in \text{Lip}([L_0, L])$, then by (13), $\tilde{f}$ satisfy Lipschitz condition in $(-\infty, +\infty)$. Firstly, we consider the types of solutions for IVP (12), (14) with $B \in (L_0, 0)$.

Proposition 4.1 (On three types of solution). Assume $(H_1) - (H_4), (H_6)$ hold and let $u \in C^1([0,\infty)) \bigcap C^2((0,\infty))$ be a solution of IVP(12), (14) with $B \in (L_0, 0)$. Then u is just one of the following three types
(1) u vanishes at some $T > 0$, and u is strictly increasing for $t \in (0, T]$;
(2) u attains a local maximum which belongs to $(0, L)$ at some point $\bar{t} > 0$ and u is strictly increasing in $(0, \bar{t})$, or u is strictly increasing in $(0, \infty)$ with $\lim_{t\to\infty} u(t) = 0$;
(3) u is strictly increasing in $(0, \infty)$ with $\lim_{t\to\infty} u(t) = L$.

Proof. Noting that (33), (13), $(H_1) - (H_4)$ imply u is strictly increasing for $t > 0$ as long as $u(t) \in (L_0, 0)$, Remark 3.1 and (H_3) indicate that u can not be a constant in any interval of $(0, \infty)$. These together with proposition 3.2 show the properties stated. The proof is complete. □

Let $I_i (i = 1, 2)$ be the set of all $B \in (L_0, 0)$ such that the corresponding solutions of IVP (12), (14) are type $(i) (i = 1, 2)$ in Proposition 4.1. It is obvious that $I_i (i = 1, 2)$ are disjoint. Then, we have the following result, and some ideas of the proof are taken from that of [15](Theorem 14 and Theorem 20).

Proposition 4.2. Assume that $\tilde{f} \in \text{Lip}((-\infty, +\infty))$, $(H_1) - (H_4)$ and (H_6) hold. Then $I_i (i = 1, 2)$ are open in $(L_0, 0)$ provided that $c_2/c_1 < 1 + F(L)/F(L_0)$.

Proof. We divide the proof into two steps.
Step 1. Let $B_0 \in I_1$ and u_0 be a solution of IVP (12), (14) with $B = B_0$. So, u_0 is the first type in Proposition 4.1. By proposition 3.1(iii), if $B \in (L_0, 0)$ is sufficiently close B_0, then the corresponding solution u of IVP (12), (14) must be the first type, as well.

Let $B_0 \in I_2$ and u_0 be a solution of IVP (12), (14) with $B = B_0$. So, u_0 is the second type in Proposition 4.1. In the case that u attains a local maximum which belongs to $(0, L)$ at some point $\bar{t} > 0$ and u is strictly increasing in $(0, \bar{t})$, then proposition 3.1(iii) and (33) guarantee that if B is sufficiently close to B_0, the corresponding solution u of IVP (12), (14) has also its first local maximum in $(0, L)$ at some point $\bar{t}_1 > 0$ and u is strictly increasing in $(0, \bar{t}_1)$.
Step 2. We are now in a position to consider the case that u_0 is strictly increasing in $(0, \infty)$ with $\lim_{t\to\infty} u_0(t) = 0$. Noting that $c_2/c_1 < 1 + F(L)/F(L_0)$, we can choose $c_0 > 0$ sufficiently small such that

$$\frac{3c_0 + (c_2 - c_1)F(L_0)}{c_1} < F(L). \tag{38}$$

Since u_0 fulfils (32) and noting that $\tilde{f}(u_0(t)) \geq 0, u_0'(t) \geq 0,$ for $t \in (0, \infty)$, we get by integration over $[0, t]$

$$\begin{aligned} c_1(F(B_0) - F(u_0(t))) &= c_1 \int_0^t \tilde{f}(u_0(s))u_0'(s)ds \leq \frac{{u_0'}^2(t)}{2} + \int_0^t \frac{p'(s)}{p(s)}{u_0'}^2(s)ds \\ &\leq c_2 \int_0^t \tilde{f}(u_0(s))u_0'(s)ds = c_2(F(B_0) - F(u_0(t))), \quad t > 0. \end{aligned}$$

