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RESONANCES OF THE SD OSCILLATOR DUE TO THE DISCONTINUOUS PHASE

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Abstract Resonance phenomena of a harmonically excited system with multiple potential well play an important role in nonlinear dynamics research. In this paper, we investigate the resonant behaviours of a discontinuous dynamical system with double well potential derived from the SD oscillator to gain better understanding of the transition of resonance mechanism. Firstly, the time dependent Hamiltonian is obtained for a Duffing type discontinuous system modelling snap-through buckling. This system comprises two subsystems connected at $x = 0$, for which the system is discontinuous. We construct a series of generating functions and canonical transformations to obtain the canonical form of the system to investigate the complex resonant behaviours of the system. Furthermore, we introduce a composed winding number to explore complex resonant phenomena. The formulation for resonant phenomena given in this paper generalizes the formulation of $n\omega_0 = m\omega$ used in the regular perturbation theory, where n and m are relative prime integers, ω_0 and ω are the natural frequency and external frequencies respectively. Understanding the resonant behaviour of the SD oscillator at the discontinuous phase enables us to further reveal the vibrational energy transfer mechanism between smooth and discontinuous nonlinear dynamical systems.

Keywords SD oscillator, resonance, generating function, canonical transformation, stochastic web, discontinuous Hamiltonian

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1. Introduction

Much attention has been paid to the resonance phenomena of harmonically excited forcing system with multiple well dynamics, [13, 9, 10], which plays an important role in the nonlinear dynamics research, e.g. [1, 5, 11]. The formulation of resonances by regular perturbation theory, Hamiltonians and KAM theory [16, 25, 21], is inconvenient to study the resonant phenomena exhibited in a harmonically excited forcing system with multiple well potentials, as it is impossible to find a pair

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of co-prime numbers m and n which satisfies the condition $m\omega_0 = n\omega$, where ω_0 and ω are the natural frequency and the external excitation frequency [17, 18], respectively.

This paper aims to study the resonant phenomena by presenting a discontinuous time dependent Hamiltonian derived from the discontinuous phase of SD oscillator [2, 3, 4] in the presence of a novel discontinuity [8, 7, 15] of Duffing type with snap-through buckling [14, 19]. This system comprises two subsystems connected at $x = 0$, where each of them exhibits regular resonance, while the overall system generates complex resonances due to its discontinuous nature. Even the formulation presented in this paper is proposed for the particular discontinuous system, it is valid for both smooth and discontinuous system with multiple potential well, such as Duffing oscillator with a double winged homoclinic orbit.

The paper is organized as follows. In Section 2, generalized generating functions and the canonical transformations are constructed to derive the canonical form of the time dependent discontinuous Hamiltonian system. In Section 3, the generalized winding number is introduced to formulate the resonances. This formulation generalizes the resonant condition described in the regular perturbation theory and reveals the complex discontinuous resonant structure of the stochastic web. Finally in Section 4, we provide summary and discussion.

2. Time dependent Hamiltonian

Consider the dimensionless SD oscillator, introduced and described in [2, 3], as follows.

$$\ddot{x} + x(1 - \frac{1}{\sqrt{x^2 + \alpha^2}}) = 0, \tag{2.1}$$

which exhibits both smooth and discontinuous dynamics depending on the value of parameter α. When $\alpha > 0$, system (2.1) is smooth exhibiting the standard dynamics of snap through and double well [2], while $\alpha = 0$, system (2.1) is discontinuous, which can be written as

$$\ddot{x} + (x - \mathrm{sign}(x)) = 0. \tag{2.2}$$

Such a formulation leads to a novel example of non-standard dynamics of discontinuous snap through buckling and a double well potential [2, 3].

If the system is excited by an external force of frequency ω and amplitude f_0, system (2.2) becomes

$$x'' + (x - \mathrm{sign}(x)) = f_0 \cos \omega\tau, \tag{2.3}$$

for which the time-dependent Hamiltonian can be obtained by letting $y = x'$,

$$H_\tau(x, y, \tau) = \frac{1}{2}y^2 + \frac{1}{2}x^2 - |x| - f_0 x \cos \omega\tau. \tag{2.4}$$

