Journal of Applied Analysis and Computation
Volume 1, Number 2, May 2011

Website:http://jaac-online.com/
pp. 173–182

ON THE NUMBER OF N–DIMENSIONAL INVARIANT SPHERES IN POLYNOMIAL VECTOR FIELDS OF $\mathbb{C}^{N+1}$

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Abstract We study the polynomial vector fields $\mathcal{X} = \sum_{i=1}^{n+1} P_i(x_1, \ldots, x_{n+1}) \frac{\partial}{\partial x_i}$ in $\mathbb{C}^{n+1}$ with $n \geq 1$. Let m_i be the degree of the polynomial P_i. We call $(m_1, \ldots, m_{n+1})$ the degree of $\mathcal{X}$. For these polynomial vector fields $\mathcal{X}$ and in function of their degree we provide upper bounds, first for the maximal number of invariant n–dimensional spheres, and second for the maximal number of n–dimensional concentric invariant spheres.

Keywords polynomial vector fields, invariant spheres, invariant circles, extactic algebraic hypersurface.

MSC(2000) 58F14, 58F22, 34C05.

1. Introduction and statement of the main results

Let $\mathcal{X}$ be the *polynomial vector field* in $\mathbb{C}^{n+1}$ defined by

$$\mathcal{X} = \sum_{i=1}^{n+1} P_i(x_1, \ldots, x_{n+1}) \frac{\partial}{\partial x_i},$$

where every P_i is a polynomial of degree m_i in the variables $x_1, \ldots, x_{n+1}$ with coefficients in $\mathbb{C}$. We say that $\mathbf{m} = (m_1, \ldots, m_{n+1})$ is the *degree* of the polynomial field, we assume without loss of generality that $m_1 \geq \cdots \geq m_{n+1}$. We recall that the *polynomial differential system* in $\mathbb{C}^{n+1}$ of degree $\mathbf{m}$ associated with the vector field $\mathcal{X}$ is written as

$$\frac{dx_i}{dt} = P_i(x_1, \ldots, x_{n+1}), \qquad i = 1, \ldots, n+1.$$

By the Darboux theory of integrability we know that the existence of a sufficiently large number of invariant algebraic hypersurfaces guarantees the existence of a first integral for the polynomial vector field $\mathcal{X}$ which can be calculated explicitly, see for instance [4, 6]. As usual $\mathbb{C}[x_1, \ldots, x_{n+1}]$ denotes the ring of all polynomials in the variables $x_1, \ldots, x_{n+1}$ and coefficients in $\mathbb{C}$. We recall that an *invariant algebraic*

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[*]The second author is supported by the grants MCYT/FEDER MTM 2008–03437, Generalitat de Catalunya 2009SGR410, and is partially supported by ICREA Academia.

hypersurface of $\mathcal{X}$ is a hypersurface $f(x_1, \ldots, x_{n+1}) = 0$ with $f \in \mathbb{C}[x_1, \ldots, x_{n+1}]$ such that there exists a polynomial $K \in \mathbb{C}[x_1, \ldots, x_{n+1}]$ satisfying $\mathcal{X} f = \nabla f \cdot \mathcal{X} = Kf$. This polynomial K is called the *cofactor* of the invariant algebraic hypersurface $f = 0$, and ∇f denotes the gradient of the function f. Note that if the vector field $\mathcal{X}$ has degree $\mathbf{m}$, then any cofactor has at most degree $m_1 - 1$. If the degree of f is 1 then the hypersurface $f = 0$ is an *invariant hyperplane*. Here we study the algebraic hypersurfaces $f = (x_1 - a_1)^2 + \ldots + (x_{n+1} - a_{n+1})^2 - r^2 = 0$ that are *invariant n–dimensional spheres.*