For $t \to \infty$ we get, by the fact that $u_0(t) \to 0, u_0'(t) \to 0$ as $t \to \infty$

$$c_1 F(B_0) \le \int_0^\infty \frac{p'(s)}{p(s)} {u_0'}^2(s) ds \le c_2 F(B_0). \tag{39}$$

Therefore we can find $b > 0$ such that

$$\int_b^\infty \frac{p'(s)}{p(s)} {u_0'}^2(s) ds < c_0. \tag{40}$$

Let $\delta > 0$ and $M = M(b, B_0, \delta)$ be the constants from Proposition 3.1(ii). Choose $\epsilon \in (0, c_0/2M)$. Assume that $B \in (L_0, 0)$ and u is a corresponding solution of IVP (12), (14). Using Proposition 3.1 and the continuity of F, we can find $\bar{\delta} \in (0, \delta)$ such that if $|B - B_0| < \bar{\delta}$, then

$$|F(B) - F(B_0)| < \frac{c_0}{c_1}, \tag{41}$$

moreover $|u_0'(t) - u'(t)| \le \epsilon$ for $t \in [0, b]$ and

$$\begin{aligned} \int_0^b \frac{p'(s)}{p(s)} |{u_0'}^2(s) - {u'}^2(s)| ds &\le \max_{t \in [0,b]} |u_0'(t) - u'(t)| \int_0^b \frac{p'(s)}{p(s)} (|u_0'(s)| + |u'(s)|) ds \\ &\le \epsilon \cdot 2M < \frac{c_0}{2M} 2M = c_0. \end{aligned} \tag{42}$$

Suppose that u is not the second type in Proposition 4.1. Then there exists $\theta > 0$ which is the first zero (in fact the only one zero) of u in $(0, \infty)$. Then there are two possibilities. If u is the first type, there is $b_0 > 0$ such that $u(b_0) = L$ and by Remark 3.1, $u'(b_0) > 0$. Then since $(p(t)u'(t))' = 0$ if $u(t) > L$, we get

$$u'(t) > 0 \ \text{ and } u(t) > L \ \text{ for } t > b_0. \tag{43}$$

If u is the third type, then

$$\sup\{u(t), t > \max\{b, b_0\}\} = L. \tag{44}$$

We now rule out possibilities (43) and (44). Integrating (32) over $[0, t]$ and using $(39) - (42)$, we get for $t > \max\{b_0, b\}$

$$\begin{aligned} c_2 F(B) + c_1 \int_0^{u(t)} \tilde{f}(u(s)) ds &\ge \int_0^\theta c(s) \tilde{f}(u(s)) u'(s) ds + \int_\theta^t c(s) \tilde{f}(u(s)) u'(s) ds \\ &= \frac{{u'}^2(s)}{2} + \int_0^t \frac{p'(s)}{p(s)} {u'}^2(s) ds \\ &\ge \int_0^b \frac{p'(s)}{p(s)} {u'}^2(s) ds \\ &= \int_0^b \frac{p'(s)}{p(s)} [{u'}^2(s) - {u_0'}^2(s)] ds + \int_0^b \frac{p'(s)}{p(s)} {u_0'}^2(s) ds \\ &> -c_0 + \int_0^\infty \frac{p'(s)}{p(s)} {u_0'}^2(s) ds - \int_b^\infty \frac{p'(s)}{p(s)} {u_0'}^2(s) ds \\ &> -2c_0 + c_1(F(B_0) - F(B)) + c_1 F(B) \\ &> -3c_0 + c_1 F(B). \end{aligned}$$

In view of (38) and the Monotonicity of F, we get

$$-\int_0^{u(t)} \tilde{f}(u(s))ds \leq \frac{3c_0 + (c_2 - c_1)F(B)}{c_1} < \frac{3c_0 + (c_2 - c_1)F(L_0)}{c_1} < F(L)$$

for $t > \max\{b_0, b\}$. Duo to (28) in Remark 2.1, we have $\sup\{u(t), t > \max\{b, b_0\}\} < L$, which contradicts (43) and (44). The proof is complete. □

We are now in a position to discuss the existence of a strictly increasing solution to BVP (12), (8) (or BVP (7), (8)) which has only one zero in $(0, \infty)$.