The first order term $|x|$ can be incorporated into the above Hamiltonian by shifting the origin from $(0, 0)$ to the equilibria $(\pm 1, 0)$, by means of the time dependent canonical transformation, [16, 19]. New variables $(x, y, \tau) \to (\overline{p}, \overline{q}, \tau)$ are defined as $\overline{p} = x - \mathrm{sign}(x)$ and $\overline{q} = y$, and the canonical generating function is constructed as

$$F_2(\bar{p}, \bar{q}, t) = y\bar{p} = y(x - \mathrm{sign}(x)). \tag{2.5}$$

This transformation maps two half-planes Σ_1 and Σ_2 in xy plane onto two half-planes $\bar{\Sigma}_1$ and $\bar{\Sigma}_2$ in $\bar{q}\bar{p}$ plane, as shown in Figure 1, and defined below

$$\begin{cases} \Sigma_1 = \{(x,y)|x>0\} \bigcup \{(x,y)|x=0, y>0\}, \\ \Sigma_2 = \{(x,y)|x<0\} \bigcup \{(x,y)|x=0, y<0\}, \\ \bar{\Sigma}_1 = \{(\bar{p},\bar{p})|\bar{p}>-1\} \bigcup \{(\bar{p},\bar{q})|\bar{p}=-1, \bar{q}>0\}, \\ \bar{\Sigma}_2 = \{(\bar{p},\bar{q})|\bar{p}<1\ \} \bigcup \{(\bar{p},\bar{q})|\bar{p}=1, \bar{q}<0\ \}. \end{cases} \tag{2.6}$$

As can be seen from the Figure 1 that the overlap for $\bar{p}\bar{q}$ plane in the region $\bar{p} \in (-1,1)$ has to be taken into account for appropriate reconstruction of the trajectory.

Hamiltonian (2.4) can be transformed into the following form in term of the new variables $\bar{p}, \bar{q}$.

$$H_\tau(\bar{p},\bar{q},\tau) = \begin{cases} \frac{1}{2}\bar{p}^2 + \frac{1}{2}\bar{q}^2 - (\bar{p}+1) f_0 \cos\omega\tau, \ (\bar{p},\bar{q}) \in \bar{\sum}_1, \\ \frac{1}{2}\bar{p}^2 + \frac{1}{2}\bar{q}^2 - (\bar{p}-1) f_0 \cos\omega\tau, \ (\bar{p},\bar{q}) \in \bar{\sum}_2. \end{cases} \tag{2.7}$$

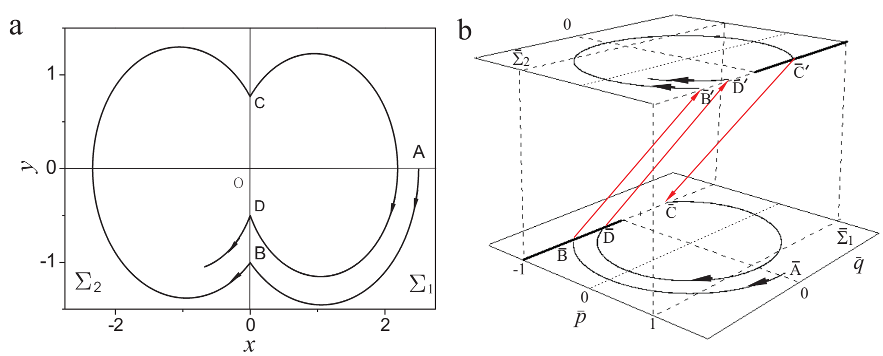

Figure 1. (Colour online) a. Trajectory in xy plane, b. trajectory and the transient at $\bar{p} = \pm 1$: the solid curve marks the part of the trajectory in $\bar{\Sigma}_1$ and $\bar{\Sigma}_2$. The trajectory starts from $\bar{A} \in \bar{\Sigma}_1$ and the transients or the jumps at $\bar{B} \to \bar{B}'$; $\bar{C}' \to \bar{C}$ and $\bar{D} \to \bar{D}'$ such that the arc $\widehat{\bar{A}\bar{B}} \in \bar{\Sigma}_1$, $\widehat{\bar{B}'\bar{C}'} \in \bar{\Sigma}_2$ and $\widehat{\bar{C}\bar{D}} \in \bar{\Sigma}_1$.

We can treat time τ and $E = -\bar{H}_\tau$ as additional co-ordinates in an extended phase space, by introducing an auxiliary parameter ξ playing the role of time [16, 19].

