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MINIMUM WAVE SPEED FOR A DIFFUSIVE COMPETITION MODEL WITH TIME DELAY

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Abstract In this paper we study mono-stable traveling wave solutions for a Lotka-Volterra reaction-diffusion competition model with time delay. By constructing upper and lower solutions, we obtain the precise minimum wave speed of traveling waves under certain conditions. Our results also extend the known results on the minimum wave speed for Lotka-Volterra competition model without delay.

Keywords Lotka-Volterra competition model, Reaction-diffusion system, Time delay, Traveling waves, Minimum wave speed.

MSC(2000) 34K, 35K.

1. Introduction

Diffusive competition models are widely used in Agricultural pest control, dispersal dynamics of populations, disease transmission dynamics, chemical reaction, and so on. Nowadays the time delay effect has been considered as an important factor in modeling the functional response, or the reaction to a population growth because in general this functional response or reaction not just depends on the current state but is an accumulated effect over a previous time period. The following is a Lotka-Volterra competition model with time delay which has been studied by many authors([1], [8]).

$$\begin{aligned} u_t(x,t) &= \Delta u(x,t) + u(x,t)\Big[1 - u(x,t) - a_1 \int_{-\sigma}^{0} v(x,t+\theta)d\eta_1(\theta)\Big] \\ v_t(x,t) &= d\Delta v(x,t) + rv(x,t)\Big[1 - v(x,t) - a_2 \int_{-\sigma}^{0} u(x,t+\theta)d\eta_2(\theta)\Big] \end{aligned} \tag{1.1}$$

where $u(x,t)$and $v(x,t)$ are densities of two populations at time t and location $x \in \mathbb{R}^n$, $\Delta = \sum_{i=1}^{n} \partial^2/\partial x_i^2$ is the Laplace operator, a_1, a_2, d, r, and σ are positive constants, for $i = 1, 2$, $\eta_i : [-\sigma, 0] \to \mathbb{R}$ is nondecreasing, and is of bounded variation with $\int_{-\sigma}^{0} d\eta_i(\theta) = 1$. The system (1.1) is a modification of the well known Lotka-Volterra competition model with the consideration of time delay effect. The system (1.1) has two equilibrium points $(1, 0)$ and $(0, 1)$ that represent the state of

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extinction of one population. One of the important and interesting problems for the system (1.1) is the existence of traveling wave solutions connecting these two equilibrium points, which gives a strong evidence of the principle of competitive exclusion in evolution ecology. In the case of bistable case, that is, both the equilibria $(1,0)$ and $(0,1)$ are stable with respect to the corresponding reaction system (equivalently $a_1 > 1$ and $a_2 > 1$), the existence of a bistable traveling was studied previously in [6]. While in the mono-stable case, i.e., the equilibria $(1,0)$ and $(0,1)$ have the opposite stability, the research on the existence of traveling wave solutions, in particular, finding exact minimum wave speed, has not yet been conducted carefully.

The purpose of this paper is to study the traveling wave solutions for mono-stable (1.1). (1.1) is mono-stable if either $a_1 < 1 < a_2$ or $a_1 > 1 > a_2$. Without loss of generality we suppose throughout this paper that

$$a_1 < 1 < a_2.$$

With this assumption, the equilibrium $(1,0)$ is stable and equilibrium $(0,1)$ is unstable. Hence it is natural to expect that there exist traveling wave fronts moving from $(0,1)$ to $(1,0)$. The existence of such mono-stable traveling wave solutions for the Lotka-Volterra competition system without delay has been studied by several authors [2], [3], [5]. In this paper, we shall extend the results for non-delay systems to our system (1.1). In particular, we will give sufficient conditions under which the minimum wave speed is precisely equal to $c_* = 2\sqrt{1-a_1}$.

This paper is organized as follows. In Section 2 we provide some preliminary results for traveling wave solutions of a monotone time-delayed reaction-diffusion system. In Section 3 we establish our main theorems on the existence and minimum wave speed of traveling wave solutions for the system (1.1). Although our approach is standard monotone iteration arguments, the construction of upper solutions for time delayed system is nontrivial.

2. Preliminaries

Consider the time delayed n-dimensional reaction-diffusion system

$$\frac{\partial u(x,t)}{\partial t} = D\Delta u(x,t) + F\left(u(x,t), \int_{-\sigma}^{0} d\eta(\theta) u(x,t+\theta)\right), \tag{2.1}$$

where $u(x,t) \in \mathbb{R}^n$, $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a smooth function, $D = \mathrm{dig}(d_1, \cdots, d_n)$ is a nonnegative and nonzero diagonal matrix, $\eta : [-\sigma, 0] \to \mathbb{R}^{n\times n}$ is of bounded variation with $\sigma > 0$. We further suppose that

H1 There is a strictly positive vector $U^+ \in \mathbb{R}^n$ such that $F(0) = F(U^+, \eta^* U^+) = 0$, where $\eta^* = \int_{-\sigma}^{0} d\eta(\theta)$.