Lemma 4.1. *Assume that $\tilde{f} \in \mathrm{Lip}((-\infty, +\infty))$ and $(H_1) - (H_6)$ are satisfied. If $c_2/c_1 < 1 + F(L)/F(L_0)$, then BVP (12), (8) (or BVP (7), (8)) possesses at least one strictly increasing solution u with just one zero and $u(0) \in (L_0, 0)$.*

Proof. The arguments are similar to that of [2](Theorem 18). First, we remark that the solution to IVP (12), (14) with $B \in (L_0, 0)$ exists and unique for $t \in (0, \infty)$ according to Proposition 3.1(i). By Proposition 3.4, 3.5 and 4.2, the set I_1 is nonempty and open in $(L_0, 0)$ and by Proposition 3.3 and 4.2, the set I_2 is also nonempty and open. Therefore the set $I_3 = (L_0, 0)\backslash(I_1 \bigcup I_2)$ is nonempty and if $B \in I_3$, then we infer from Proposition 4.1 that the corresponding solution of IVP (12), (14) is a solution with the required properties. The proof is complete. □

Lemma 4.2. *Assume that $\tilde{f} \in \mathrm{Lip}((-\infty, +\infty))$, $(H_1) - (H_5)$ and (11) are satisfied. If $f'(0) < 0$, then BVP (12), (8) (or BVP (7), (8)) possesses at least one strictly increasing solution u with just one zero and $u(0) \in (L_0, 0)$.*

Proof. The arguments are similar to that of Lemma 4.1. Therefore, we just briefly sketch it. It follows from Proposition 3.2 that in the second type of Proposition 4.1, the case that the solution u of IVP (12), (14) is strictly increasing in $(0, \infty)$ with $\lim_{t\to\infty} u(t) = 0$ is impossible. Therefore, same arguments as Step 1 in the proof of proposition 4.2 show that $I_i (i = 1, 2)$ are nonempty. The rest of the proof is the same as that of Lemma 4.1 and the proof is complete. □

Remark 4.1. If u is a solution obtained in Lemma 4.1 and Lemma 4.2, we can conclude that $u'(t) > 0$ on any bounded interval of $(0, \infty)$. In fact, if $u'(t_0) = 0$ at some $t_0 \in (0, \infty)$, noting that $u(t_0) \neq 0$ by Remark 3.1, we have

$$0 = (p(t_0)u'(t_0))' = c(t_0)p(t_0)\tilde{f}(u(t_0)) \neq 0,$$

a contradiction.

5. Existence results for BVP (7), (8) with non-Lipschitz nonlinearities

If f satisfies (H_2), it is obvious that $\tilde{f}$ satisfies linear growth condition in $(-\infty, +\infty)$ by (13), that is $\forall x \in R$, $|\tilde{f}(x)| \leq K(1 + |x|)$. Firstly, note that the following lemma gives a nice approximation of continuous functions by Lipschitz functions.

Lemma A. ([10]) *Let $\tilde{f} : R \to R$ be a continuous function with linear growth, that is there is a constant $K < \infty$ such that $\forall x \in R$, $|\tilde{f}(x)| \leq K(1 + |x|)$. Then the sequence of functions*

$$\tilde{f}_n(x) = \inf_{y\in R}\{\tilde{f}(y) + n|x - y|\} \tag{45}$$

is well defined for $n \geq K$ and it satisfies
(i) linear growth: $\forall x \in R$, $|\tilde{f}_n(x)| \leq K(1+|x|)$;
(ii) monotonicity in n: $\forall x \in R$, $\tilde{f}_n(x)$ is increasing respect to n and $\tilde{f}_n(x) \leq \tilde{f}(x)$;
(iii) Lipschitz condition: $\forall x, y \in R$, $|\tilde{f}_n(x) - \tilde{f}_n(y)| \leq n|x-y|$;
(iv) strong convergence: if $x_n \to x$ then $\tilde{f}_n(x_n) \to \tilde{f}(x)$ as $n \to \infty$.

Therefore, according to Lemma A and (H_2), we can construct a sequence of functions $\{\tilde{f}_n, n \geq K\}$ given by (45) and combing with (H_3), we have following properties of $\tilde{f}_n$.