The new Hamiltonian of the forced system with the canonical generating function $F_2 = \bar{p}\bar{q} + E\tau$ in the extended phase space is

$$\bar{H}(\bar{p},\bar{q},E,\tau) = \begin{cases} \frac{1}{2}\bar{p}^2 + \frac{1}{2}\bar{q}^2 + E - (\bar{p}+1) f_0 \cos\omega\tau, \ (\bar{p},\bar{q}) \in \bar{\sum}_1, \\ \frac{1}{2}\bar{p}^2 + \frac{1}{2}\bar{q}^2 + E - (\bar{p}-1) f_0 \cos\omega\tau, \ (\bar{p},\bar{q}) \in \bar{\sum}_2. \end{cases} \tag{2.8}$$

Each of the branches of Eq.(2.8) is similar to an autonomous two degrees-of-freedom and it is non-integrable due to a lack of the integral factor. The first two terms of the extended Hamiltonian represent a simple harmonic oscillator, whose corresponding action and angle variables (J, θ) are introduced and defined as $\bar{p} = \sqrt{2J}\sin\theta, \bar{q} = \sqrt{2J}\cos\theta$, see [16] for details. This maps $\bar{\Sigma}_1 \to \Xi_1$ and $\bar{\Sigma}_2 \to \Xi_2$, which leads to

$$H_J(J, E, \theta, \tau) = \begin{cases} J + E - (\sqrt{2J}\sin\theta + 1) f_0 \cos\omega\tau, \ (J, \theta) \in \Xi_1, \\ J + E - (\sqrt{2J}\sin\theta - 1) f_0 \cos\omega\tau, \ (J, \theta) \in \Xi_2. \end{cases} \tag{2.9}$$

To obtain the action angle variables, canonical transformation is made via generating function $\bar{F}_2(J_\theta, J_\varphi; \theta, \varphi) = \varphi J_\varphi + \theta J_\theta$, such that $(J, \theta, E, \tau) \to (J_\theta, \theta, J_\varphi, \varphi)$ defined by $J_\theta = J, \theta = \theta, J_\varphi = \frac{E}{\omega}$ and $\varphi = \omega\tau$. The Hamiltonian is then in the form

$$\begin{aligned} \hat{H}_J(J_\theta, J_\varphi; \theta, \varphi) &= \hat{H}_J^0(J_\theta, J_\varphi) + \hat{H}_J(J_\theta, J_\varphi; \theta, \varphi) \\ &= \omega_\theta J_\theta + \omega J_\varphi - \begin{cases} \left(\sqrt{2J_\theta}\sin\theta + 1\right) f_0 \cos\varphi, \ (J_\theta, \theta) \in \Xi_1, \\ \left(\sqrt{2J_\theta}\sin\theta - 1\right) f_0 \cos\varphi, \ (J_\theta, \theta) \in \Xi_2. \end{cases} \end{aligned} \tag{2.10}$$

This extended four dimensional system has only three independent variables. One of them is the angle $\varphi = (\omega\tau) \bmod (2\pi)$. The Poincaré section for $\varphi = 0$ is equivalent to $T = 2\pi/\omega$.

Two frequencies for the two angle function (θ, φ) can be introduced to characterise the extended tori for the unperturbed part of the Hamiltonian and written as

$$\omega_\theta = \frac{\partial \hat{H}_J^0}{\partial J_\theta} = \omega_0 = 1, \ \ \omega_\varphi = \frac{\partial \hat{H}_J^0}{\partial J_\varphi} = \omega, \tag{2.11}$$

where ω_φ and ω_θ are the forcing and natural frequencies, respectively. The dynamic behaviour of the forced system depends on the ratio of these two frequencies.

If there are co-prime integers m and n, such that the frequencies are commensurate, satisfying $m\omega_\varphi - n\omega_\theta = 0$. The ratio $\alpha = \omega_\theta/\omega_\varphi = m/n$ is defined as the winding number, or an $m:n$ primary resonance occurs [19, 23].

An $m:n$ resonance is the closed orbit on the torus (φ, θ), m turns in the short way and n turns in the long way. On the Poincaré section (or stroboscopic map, $t = t_0 + k\frac{2\pi}{\omega}$), this resonance is observed as a series of fixed points. The stable series of fixed points are surrounded by the corresponding closed orbits in the neighborhood of the fixed points. These closed orbits make the island chains representing the quasi-periodic motions. The unstable series of fixed points are always associated with chaotic orbit connecting the corresponding islands. In addition to the resonances, the system also exhibits generic KAM curves densely covering the closed curve in a long period of time. There exists a special chaotic orbit filling in the finite region between the separatrices of the lower order of resonances, forming the stochastic web [16, 12].