We look for a traveling wave front of (2.1) connecting the equilibrium points 0 and U^+, i.e., a solution of the form $u(x,s) = U(x \cdot k + cs)$ satisfying the boundary condition

$$U(-\infty) = 0, \qquad U(\infty) = U^+, \tag{2.2}$$

where $k \in \mathbb{R}^n$ is a unit vector and $c \in \mathbb{R}$ is a wave speed. A straightforward substitution yields that $U(t)$ with $t = x \cdot k + cs$ satisfying the system

$$c\dot{U}(t) = D\ddot{U}(t) + F\Big(U(t), \int_{-\sigma}^{0} d\eta(\theta)U(t+c\theta)\Big). \tag{2.3}$$

Let $\mathcal{R}$ be a rectangle region:

$$\mathcal{R} = \big\{0 \leq u \leq U^+\big\}$$

We further suppose that

H2 $\eta(\theta)$ is non-decreasing and $\eta^* : \mathcal{R} \to \mathcal{R}$.

H3 the function $F(u,v) = (F_1(u,v), \cdots, F_n(u,v))$ satisfies the monotone condition in $\mathcal{R} \times \mathcal{R}$. That is, for $u, v \in \mathcal{R}$,

$$\begin{aligned} \frac{\partial F_i(u,v)}{\partial u_j} &\geq 0, \quad i,j = 1, \cdots, n, i \neq j, \\ \frac{\partial F_i(u,v)}{\partial v_j} &\geq 0, \quad i,j = 1, \cdots . \end{aligned} \tag{2.4}$$

H4 $F(u, \eta^* u) \neq 0$ for all $0 \ll u \ll U^+$ and there is a positive vector $h \in \mathbb{R}^n$ and a positive number s_0 such that $F(sh, s\eta^* h) \geq 0$ for all $0 < s \leq s_0$. (Here for $u = (u_1, \cdots, u_n), v = (v_1, \cdots, v_n) \in \mathbb{R}^n$, $u \ll v$ if $u_i < v_i$ for $i = 1, \cdots, n$.)

Theorem 2.1. *Under Assumptions* **H1** - **H4***, there is a* $c_* \geq 0$ *such that the system (2.1) has a nonnegative traveling wave solution connecting* 0 *and* U^+ *if and only if* $c \geq c_*$. *Here* c_* *is called the minimum wave speed.*

We shall omit the proof of this theorem because the proof is essentially similar to the proof for non-delay system (see Theorem 4.2 in [7]).

By the monotone condition (2.4), we can pick a sufficiently large number ρ such that the function

$$F^\rho(U,V) = \rho U + F(U,V)$$

is monotone increasing for $U, V \in \mathcal{R}$. It is well known that $U(t) = \big(U_1(t), \cdots, U_n(t)\big)$ is a bounded solution of (2.3) if and only if for $i = 1, \cdots, n$,

$$\begin{aligned} U_i(t) &= \frac{1}{d_i(\beta_i - \alpha_i)} \bigg[\int_{-\infty}^{t} e^{\alpha_i(t-s)} F_i^\rho\big(U(s), \textstyle\int_{-\sigma}^{0} d\eta(\theta)U(s+c\theta)\big)ds + \\ &\qquad \int_{t}^{\infty} e^{\beta_i(t-s)} F_i^\rho\big(U(s), \textstyle\int_{-\sigma}^{0} d\eta(\theta)U(s+c\theta)\big)ds \bigg] \\ &\stackrel{\text{def}}{=} \mathcal{F}_i(U)(t), \end{aligned} \tag{2.5}$$

where $\alpha_i < 0$ and $\beta_i > 0$ are two roots of the equation

$$d_i\lambda^2 - c\lambda - \rho = 0.$$

Definition 2.1. A function $U \in C(\mathbb{R}; \mathbb{R}^n)$ is an upper (lower) solution of (2.5) if

$$U_i(t) \geq \mathcal{F}_i(U)(t) \quad \big(U_i(t) \leq \mathcal{F}_i(U)(t)\big), \quad t \in \mathbb{R}.$$

It is obvious that, if $U \in C^2(\mathbb{R}, \mathbb{R}^n)$ and

$$c\dot{U}(t) \le D\ddot{U}(t) + F\big(U(t), \int_{-\sigma}^{0} d\eta(\theta)U(t+c\theta)\big), \quad t \in \mathbb{R},$$

then $U(t)$ is a lower solution of (2.5).

In general, finding an upper solution of (2.5) is more difficult. The following lemma provides a way for construction of an upper solution.

Lemma 2.1. *Suppose $c \ge 0$ in (2.5). Let $U = (U_1, \cdots, U_n) \in C(\mathbb{R}, \mathcal{R})$. If $U(t)$ is nondecreasing and there are constants $t_1, \cdots, t_n$ such that $\dot{U}_i(t)$ is continuous on $(-\infty, t_i]$ and $\ddot{U}_i(t)$ is piecewise continuous on $(-\infty, t_i]$, in addition,*

$$\begin{aligned} c\dot{U}_i &\ge d_i\ddot{U}_i(t) + F_i\big(U_i(t), \textstyle\int_{-\sigma}^{0} d\eta(\theta)U(t+c\theta)\big), \quad t \in (-\infty, t_i] \\ U_i(t) &= U_i^+, \qquad t \ge t_i \\ & i = 1, 2, \cdots, n, \end{aligned} \tag{2.6}$$

then $U(t)$ is an upper solution of (2.5).