Proposition 5.1. If (H_2) and (H_3) are satisfied, then there exist $n_0 \in N(n_0 > K)$ and sequences $\{\bar{B}_n\}, \{\epsilon_n\}, \{L_n\}$ such that for any $n \geq n_0$, we have
(a) $L_0 < \bar{B}_n \leq \bar{B}_{n+1} \leq \cdots \leq \bar{B} < \epsilon_n \leq \epsilon_{n+1} \leq \cdots \leq 0,\ L \leq L_{n+1} \leq L_n \leq \cdots \leq L+1$ and $\bar{B}_n \to \bar{B}, \epsilon_n \to 0, L_n \to L$ as $n \to \infty$;
(b) $\tilde{f}_n(\epsilon_n) = \tilde{f}_n(x) = 0$ for $x \in (-\infty, L_0] \bigcup [L_n, +\infty)$ and $(x-\epsilon_n)\tilde{f}_n(x) < 0$ for $x \in (L_0, L_n) \backslash \{\epsilon_n\}$, moreover we have $\tilde{F}_n(\bar{B}_n) = \tilde{F}_n(L_n)$, where

$$\tilde{F}_n(x) = -\int_{\epsilon_n}^{x} \tilde{f}_n(z)dz, \quad x \in [L_0, L_n]. \tag{46}$$

Proof. From now on, we always suppose that $n \geq K$. We only need to show that $\tilde{f}_n(x) = 0$ for $x \in (-\infty, L_0] \bigcup [L_n, +\infty)$ and $\tilde{f}_n(x) > 0$ for $x \in (L_0, \bar{B})$, the existence of $\{\bar{B}_n\}, \{\epsilon_n\}$ and the other properties are obvious according to (ii) and (iv) in Lemma A and (H_3) for n sufficiently large.

We first show the existence of $\{L_n\}$ and $\tilde{f}_n(x) = 0$ for $x \in [L_n, +\infty)$. We note that, for n sufficiently large, there exists $x \in [L, L+1]$ such that $\tilde{f}_n(x) = 0$. In fact, since $|\tilde{f}_n(x)| \leq K, x \in R$, we can choose $x_0 \in (L, L+1]$ and $n \geq (K+1)/(x_0 - L)$. Since Lemma A(ii) and $\tilde{f}(x) = 0$ for $x \geq L$, we can therefore suppose by contradiction that $\tilde{f}_n(x) < 0$ for any $x \in [L, L+1]$, and thus, by the fact that $\tilde{f}(y) \geq 0, \forall y \in R\backslash(0, L)$ and (13), (45), we have

$$0 > \tilde{f}_n(x) = \inf_{y \in (0,L)} \{\tilde{f}(y) + n|x-y|\}, \quad x \in [x_0, L+1]. \tag{47}$$

On the other hand, for $n \geq (K+1)/(x_0 - L)$, we obtain

$$\tilde{f}_n(y) + n(x-y) \geq -K + \frac{K+1}{x_0 - L}|x-y| \geq 1, \quad x \in [x_0, L+1],\ y \in (0, L),$$

which contradicts to (47).

We now show that if there exists $x_0 \in [L, L+1]$ such that $\tilde{f}_n(x_0) = 0$, then $\tilde{f}_n(x) = 0$ for $x \in [x_0, +\infty)$. For $y \in R\backslash(0, L)$, we have

$$\tilde{f}(y) + n|x-y| \geq 0, \quad x \in [x_0, +\infty), \tag{48}$$

and for $y \in (0, L)$, we have

$$\tilde{f}(y) + n|x-y| \geq \tilde{f}(y) + n|x_0 - y|, \quad x \in [x_0, +\infty),$$

which means that

$$\inf_{y \in (0,L)} \{\tilde{f}(y) + n|x-y|\} \geq \inf_{y \in (0,L)} \{\tilde{f}(y) + n|x_0 - y|\} = \tilde{f}_n(x_0) = 0, \quad x \in [x_0, +\infty).$$

Therefore, this together with (48) implies that

$$\tilde{f}_n(x) = \inf_{y\in R}\{\tilde{f}(y) + n|x-y|\} \geq 0, \quad x \in [x_0, +\infty). \tag{49}$$

On the other hand, according to (H_3) and Lemma A(ii), we have $\tilde{f}_n(x) \leq \tilde{f}(x) = 0$ for $x \in [x_0, +\infty)$. Combing this with (49), the result follows.