3. Generalized winding number and resonances

Even the resonant and the winding number have been defined in regular perturbation theory, one cannot find the relative prime integers m and n such that $m\omega_\varphi - n\omega_\theta = 0$ for each resonance in system (2.10). As shown in Figure 2, the resonances cannot be described as $m\omega_\varphi - n\omega_\theta = 0$ for any co-prime integers m and

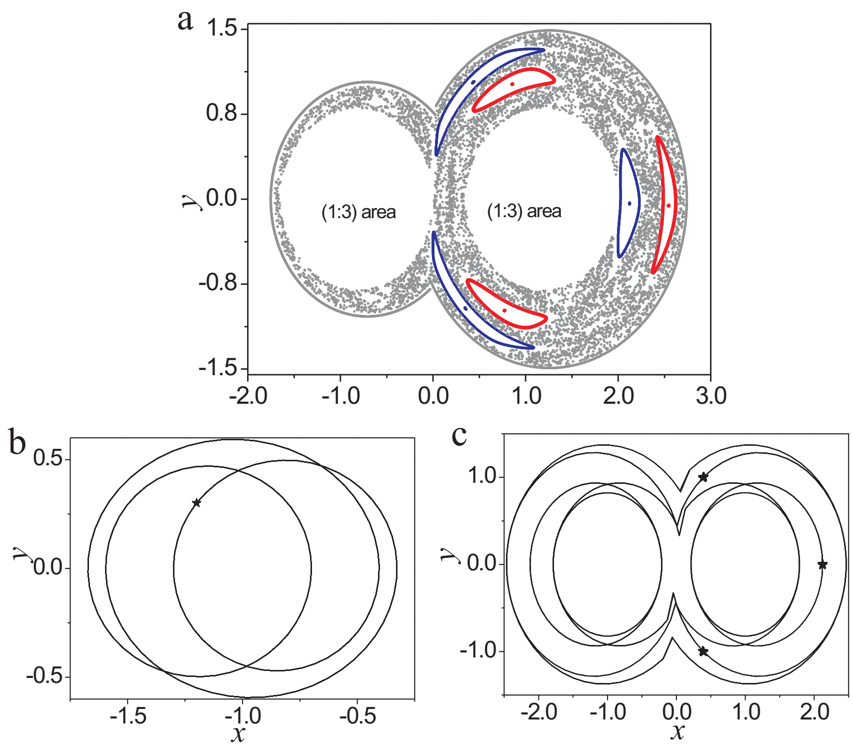

Figure 2. (Colours online) Stochastic web and trajectories for $f_0 = 0.2, \omega = \frac{1}{3}$: a. Stochastic web showing a pair of harmonic motion areas of $(1:3)$ for each of the single system, and the pair of composite resonant solution with the winding number $(3:7)$ marked with blue and red, respectively, b. resonant trajectories for one of the $(1:3)$ and c. the $(3:7)$ respectively.

n for $\omega_\theta = 1$ and $\omega_\varphi = \frac{1}{3}$. To understand these resonances, the resonant trajectories in this system can be divided into two types: the first type are the trajectories which belong to either of the two half planes, while the other type are located in the opposite half plane, respectively.

For the perturbation $\hat{H}_J(J_\theta, J_\varphi; \theta, \varphi) \neq 0$ in Hamiltonian (2.10), the periodic trajectory can be divided into pieces by the pair of successive transient points. Each of these pieces is composed of two half circles located in their half planes. Suppose that there are $2k$ transient points and i circles and j twists for angle variables ω and θ in each piece of the trajectory. We say that (i, j) is the piece winding number for the corresponding piece of the trajectory. The composed winding number for the combined resonant orbit can be formulated in the following way.