Proof. Since $U_i(t)$ is non-decreasing, $\dot{U}_i(t_i) \ge 0$, here $\dot{U}_i(t_i)$ is the left derivative. Define $\bar{U} = (\bar{U}_1, \cdots, \bar{U}_n)$ by

$$\bar{U}_i(t) = \begin{cases} U_i(t), & t \le t_i \\ U_i^+ + \dot{U}_i(t_i)(t - t_i), & t > t_i. \end{cases} \tag{2.7}$$

Then $U(t) \le \bar{U}(t)$, $t \in \mathbb{R}$. For $i = 1, \cdots, n$, let

$$d_i\ddot{\bar{U}}_i(t) - c\dot{\bar{U}}_i(t) - \rho\bar{U}_i(t) = -h_i(t), \quad t \in \mathbb{R}. \tag{2.8}$$

Then (2.6)- (2.8) yield that

$$h_i(t) \quad \ge \rho U_i(t) + F_i\big(U(t), \textstyle\int_{-\sigma}^{0} d\eta_i(\theta)U(t+c\theta)\big), \quad t \in (-\infty, t_i]. \tag{2.9}$$

Recall that $F_i(U^+, \eta^* U^+) = 0$ and $U(t) \le U^+$. The monotonicity of $F^\rho(U, V)$ implies that

$$\rho U_i^+ = \rho U_i^+ + F_i\big(U^+, \int_{-\sigma}^{0} d\eta(\theta)U^+\big) \ge F_i^\rho\big(U(t), \int_{-\sigma}^{0} \eta(\theta)U(t+c\theta)\big). \tag{2.10}$$

Hence for $t > t_i$,

$$\begin{aligned} h_i(t) &= \dot{U}_i(t_i) + \rho\bar{U}_i(t) \\ &\ge \rho U_i^+ \\ &= \rho U_i^+ + F_i(U^+, \eta^* U^+) \\ &\ge \rho U_i(t) + F_i\big(U(t), \textstyle\int_{-\sigma}^{0} d\eta(\theta)U(t+c\theta)\big), \quad t > t_i. \end{aligned} \tag{2.11}$$

From (2.9) and (2.11) it follows that for $t \le t_i$,

$$\begin{aligned} U_i(t) &= \bar{U}_i(t) \\ &= \frac{1}{d_i(\beta_i - \alpha_i)}\Big[\int_{-\infty}^{t} e^{\alpha_i(t-s)}h_i(s)ds + \int_{t}^{\infty} e^{\beta_i(t-s)}h_i(s)ds\Big] \\ &\ge \mathcal{F}_i(U)(t). \end{aligned} \tag{2.12}$$

Now for $t > t_i$, by (2.10) we have

$$\begin{aligned} U_i(t) &= U_i^+ \\ &= \frac{1}{d_i(\beta_i - \alpha_i)} \Big[\int_{-\infty}^{t} e^{\alpha_i(t-s)} \rho U_i^+ ds + \int_{t}^{\infty} e^{\beta_i(t-s)} \rho U_i^+ ds \Big] \\ &\geq \mathcal{F}_i(U)(t). \end{aligned} \tag{2.13}$$

(2.12) and (2.13) therefore imply that $U(t)$ is an upper solution. □

The following lemma is an immediate consequence of monotone iteration argument.

Lemma 2.2. *Under Assumptions* **H1 - H4***, if (2.5) has a nondecreasing lower solution $U_0(t)$ and a nondecreasing upper solution $U^0(t)$ such that*

$$0 \ll U_0(t) \leq U^0(t) \leq U^+, \qquad t \in \mathbb{R},$$

then the system (2.1) has a traveling wave solution of wave speed c connecting 0 *and U^+.*

3. Minimum wave speed of traveling waves for competition model

Let us now turn to the time delayed competition system (1.1). If we let

$$w(x,t) = 1 - v(x,t),$$

then (1.1) is transformed to the system

$$\begin{aligned} u_t(x,t) &= \Delta u(x,t) + u(x,t)[1 - u(x,t) - a_1(1 - \int_{-\sigma}^{0} w(x,t+\theta) d\eta_1(\theta))] \\ w_t(x,t) &= d\Delta w(x,t) + r(1 - w(x,t))[a_2 \textstyle\int_{-\sigma}^{0} u(x,t+\theta) d\eta_2(\theta) - w(x,t)] \end{aligned} \tag{3.1}$$

The equilibrium points $(0,1)$ and $(1,0)$ are transformed to the equilibrium points $(0,0)$ and $(1,1)$ of (3.1), respectively. For system (3.1), the corresponding reaction function $F = (F_1, F_2)$ are given by

$$\begin{aligned} F_1\big(u, \textstyle\int_{-\sigma}^{0} w(t+\theta) d\eta_1(\theta)\big) &= u\Big[1 - a_1 - u + a_1 \displaystyle\int_{-\sigma}^{0} w(t+\theta) d\eta_1(\theta)\Big], \\ F_2\big(\textstyle\int_{-\sigma}^{0} u(t+\theta) d\eta_2(\theta), w\big) &= r(1-w)\Big[a_2 \displaystyle\int_{-\sigma}^{0} u(t+\theta) d\eta_2(\theta) - w\Big]. \end{aligned} \tag{3.2}$$