For n sufficiently large, let L_n be the first zero point of $\tilde{f}_n$ on $[L, L+1]$, we have $\tilde{f}_n(x) = 0$ for $x \in [L_n, +\infty)$. Furthermore, by Lemma A(ii)(iv), we can get that $L \leq L_{n+1} \leq L_n \leq \cdots \leq L+1$ and $L_n \to L$ as $n \to \infty$.

We are now in a position to prove that $\tilde{f}_n(L_0) = 0$ for n sufficiently large. Suppose that $\tilde{f}_n(L_0) \neq 0$, then $\tilde{f}_n(L_0) < 0$ by (H_3) and Lemma A(ii). Therefore, since $|\tilde{f}_n(x)| \leq K, x \in R$ and for $n \geq (K+1)/L_0$ we have

$$\begin{aligned}
0 > \tilde{f}_n(L_0) &= \inf_{y\in R}\{\tilde{f}(y) + n|L_0 - y|\} \\
&= \inf_{y\in(0,L)}\{\tilde{f}(y) + n|L_0 - y|\} \\
&\geq -K + \frac{K+1}{L_0}L_0 \\
&= 1,
\end{aligned}$$

a contradiction.

Our task is now to prove $\tilde{f}_n(x) = 0$ for $x \in (-\infty, L_0]$. In fact, noting that $\tilde{f}_n(L_0) = 0$, by using the similar arguments as we did to obtain (49), we can get that

$$\tilde{f}_n(x) = \inf_{y\in R}\{\tilde{f}(y) + n|x-y|\} \geq 0, \quad x \in (-\infty, L_0].$$

Therefore by Lemma A(ii), we have

$$0 \leq \tilde{f}_n(x) \leq \tilde{f}(x) = 0, \quad x \in (-\infty, L_0]$$

and the result follows.

Finally, we will show that $\tilde{f}_n(x) > 0$ for $x \in (L_0, \bar{B})$. For any fixed $x_0 \in (L_0, \bar{B})$, we can find $\delta > 0$ such that $[x_0 - \delta, x_0 + \delta] \subset (L_0, \bar{B})$. Therefore, for any $y \in (-\infty, 0]\backslash[x_0 - \delta, x_0 + \delta]$, we have

$$\tilde{f}(y) + n|x_0 - y| \geq \delta > 0, \tag{50}$$

and for any $y \in [x_0 - \delta, x_0 + \delta]$, we have

$$\inf_{y\in[x_0-\delta,x_0+\delta]}\{\tilde{f}(y) + n|x_0 - y|\} \geq \min_{y\in[x_0-\delta,x_0+\delta]}\{\tilde{f}(y)\} > 0. \tag{51}$$

On the other hand, for $n > (K+1)/|\bar{B}|$, we have

$$\tilde{f}(y) + n|x_0 - y| \geq -K + \frac{K+1}{|\bar{B}|}|\bar{B}| > 0, \quad y \in (0, \infty). \tag{52}$$

The result follows from (50)$-$(52) and the proof is complete. □

Proposition 5.2. Suppose that (H_2) and (H_3) are satisfied. For any $x_0 \in R, \delta > 0$, if there exists $K_1 > 0$ such that

$$|\tilde{f}(x) - \tilde{f}(y)| \leq K_1|x-y|, \qquad \forall x, y \in [x_0 - 2\delta, x_0 + 2\delta],$$

then for $n > K_1$ sufficiently large, we have $\tilde{f}_n(x) = \tilde{f}(x), \forall x \in [x_0 - \delta, x_0 + \delta]$.