Assuming that $(m : n)$ is the co-prime integers satisfying $m\omega_\varphi - n\omega_\theta = 0$, we split both m and n into $m_1, m_2, \cdots, m_k$ and $n_1, n_2, \cdots, n_l$. All the possible piece winding numbers are $(m_{k_i} : n_{l_j})$, $m_{k_i} = 0, 1, \cdots, m$ and $n_{l_j} = 0, 1, \cdots, n$. The

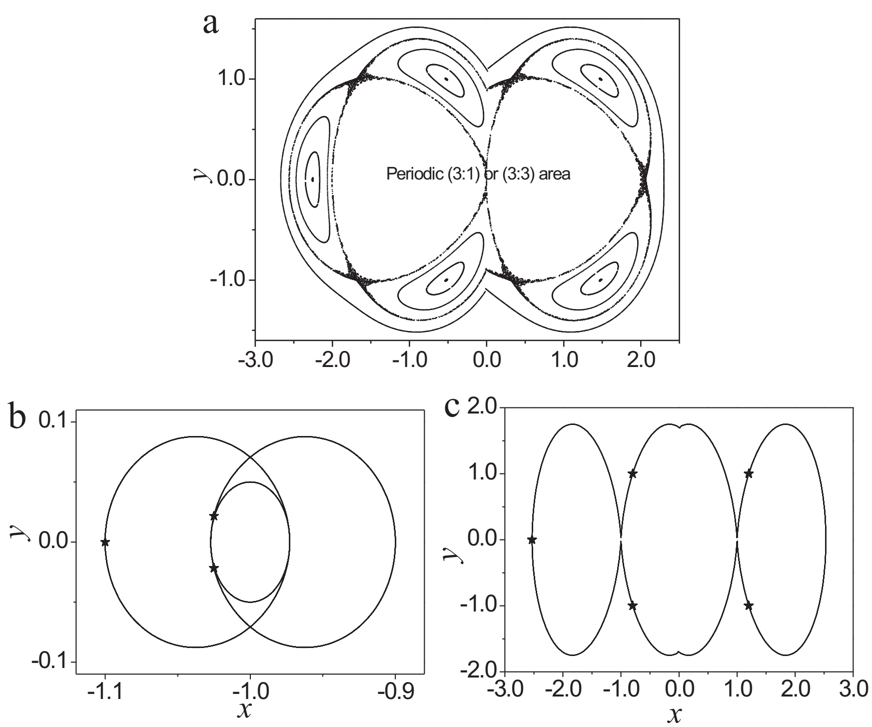

Figure 3. a. Resonant layer separatrix for $f_0 = 0.4, \omega = 3$ showing the pair of harmonic motion areas for both (3 : 1) and (3 : 3) resonance and the resonant solution with winding number (5 : 1), b. one of the trajectory of winding number (3 : 3) and c. the resonant trajectories of winding number (5 : 1). respectively.

composed resonant co-relative number, or the winding number, can be defined as:

$$(M : N) = \sum_{i=1}^{k} (m_{k_i} : n_{l_i}) = \left(\sum_{i=1}^{k} m_{k_i} : \sum_{i=1}^{k} n_{l_i} \right) \tag{3.1}$$

where $(m_{k_i} : n_{l_i})$ are the segment resonant number of the piece of the trajectory.

Figure 2a. for parameters taken as $f_0 = 0.2, \omega_\varphi = \frac{1}{3}$ shows the stochastic or resonant web [6, 12] with a chaotic orbit, marked grey, surrounding a pair of harmonic motion areas of (1 : 3) for each of the subsystem, and the pair of composite resonant solutions of (3 : 7), the corresponding islands for the quasi-periodic solutions are marked in blue and red respectively. Figure 2b. displays one of the (1 : 3) and Figure 2c is one of the (3 : 7) resonant trajectories.

Figure 3a. shows a resonant separatrix for $f_0 = 0.2, \omega_\varphi = 3$ for the harmonic motion areas for both (3 : 1) and (3 : 3) and the resonant solution of winding number (5 : 1). Figure 3b. shows one of the (3 : 3) harmonic solutions, and Figure 3c. depicts the trajectory of resonance of winding number (5 : 1).