A straightforward verification indicates that, under the assumption of $a_1 < 1 < a_2$, the functions F_1 and F_2 expressed in (3.2) satisfy all Assumptions **H1 - H4**. Let

$$\begin{aligned} u(x,s) &= U(k \cdot x + cs), \\ w(x,s) &= W(k \cdot x + cs) \end{aligned} \tag{3.3}$$

be a traveling wave solution connecting the equilibrium (0,0) and (1,1). Substituting (3.3) into (3.1) and letting $t = x + cs$ yield that

$$\begin{aligned} c\dot{U}(t) &= \ddot{U}(t) + U(t)[1 - U(t) - a_1(1 - \textstyle\int_{-\sigma}^{0} W(t + c\theta)d\eta_1(\theta)] \\ c\dot{W}(t) &= d\ddot{W}(t) + r[1 - W(t)][a_2 \textstyle\int_{-\sigma}^{0} U(t + c\theta)d\eta_2(\theta) - W(t)] \end{aligned} \tag{3.4}$$

with the boundary condition

$$\begin{aligned} U(-\infty) &= W(-\infty) = 0 \\ U(\infty) &= W(\infty) = 1. \end{aligned} \tag{3.5}$$

The linearizion of (3.4) at $(0, 0)$ is

$$\begin{aligned} c\dot{U} &= \ddot{U} + (1 - a_1)U \\ c\dot{W} &= d\ddot{W} + ra_2 \int_{-\sigma}^{0} U(t + c\theta)d\eta_2(\theta) - rW. \end{aligned} \tag{3.6}$$

In order that (3.4) has a nonnegative solution $(U(t), W(t))$ connecting $(0, 0)$ and $(1, 1)$, it is necessary that (3.6)) has at least one positive eigenvalue. The first equation of (3.6) implies that the eigenvalue λ is the solution of the equation

$$c\lambda = \lambda^2 + (1 - a_1)$$

or

$$\lambda = \frac{c \pm \sqrt{c^2 - 4(1 - a_1)}}{2}.$$

Hence λ is real and positive if and only if $c^2 \geq 4(1 - a_1)$ and $c > 0$, equivalently $c \geq 2\sqrt{1 - a_1}$. That is, a necessary condition that (3.1) has a nonnegative traveling wave solution connecting $(0, 0)$ and $(1, 1)$ is that the wave speed $c \geq 2\sqrt{1 - a_1}$. In other words, $2\sqrt{1 - a_1}$ is a lower bound of the minimum wave speed. The interesting question is whether the minimum wave speed is equal to $2\sqrt{1 - a_1}$. We shall find sufficient conditions that guarantees that $2\sqrt{1 - a_1}$ is the minimum wave speed.

Throughout the rest of the paper we let

$$\lambda_0 = \sqrt{1 - a_1}, \qquad c_* = 2\sqrt{1 - a_1} = 2\lambda_0.$$

The above equalities implies that

$$\lambda_0 c_* = 2(1 - a_1).$$

We show the existence of traveling wave solution by constructing of upper and lower solutions. We shall construct upper and lower solutions for the cases $d \leq 2$ and $d > 2$ separately.

3.1. Minimum wave speed under condition $d \leq 2$.

For the case of $d \leq 2$, we first establish the following lemma.

Lemma 3.1. *Suppose that*

[**A1**] $\quad a_1 \left[\int_{-\sigma}^{0} e^{2(1-a_1)\theta} d\eta_1(\theta)\right] \left[\int_{-\sigma}^{0} e^{2(1-a_1)\theta} d\eta_2(\theta)\right] \leq \dfrac{r+(2-d)(1-a_1)}{ra_2}.$

Let $(U^0(t), W^0(t))$ *be defined as follows:*

$$U^0((t) = \begin{cases} u_0 e^{\lambda_0 t}, & t \leq t_1 \\ 1 & t > t_1. \end{cases} \tag{3.7}$$

$$W^0((t) = \begin{cases} w_0 e^{\lambda_0 t}, & t \leq t_2 \\ 1 & t > t_2 \end{cases} \tag{3.8}$$

where

$$\begin{aligned} u_0 &= \frac{r+(2-d)(1-a_1)}{ra_2}, \\ w_0 &= \int_{-\sigma}^{0} d\eta_2(\theta) e^{2(1-a_1)\theta} = \int_{-\sigma}^{0} d\eta_2(\theta) e^{\lambda_0 c_* \theta}, \\ t_1 &= \frac{1}{\lambda_0} \ln\left(\frac{1}{u_0}\right), \quad t_2 = \frac{1}{\lambda_0} \ln\left(\frac{1}{w_0}\right). \end{aligned} \tag{3.9}$$

Then $(U^0(t), W^0(t))$ *is an upper solution of (3.4) with* $c = c_* = 2(1-a_1)$.