Proof. According to the boundness of $\tilde{f}$, we have for $x \in [x_0 - \delta, x_0 + \delta]$ and n sufficiently large

$$\inf_{y \in R \setminus [x_0 - 2\delta, x_0 + 2\delta]} \{\tilde{f}(y) + n|x - y|\} \geq \tilde{f}(x_0) \geq \inf_{y \in [x_0 - 2\delta, x_0 + 2\delta]} \{\tilde{f}(y) + n|x - y|\}.$$

Therefore, for $x \in [x_0 - \delta, x_0 + \delta]$ and $n > K_1$ sufficiently large, we have by (45)

$$\begin{aligned}
\tilde{f}_n(x) &= \inf_{y \in [x_0 - 2\delta, x_0 + 2\delta]} \{\tilde{f}(y) + n|x - y|\} \\
&\geq \inf_{y \in [x_0 - 2\delta, x_0 + 2\delta]} \{\tilde{f}(x) - K_1|x - y| + n|x - y|\} \\
&\geq \inf_{y \in [x_0 - 2\delta, x_0 + 2\delta]} \{\tilde{f}(x)\} \\
&= \tilde{f}(x), \quad \forall x \in [x_0 - \delta, x_0 + \delta].
\end{aligned}$$

This together with Lemma A(ii) implies $\tilde{f}(x) = \tilde{f}_n(x),\ \forall x \in [x_0 - \delta, x_0 + \delta]$. The proof is complete. □

According to Proposition 5.1 and 5.2, we can consider the auxiliary problems for n sufficiently large

$$(p(t)u')' = c(t)p(t)\tilde{f}_n(u), \quad t \in [0, \infty), \tag{53}$$

$$u'(0) = 0, \quad u(+\infty) = L_n. \tag{54}$$

We are now in a position to prove our main results given in Introduction.

Proofs of Theorem 1.1 and Theorem 1.2. According to Lemma A(iv) and Proposition 5.1, we can get that for n sufficiently large, there exists a constant $\bar{b} > 0$, such that

$$\int_0^{\bar{b}} p(t)dt > \frac{1 + 2c_2\tilde{F}_n(L_n)}{2c_1(\tilde{F}_n(L_0) - \tilde{F}_n(L_n))} \int_{\bar{b}}^{\bar{b} - L_0 + L_n} p(t)dt,$$

where c_2 and $\tilde{F}_n$ are given by (H_1) and (46), respectively.

If $c_2/c_1 < 1 + F(L)/F(L_0)$, Lemma A(iv) and Proposition 5.1 imply that for n sufficiently large, we have $c_2/c_1 < 1 + F(L_n)/F(L_0)$. On the other hand, if $f'(0)$ exists and $f'(0) < 0$, Proposition 5.2 implies that there exists $\delta > 0$ such that for n sufficiently large, $\tilde{f}_n(x) = \tilde{f}(x)$, $\forall x \in [-\delta, \delta]$, and thus by Proposition 5.1, we have $\epsilon_n = 0, \tilde{f}_n'(\epsilon_n) = 0$.

Therefore, according to Lemma 4.1, Lemma 4.2 and Proposition 5.1, we get that for any $n > K$ sufficiently large, BVP (53), (54) possesses at least one strictly increasing solution $u_n \in C^1([0, \infty)) \bigcap C^2((0, \infty))$ with just one zero and $u_n(0) \in (L_0, \epsilon_n)$.

It is obvious that u_n is bounded uniformly in n. We now show that the sequence $\{u_n\}$ is equicontinuous on any bounded interval. Noting that (53) and (54) imply that u_n satisfies the following integral equation

$$u_n(t) = u_n(0) + \int_0^t \frac{1}{p(s)} \int_0^s c(\tau)p(\tau)\tilde{f}_n(u_n(\tau))d\tau ds, \quad t \in [0, \infty). \tag{55}$$

For any bounded interval $[0, b]$ and $0 \le t_1 < t_2 \le b$, we have by the boundness of $c, \tilde{f}_n$ and the monotonicity of p

$$u_n(t_2) - u_n(t_1) = \int_{t_1}^{t_2} \frac{1}{p(s)} \int_0^s c(\tau)p(\tau)\tilde{f}_n(u_n(\tau))d\tau ds \le C(t_2 - t_1),$$

where C depends on b. Therefore, take a sequence $\{T_k\}_{k\ge1}$ such that $T_k < T_{k+1}$ and $T_k \to \infty$ as $k \to \infty$, we conclude that the sequence $\{u_n\}$ is equicontinuous and uniformly bounded on every interval $[0, T_k]$. Hence, it has a uniformly convergent subsequence on every $[0, T_k]$.