Figure 4 plots the structures of the resonance webs or a chaos sea [2, 6, 12] with a chaotic trajectory, plotted in grey, entering different resonant layers of the

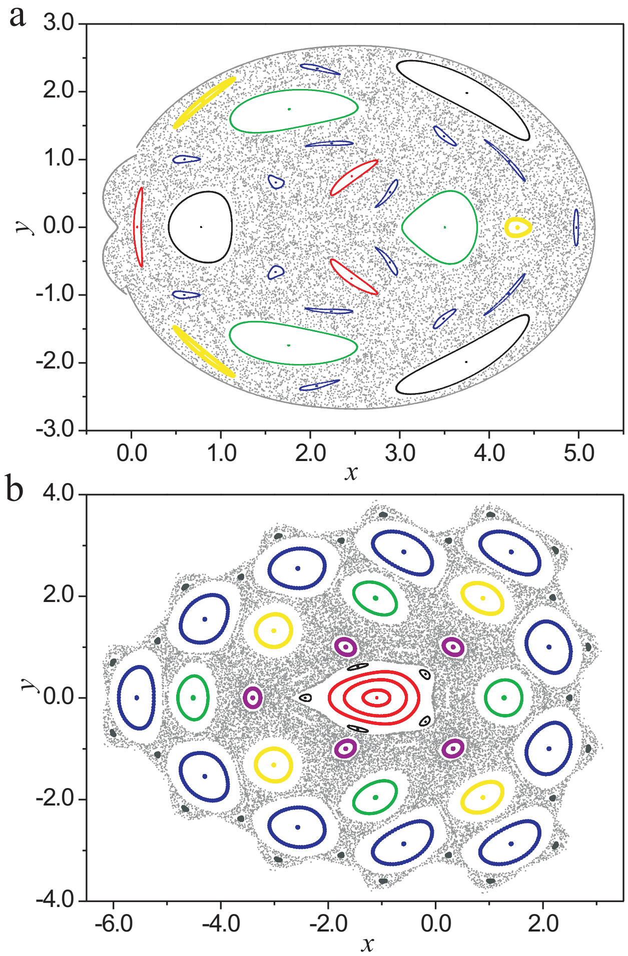

Figure 4. (Colours online) Chaotic seas comprising a chaotic trajectory, marked in grey, and the island chains: a. chaos sea with two pairs of prime resonances island chains of winding number $(3:8)$ for parameter $f_0 = 1.2, \omega_\varphi = \frac{1}{3}$, marked in green and black, red and yellow, respectively and the islands chain of the second resonance of $(15:39)$ plotted in blue; b. chaos sea for $f_0 = 10, \omega_\varphi = 3$ with a prime resonance of winding number $(1:1)$, marked in red, a $(5:5)$ resonant island chain, marked in purple, a pair of $(4:4)$ resonant island chains, marked in yellow and green respectively, the chain of $(11:11)$, marked in blue, and the second resonant island chains of $(5:5)$ near the $(1:1)$ plotted in black and the outer one of $(17:25)$, plotted in dark grey.

corresponding resonant islands. Figure 4a computed for $f_0 = 1.2, \omega_\varphi = \frac{1}{3}$ presents two pairs of prime resonances with winding number $(3:8)$, the corresponding islands are marked in red, yellow, green and black, respectively, and the islands of the second resonance of $(15:39)$ are marked in blue. Figure 4b for $f_0 = 10, \omega_\varphi = 3$ demonstrates a prime resonance of winding number $(1:1)$, the corresponding resonant islands is marked in red, the (5 : 5) resonant islands, marked in purple, a pair of (4 : 4) resonant islands, marked as yellow and green respectively. The resonance of (11 : 11) and the corresponding islands are blue. In addition to the prime resonance, the second resonances are also formulated with the composed winding number, which is not multiple of the prime resonance. The second resonant islands of (5 : 5) near the (1 : 1) resonance are coloured in black, and the outer second resonance of (17 : 25) are plotted in dark grey.

4. Summary and discussion

We have presented a novel discontinuous system derived from the SD oscillator at the discontinuous phase, for which the resonant behaviour has been investigated by introducing the generalised winding number and the generalised canonical generating functions. The time dependent Hamiltonian for the discontinuous system with snap-through bucking has been derived, which enables us to understand such a resonant motion by a composed winding number consisting of multiple phases, each of which is confined within one of the separate resonant area (half plane, as described in Figure 1), satisfying the regular resonant condition. Although the formulation derived in this paper concerns the particular discontinuous system, it is valid for any nonlinear nonlinear dynamic systems with multiple well potentials. The results presented herein also showed the resonant structure of stochastic webs with a stochastic trajectory entering different resonant layers of resonant island chains. These complex resonant behaviours observed may inspire further investigations into resonant synchronization [22, 20] and on mechanisms of vibrational power flow transfer [24] of nonlinear smooth and discontinuous dynamical systems.

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