Proof. It is obvious that $U^0(t) \leq u_0 e^{\lambda_0 t}$, $W^0(t) \leq w_0 e^{\lambda_0 t}$ for all $t \in \mathbb{R}$. Hence for $t \leq t_1$,

$$\begin{aligned} &U^0(t) - a_1 \textstyle\int_{-\sigma}^{0} W^0(t+c_*\theta) d\eta_1(\theta) \\ &\geq \left[u_0 - a_1 w_0 \textstyle\int_{-\sigma}^{0} e^{\lambda_0 c_* \theta} d\eta_1(\theta)\right] e^{\lambda_0 t} \\ &= \left[\frac{r+(2-d)(1-a_1)}{ra_2} - a_1 \int_{-\sigma}^{0} e^{2(1-a_1)\theta} d\eta_1(\theta) \int_{-\sigma}^{0} e^{2(1-a_1)\theta} d\eta_2(\theta)\right] \\ &\geq 0 \end{aligned} \tag{3.10}$$

Noting that

$$\ddot{U}^0(t) - c_* \dot{U}^0(t) + (1-a_1) U^0(t) = 0, \quad t \leq t_1, \tag{3.11}$$

from (3.10) and (3.11) it follows that

$$\begin{aligned} &\ddot{U}^0(t) - c_* \dot{U}^0(t) + U^0(t) \left[1 - U^0(t) - a_1 + a_1 \textstyle\int_{-\sigma}^{0} W^0(t+c\theta) d\eta_1(\theta)\right] \\ &\quad = \ddot{U}^0(t) - c_* \dot{U}^0(t) + (1-a_1) U^0(t) \\ &\qquad - U^0(t) \left[U^0(t) - a_1 \textstyle\int_{-\sigma}^{0} W^0(t+c_*\theta) d\eta_1(\theta)\right] \\ &\quad = -U^0(t) \left[U^0(t) - a_1 \textstyle\int_{-\sigma}^{0} W^0(t+c_*\theta) d\eta_1(\theta)\right] \\ &\quad \leq 0. \end{aligned} \tag{3.12}$$

Similarly, we have $U^0(t) \le u_0 e^{\lambda_0 t}$ for $t \in \mathbb{R}$. Hence, for $t \le t_2$,

$$\begin{aligned} & r\Big[a_2 \int_{-\sigma}^{0} U^0(t+c_*\theta)d\eta_2(\theta) - W^0(t)\Big] \\ & \quad \le r\Big[a_2 u_0 \int_{-\sigma}^{0} e^{\lambda_0 c_* \theta} d\eta_2(\theta) - w_0\Big] e^{\lambda_0 t} \\ & \quad = (2-d)(1-a_1) w_0 e^{\lambda_0 t} \end{aligned} \tag{3.13}$$

Since $1 - W^0(t) \ge 0$ and

$$d\ddot{W}^0(t) - c_* \dot{W}^0(t) = w_0(d-2)(1-a_1)e^{\lambda_0 t}, \qquad t \le t_2,$$

we have

$$\begin{aligned} & d\ddot{W}^0(t) - c_*\dot{W}^0(t) + r[1-W^0(t)]\Big[a_2 \int_{-\sigma}^{0} U^0(t+c_*\theta)d\eta_2(\theta) - W^0(t)\Big] \\ & \le w_0(d-2)(1-a_1)e^{\lambda_0 t} + [1-W^0(t)] w_0 (2-d)(1-a_1)e^{\lambda_0 t} \\ & = -(2-d)(1-a_1)[W^0(t)]^2 \\ & \le 0. \end{aligned} \tag{3.14}$$

Hence $(U^0(t), W^0(t))$ is an upper solution by Lemma 2.2. □

Next we shall construct a lower solution of (3.4). First, it is well known that the following boundary value problem

$$\begin{aligned} & \ddot{u} - c_*\dot{u} + u(1-a_1-u) = 0, \\ & u(-\infty) = 0, \qquad u(\infty) = 1 - a_1 \end{aligned} \tag{3.15}$$

has a monotone increasing solution $U_0(t)$. Moreover, $U_0(t)$ has the property

$$U_0(t) = e^{\lambda_0 t} + o(e^{\lambda_0 t}) \quad \text{as } t \to -\infty. \tag{3.16}$$

Lemma 3.2. *Let $W_0(t) \equiv 0$. Then $(U_0(t), W_0(t))$ is a lower solution of (3.4). Moreover, let $\mathcal{F}(U,W)$ be the integral operator defined in (2.5) that is associated to the system (3.4). Then $0 \ll \mathcal{F}(U_0, W_0)(t)$ for $t \in \mathbb{R}$.*

Proof. The lemma follows a straightforward verification. □

Theorem 3.1. *Suppose that $d \le 2$ and Condition [**A1**] is satisfied. Then the system (3.1) has a nonnegative traveling wave solution connecting equilibria $(0,0)$ and $(1,1)$ if and only if $c \ge c_*$. That is, the minimum wave speed is $c_* = 2\sqrt{1-a_1}$.*