So let $\{u^1_{n_i}\}$ be a subsequence of $\{u_n\}$ that converges on $[0, T_1]$. Consider this subsequence on $[0, T_2]$ and select a further subsequence $\{u^2_{n_i}\}$ of $\{u^1_{n_i}\}$ that converges uniformly on $[0, T_2]$. Repeat this procedure for all k, and then take a diagonal sequence $\{u_{n_i}\}$, which consists of $u^1_{n_1}, u^2_{n_2}, u^3_{n_3}, \cdots$. Since the diagonal sequence $u^p_{n_p}, u^{p+1}_{n_{p+1}}, \cdots$ is a subsequence of $\{u^p_{n_i}\}$ for any $p \ge 1$, it follows that it converges uniformly on any bounded interval to a function u. Without loss of generality, we still denote $\{u_{n_i}\}$ by $\{u_n\}$.

Expressing u_n from (55) and in view of Lemma A(iv), we conclude that for $n \to \infty$

$$u(t) = u(0) + \int_0^t \frac{1}{p(s)} \int_0^s c(\tau)p(\tau)\tilde{f}(u(\tau))d\tau ds, \quad t \in [0, \infty),$$

which means that $u \in C^1([0, \infty)) \bigcap C^2((0, \infty))$ and satisfies (12).

Our task is now to show that u satisfies (8) and u is strictly increasing having just one zero in $(0, \infty)$. Expressing u'_n by the integral equation

$$u'_n(t) = \frac{1}{p(t)} \int_0^t c(s)p(s)\tilde{f}_n(u_n(s))ds,$$

we conclude that the sequence $\{u'_n\}$ converges uniformly on bounded intervals. Suppose the $u'_n \to v$ as $n \to \infty$. Note that for $t \in [0, \infty)$,

$$u_n(t) = u_n(0) + \int_0^t u'_n(s)ds.$$

Let $n \to \infty$ in the above, we obtain

$$u(t) = u(0) + \int_0^t v(s)ds, \qquad \text{for } t \in [0, \infty).$$

This shows that $v(t) = u'(t)$ for $t \in [0, \infty)$ and thus $u'_n \to u'$. Since $u'_n(0) = 0$, we have $u'(0) = u'_n(0) = 0$.

We now show that u is strictly increasing in $(0, \infty)$. In fact, according to Remark 4.1, we know that $u'_n(t) > 0$ on any bounded interval of $(0, \infty)$. Therefore $u'_n \to u'$, as $n \to \infty$ implies that $u'(t) > 0$ on any bounded interval of $(0, \infty)$, i.e. u is strictly increasing in $(0, \infty)$.

It remains to prove that $u(0) \in [L_0, 0)$. Since $u_n \to u$ as $n \to \infty$ and $u_n(0) \in (L_0, \epsilon_n) \subseteq (L_0, 0)$, we get that $u(0) \in [L_0, 0]$. On the other hand, we note that $u(0) \ne 0$. In fact, if $u(0) = 0$, then according to $u' > 0$ on any bounded interval of $(0, \infty)$ and (H_3), we have

$$(p(t)u'(t))' = c(t)p(t)\tilde{f}(u(t)) < 0, \quad t \in (0, \infty),$$

which means that $p(t)u'(t)$ is strictly decreasing in $(0,\infty)$. Noting that $p(0)u'(0)=0$ and $p(t)>0, t\in(0,\infty)$ lead to $u'(t)<0$ on any bounded interval of $(0,\infty)$, a contradiction.

Finally, we need to show that $\lim_{t\to\infty}u(t)=L$ and $u(t)\neq L$ for $t\in(0,\infty)$. In fact, $\lim_{t\to\infty}u(t)=L$ is obvious since $\lim_{t\to\infty}u_n(t)=L_n$ and $u_n\to u$, $L_n\to L$ as $n\to\infty$. Suppose that there exists $t_1\in(0,\infty)$ such that $u(t_1)=L$. There are two possibilities. If $u'(t_1)\leq 0$, contradicts to $u'(t)>0$ on any bounded interval of $(0,\infty)$. If $u'(t_1)>0$, the fact that u is strictly increasing in $(0,\infty)$ implies $u(t)>L$ for $t>t_1$, contradicts to $\lim_{t\to\infty}u(t)=L$. The proof is complete. □

Appendix

In this part, assume that (H_4) is satisfied. We shall prove that (11) is a sufficient condition for (H_6). In fact, if (11) holds, as $t\to\infty$, the boundness of $p'(t)/p(t)$ is obvious. We are now in a position to prove (10). For convenience, we denote $L-L_0>0$ by ξ and (10) can be rewritten as

$$\int_0^{\bar{b}} p(t)dt > F_0\int_{\bar{b}}^{\bar{b}+\xi} p(t)dt. \tag{56}$$

We will show that (56) holds as long as $\bar{b}$ sufficiently large.

In the case that $|p(t)|\leq C$ for $t\in[0,\infty)$, we can choose $\delta\in(0,\bar{b})$ and we have

$$\frac{\int_0^{\bar{b}} p(t)dt}{\int_{\bar{b}}^{\bar{b}+\xi} p(t)dt} \geq \frac{\int_\delta^{\bar{b}} p(t)dt}{C\xi} \geq \frac{p(\delta)}{C\xi}(\bar{b}-\delta)\to+\infty, \quad \text{as } \bar{b}\to+\infty,$$

and (56) follows as long as $\bar{b}$ sufficiently large.

Our task is now to consider the case that p is unbounded in $(0,\infty)$, i.e. $p(t)\to\infty$ as $t\to\infty$. Choose $M=(2+F_0)\xi$, according to (11), there exists $t_0>0$ such that

$$p(t)\geq Mp'(t), \quad t>t_0,$$

and thus

$$\int_{t_0}^{t} p(s)ds \geq \int_{t_0}^{t} Mp'(s)ds = Mp(t)-Mp(t_0), \quad t>t_0. \tag{57}$$

Noting that

$$\int_{t_0}^{t} p(s)ds = \int_0^t p(s)ds - \int_0^{t_0} p(s)ds, \quad t>t_0. \tag{58}$$

Combing (57) and (58), we get that

$$\int_0^t p(s)ds \geq Mp(t)-Mp(t_0)+\int_0^{t_0} p(s)ds, \quad t>t_0. \tag{59}$$

On the other hand, we have

$$\int_0^t p(s)ds = \int_0^{t+\xi} p(s)ds - \int_t^{t+\xi} p(s)ds, \quad t>t_0. \tag{60}$$

Noting that (59) holds for any $t > t_0$, together with (60), we obtain

$$\int_0^t p(s)ds \geq Mp(t+\xi) - Mp(t_0) + \int_0^{t_0} p(s)ds - \int_t^{t+\xi} p(s)ds, \quad t > t_0.$$

Therefore, we have

$$\begin{aligned}
\frac{\int_0^t p(s)ds}{\int_t^{t+\xi} p(s)ds} &\geq \frac{Mp(t+\xi) - Mp(t_0) + \int_0^{t_0} p(s)ds - \int_t^{t+\xi} p(s)ds}{\int_t^{t+\xi} p(s)ds} \\
&\geq \frac{Mp(t+\xi) - Mp(t_0)}{\int_t^{t+\xi} p(s)ds} - 1 \\
&\geq \frac{Mp(t+\xi) - Mp(t_0)}{\xi p(t+\xi)} - 1 \\
&= \frac{M}{\xi} - \frac{Mp(t_0)}{\xi p(t+\xi)} - 1 \\
&= F_0 + 1 - \frac{Mp(t_0)}{\xi p(t+\xi)}, \quad t > t_0.
\end{aligned}$$

Since $p(t+\xi) \to +\infty$ as $t \to +\infty$, we can choose $t > t_0$ sufficiently large such that

$$\frac{\int_0^t p(s)ds}{\int_t^{t+\xi} p(s)ds} > F_0, \quad t > t_0. \tag{61}$$

Replacing t in (61) by $\bar{b}$ and (56) follows. The proof is complete. ☐

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