Proof. By Lemmas 1 and 2 we have an upper solution (U^0, W^0) and a lower solution (U_0, W_0) of (3.4) with wave speed $c = c_*$. Note that both $U^0(t)$ and $U_0(t)$ are monotone increasing and any translation $U_0(t+\tau)$ of $U_0(t)$ is a solution of (3.15). By (3.16), without loss of generality we can suppose $U^0(t) > U_0(t)$ for all $t \in \mathbb{R}$. Hence $(U_0(t), W_0(t)) \le (U^0(t), W^0(t))$, for $t \in \mathbb{R}$. Noticing that $\mathcal{F}(U,W)$ is a monotone operator for $U, W \in C(\mathbb{R}, \mathcal{R})$, Lamma 3.2 implies that

$$0 \ll \mathcal{F}(U_0, W_0)(t) = (U_1(t), W_1(t)) \le (U^0(t), W^0(t)), \quad t \in \mathbb{R}. \tag{3.17}$$

It is obvious that $(U_1(t), W_1(t))$ is a lower solution of (3.4) with $c = c_*$. Therefore Theorem 3 follows from (3.17), Theorem 2.1 and Lemmas 2.3. □

3.2. Minimum wave speed under condition $d > 2$.

Now let us study the case when $d > 2$. We suppose $r-(d-2)\lambda_0^2 > 0$ and define the following numbers and functions which will be used to construct an upper solution.

$$\begin{aligned}
K_i &= \int_{-\sigma}^0 e^{2(1-a_1)\theta}[-(1-a_1)\theta+1]d\eta_i(\theta),\\
\beta &= \frac{r-(d-2)\lambda_0^2}{ra_2}{K_2}^{-1},\\
w_0 &= \frac{\lambda_0(d-2)}{2(d-1)}e^{2/(d-2)},\\
t_* &= -\frac{2}{\lambda_0(d-2)},\\
\xi(t) &= (-t+\frac{2}{\lambda_0})e^{\lambda_0 t},\\
\mu(t) &= \Big(-t+\frac{2}{\lambda_0}\Big)^2 e^{\lambda_0 t}.
\end{aligned} \tag{3.18}$$

Straightforward computations yield that

$$\begin{aligned}
\dot{\xi}(t) &= (1-\lambda_0 t)e^{\lambda_0 t},\\
\ddot{\xi}(t) &= -\lambda_0^2 t e^{\lambda_0 t},\\
\ddot{\mu}(t) &= (-\lambda_0 t)\Big(-t+\frac{2}{\lambda_0}\Big)e^{\lambda_0 t}.
\end{aligned} \tag{3.19}$$

From (3.19)) it follows that

$$d\ddot{\xi}(t) - c_*\dot{\xi}(t) = (d-2)\lambda_0^2\xi(t) - 2\lambda_0(d-1)e^{\lambda_0 t}, \tag{3.20}$$

and $\mu(t)$ is monotone increasing for $t \le 0$. Moreover, by the definition of t_*, we have

$$\begin{aligned}
&w_0\xi(t) < w_0\xi(t_*) = w_0(-t_*+\frac{2}{\lambda_0})e^{\lambda_0 t_*} = 1, \quad t \le t_*,\\
&w_0\mu(t_*)(d-2)\lambda_0^2 = 2\lambda_0(d-1).
\end{aligned} \tag{3.21}$$

In addition, we have

$$\begin{aligned}
&r - ra_2\beta\int_{-\sigma}^0 e^{\lambda_0 c_*\theta}d\eta_2(\theta)\\
&\ge r - ra_2\beta\int_{-\sigma}^0 e^{\lambda_0 c_*\theta}(-\frac{\lambda_0 c_*\theta}{2}+1)d\eta_2(\theta)\\
&= r - ra_2\frac{r-(d-2)\lambda_0^2}{ra_2} = (d-2)\lambda_0^2
\end{aligned} \tag{3.22}$$

(3.22) yields that

$$
\begin{aligned}
& r[\xi(t) - a_2\beta \textstyle\int_{-\sigma}^{0} \xi(t+c_*\theta) d\eta_2(\theta)] \\
& = r\left[\xi(t) - a_2\beta \int_{-\sigma}^{0} e^{\lambda_0 c_* \theta} d\eta_2(\theta)(-t)e^{\lambda_0 t}\right. \\
& \quad \left. -a_2\beta \int_{-\sigma}^{0} e^{\lambda_0 c_* \theta}(-\frac{\lambda_0 c_* \theta}{2}+1) d\eta_2(\theta) \frac{2}{\lambda_0} e^{\lambda_0 t}\right] \\
& = \left[r - ra_2\beta \int_{-\sigma}^{0} e^{\lambda_0 c_* \theta} d\eta_2(\theta)\right](-t)e^{\lambda_0 t} \\
& \quad + \left[r - ra_2\beta \int_{-\sigma}^{0} e^{\lambda_0 c_* \theta}(-\frac{\lambda_0 c_* \theta}{2}+1) d\eta_2(\theta)\right] \frac{2}{\lambda_0} e^{\lambda_0 t} \\
& \geq (d-2)\lambda_0^2(-t+\frac{2}{\lambda_0})e^{\lambda_0}t \\
& = (d-2)\lambda_0^2 \xi(t)
\end{aligned}
\tag{3.23}
$$

Now we define the function $W^0(t)$ as

$$
W^0(t) = \begin{cases} w_0\xi(t), & -\infty < t \leq t_* \\ 1, & t > t_* \end{cases} \tag{3.24}
$$

and we define $U^0(t)$ depending on two cases:
Case 1. If there is a $t_1 \leq 0$ such that

$$w_0\beta\xi(t_1) = 1,$$

then we define

$$
U^0(t) = \begin{cases} w_0\beta\xi(t), & t \leq t_1 \\ 1, & t > t_1 \end{cases} \tag{3.25}
$$

Case 2. If $w_0\beta\xi(t) < 1$ for all $t \leq 0$. We define

$$
U^0(t) = \begin{cases} w_0\beta\xi(t), & t \leq 0 \\ w_0\beta\xi(0) + w_0\beta\dot{\xi}(0)t, & 0 < t \leq t_1 \\ 1, & t > t_1, \end{cases} \tag{3.26}
$$

where

$$t_1 = \frac{1 - w_0\beta\xi(0)}{w_0\beta\dot{\xi}(0)}.$$

Lemma 3.3. *Suppose that*

$$[\mathbf{A2}]\ \ \frac{r-(d-2)\lambda_0^2}{ra_2K_2} \geq \max\left\{a_1K_1, \frac{(d-1)}{2(d-2)}e^{-2/(d-2)}, \frac{a_1(d-1)}{(d-2)}e^{-2/(d-2)}\right\}.$$

Then $(U^0(t), W^0(t))$ *defined by (3.24), (3.25) or (3.24), (3.26) is an upper solution.*

Proof. For either Case 1 or Case 2, we have

$$U^0(t) \leq w_0\beta\xi(t), \qquad t \leq 0.$$

Hence by (3.20), (3.21), (3.23), for $t \leq t_*$,

$$\begin{aligned}
&d\ddot{W}^0(t) - c_*\dot{W}^0(t) - r(1-W^0(t))(W^0(t) - a_2\int_{-\sigma}^0 U^0(t+c_*\theta)d\eta_2(\theta))\\
&\quad = w_0[d\ddot{\xi}(t) - c_*\dot{\xi}(t) - (1-w_0\xi(t))r(\xi(t) - a_2\textstyle\int_{-\sigma}^0 \xi(t+c_*\theta)d\eta_2(\theta))]\\
&\quad \leq w_0[(d-2)\lambda_0^2\xi(t) - 2\lambda_0(d-1)e^{\lambda_0 t} - (1-w_0\xi(t))(d-2)\lambda_0^2\xi(t)]\\
&\quad = w_0[-2\lambda_0(d-1)e^{\lambda_0 t} + w_0(d-2)\lambda_0^2\xi^2(t)]\\
&\quad = w_0[-2\lambda_0(d-1) + w_0(d-2)\lambda_0^2\mu(t)]e^{\lambda_0 t}\\
&\quad \leq w_0[-2\lambda_0(d-1) + w_0(d-2)\lambda_0^2\mu(t_*)]e^{\lambda_0 t_*}\\
&\quad = 0.
\end{aligned} \tag{3.27}$$

Now for $U^0(t)$ with Case 1, since $W^0(t) \leq w_0\xi(t)$ for $t \leq t_1$, for $t \leq t_1$,

$$\begin{aligned}
&\ddot{U}^0(t) - c_*\dot{U}^0(t) + U^0(t)[1 - a_1 - U^0(t) + a_1\textstyle\int_{-\sigma}^0 W^0(t+c_*\theta)d\eta_1(\theta)]\\
&\quad = \ddot{U}^0(t) - c_*\dot{U}^0(t) + \lambda_0^2U^0(t) - U^0(t)[U^0(t) - a_1\textstyle\int_{-\sigma}^0 W^0(t+c_*\theta)d\eta_1(\theta)]\\
&\quad \leq -w_0^2\beta\xi(t)[\beta\xi(t) - a_1\textstyle\int_{-\sigma}^0(-t-c_*\theta+\dfrac{2}{\lambda_0})e^{\lambda_0c_*\theta}d\eta_1(\theta)e^{\lambda_0 t}]\\
&\quad = -w_0^2\beta\xi(t)\Big\{[\beta - a_1\textstyle\int_{-\sigma}^0 e^{\lambda_0c_*\theta}d\eta_1(\theta)](-t)e^{\lambda_0 t} +\\
&\qquad \dfrac{2}{\lambda_0}[\beta - a_1\textstyle\int_{-\sigma}^0 e^{\lambda_0c_*\theta}(-\dfrac{\lambda_0c_*\theta}{2}+1)d\eta_1(\theta)]e^{\lambda_0 t}\Big\}
\end{aligned} \tag{3.28}$$

By the assumption [**A2**], we have

$$\begin{aligned}\beta - a_1\int_{-\sigma}^{0} e^{\lambda_0 c_*\theta} d\eta_1(\theta) \geq\ & \beta - a_1\int_{-\sigma}^{0} e^{\lambda_0 c_*\theta}(-\frac{\lambda_0 c_*\theta}{2}+1)d\eta_1(\theta)\\ &= \frac{1}{K_2}[\frac{r-(d-2)\lambda_0^2}{ra_2} - a_1K_1K_2]\\ &\geq 0.\end{aligned} \tag{3.29}$$

(3.28) and (3.29) immediately imply that, for $t \leq t_1$,

$$\ddot{U}^0(t) - c_*\dot{U}^0(t) + U(t)\big[1 - a_1 - U^0(t) + a_1\int_{-\sigma}^{0} W^0(t+c_*\theta)d\eta_1(\theta)\big] \leq 0. \tag{3.30}$$

For Case 2, first it is obvious that (3.30) still holds for $t \leq 0$. Let us first suppose that $a_1 \leq \frac{1}{2}$, then by Assumption [**A2**] and definitions of β and w_0 (see 3.18), we have

$$U^0(0) = w_0\beta\xi(0) = w_0\beta\frac{2}{\lambda_0} \geq \frac{1}{2}. \tag{3.31}$$

Moreover,

$$2\lambda_0\dot{U}^0(0) = 2\lambda_0 w_0\beta \geq \frac{\lambda^2}{2} = \frac{1-a_1}{2} \geq \frac{1}{4}\,. \tag{3.32}$$

By the definition of $U^0(t)$, $\ddot{U}^0(t) = 0$ and $1 \geq U^0(t) \geq U(0) \geq \frac{1}{2}$ for $0 < t \leq t_1$. Thus (3.31) and (3.32) imply that for all $t > 0$ with $U^0(t) \leq 1$,

$$\begin{aligned}&\ddot{U}^0(t) - c_*\dot{U}^0(t) + U^0(t)(1 - U^0(t) - a_1[1 - \textstyle\int_{-\sigma}^{0} W^0(t+c_*\theta)d\eta_1(\theta)])\\ &\leq -\frac{1}{4} + U(t)(1-U(t))\\ &\leq -\frac{1}{4} + \frac{1}{2}(1-\frac{1}{2})\\ &= 0.\end{aligned} \tag{3.33}$$

Next suppose $a_1 > \frac{1}{2}$, then from Assumption [**A2**] and definitions of w_0 and β it follows that for $0 < t \leq t_1$,

$$1 \geq U^0(t) \geq U^0(0) = w_0\beta\frac{2}{\lambda_0} \geq a_1. \tag{3.34}$$

(3.34) yields that

$$U^0(t)(1-U^0(t)) \leq a_1(1-a_1). \tag{3.35}$$

Note that $2c_*\dot{U}^0(0) = (1-a_1)U^0(0)$. From (3.34) and (3.35) we deduce that for $0 < t \leq t_1$,

$$
\begin{aligned}
&\ddot{U}^0(t) - c_*\dot{U}^0(t) + U^0(t)\Big[1 - U^0(t) - a_1\big(1 - \textstyle\int_{-\sigma}^0 W(t + c_*\theta)d\eta_1(\theta)\big)\Big] \\
&\quad \le -2c_*\dot{U}^0(0) + U^0(t)[1 - U^0(t)] \\
&\quad \le -(1 - a_1)U^0(0) + a_1(1 - a_1) \\
&\quad \le 0.
\end{aligned} \tag{3.36}
$$

Hence $(U^0(t), W^0(t))$ is an upper solution by Lemma 2.2. □

By using the same argument for the proof of Theorem 3.3 we immediately have the following

Theorem 3.2. *If $d > 2$ and Assumption* **A2** *hold, then the system (3.1) has a nonnegative traveling wave solution connecting equilibria $(0,0)$ and $(1,1)$ if and only if $c \ge c_*$. That is, the minimum wave speed is $c_* = 2\sqrt{1 - a_1}$.*

Remark. Note that, if

$$\eta_1(\theta) = \eta_2(\theta) = 0, \quad \theta \in [-\sigma, 0), \qquad \eta_1(0) = \eta_2(0) = 1,$$

then (1.1) is reduced to a non-delay system. In this case, we have $K_1 = K_2 = 1$. Moreover, $d > 2$ implies that $\dfrac{d-2}{d-1} < 1$. Hence Assumption **A2** is reduced to

[**A2**'] $\quad \dfrac{r - (d-2)(1 - a_1)}{ra_2} \ge \max\left\{a_1, \dfrac{d-1}{2(d-2)}e^{-2/(d-2)}\right\}.$

We have

$$
\begin{aligned}
(d-2)^2 e^{2/(d-2)} &\ge (d-2)^2\Big[1 + \frac{2}{d-2} + \frac{2}{(d-2)^2}\Big] \\
&= (d-2)^2 + 2(d-2) + 2 \\
&= d^2 - 2d + 2 \\
&> (d-1)^2.
\end{aligned} \tag{3.37}
$$

From (3.37) it follows that

$$\frac{d-1}{2(d-2)}e^{-2/(d-2)} < \frac{d-2}{2(d-1)}. \tag{3.38}$$

(3.38) implies that Condition [**A2**'] is an improvement of Condition[**C4**]

$$\frac{r - (d-2)(1 - a_1)}{ra_2} \ge \max\left\{a_1, \frac{d-2}{2(d-1)}\right\}$$

given in [3].

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