Now we introduce one of the best tools in order to look for invariant algebraic hypersurfaces. Let W be a vector subspace of the space of polynomials $\mathbb{C}[x_1, \ldots, x_{n+1}]$ generated by the polynomials $v_1, \ldots, v_l$. The *extactic algebraic hypersurface of $\mathcal{X}$ associated with W* is

$$\varepsilon_W(\mathcal{X}) = \det \begin{pmatrix} v_1 & v_2 & \cdots & v_l \\ \mathcal{X}(v_1) & \mathcal{X}(v_2) & \cdots & \mathcal{X}(v_l) \\ \vdots & \vdots & \cdots & \vdots \\ \mathcal{X}^{l-1}(v_1) & \mathcal{X}^{l-1}(v_2) & \cdots & \mathcal{X}^{l-1}(v_l) \end{pmatrix} = 0, \tag{1.1}$$

where $\{v_1, \ldots, v_l\}$ is a basis of W, $l = \dim(W)$ is the dimension of W and $\mathcal{X}^j(v_i) = \mathcal{X}^{j-1}(\mathcal{X}(v_i))$. From the properties of the determinants and of the derivation we know that the definition of extactic algebraic hypersurface is independent of the chosen basis of W.

The notion of extactic algebraic hypersurface $\varepsilon_W(\mathcal{X})$ is important here for two reasons. First it allows to detect when an algebraic hypersurface $f = 0$ with $f \in W$ is invariant by the polynomial vector field $\mathcal{X}$, see the next proposition proved in [3] for polynomial vector fields in $\mathbb{C}^2$; the extension to $\mathbb{C}^{n+1}$ is easy and it is presented here. Second $\varepsilon_W(\mathcal{X})$ is also important because it allows to define and compute easily the multiplicity of an invariant algebraic hypersurface.

Proposition 1.1. Let $\mathcal{X}$ be a polynomial vector field in $\mathbb{C}^{n+1}$ and let W be a finitely generated $\mathbb{C}$–vector subspace of $\mathbb{C}[x_1, \ldots, x_{n+1}]$ with $\dim(W) > 1$. Then every algebraic invariant hypersurface $f = 0$ for the vector field $\mathcal{X}$, with $f \in W$, is a factor of the polynomial $\varepsilon_W(\mathcal{X})$.

The number of invariant straight lines for polynomial vector fields in $\mathbb{C}^2$ has been studied by several authors, see [1]. We know that for polynomial vector fields of degree $(2, 2)$ the maximal number of invariant straight lines is 5. Xiang Zhang [8] and Sokulski [7] proved that the maximal number of real invariant straight lines for polynomial vector fields in $\mathbb{R}^2$ of degrees (3,3) and (4,4) are 8 and 9, respectively. Later on Llibre and Medrado in [5] generalized these results to invariant hyperplanes of polynomial vector fields in $\mathbb{C}^{n+1}$.

We want to know the maximum number of n-dimensional invariant spheres that a polynomial vector field in $\mathbb{C}^{n+1}$ can have. We begin this study by analyzing first the case $\mathbb{C}^2$. That is, given a polynomial vector field $\mathcal{X} = P\dfrac{\partial}{\partial x} + Q\dfrac{\partial}{\partial y}$ with P and Q polynomials in the variables x and y of degree $\mathbf{m} = (m_1, m_2)$, $m_1 \geq m_2$ and $f = f(x, y) = (x - a)^2 + (y - b)^2 - r^2 = 0$ a circle with center (a, b) and radius $r > 0$, we find an upper bound for the maximum number of invariant circles $f = 0$ of the polynomial vector field $\mathcal{X}$, and we determine if this bound is reached or not in the class of all polynomial vector fields with degree (m_1, m_2).

The *multiplicity of an invariant circle* $f = (x-a)^2 + (y-b)^2 - r^2 = 0$ is the greatest positive integer k such that f^k divides the polynomial $\varepsilon_W(\mathcal{X})$ with W generated by $1, x^2, x, y^2, y$. When we study the maximum number of concentric invariant circles of a polynomial vector field $\mathcal{X}$, doing a translation of the center, we can assume without loss of generality, that the center is at the origin. Then the *multiplicity of an invariant circle centered at the origin* $f = x^2 + y^2 - r^2 = 0$ is the greatest positive integer k such that f^k divides the polynomial $\varepsilon_W(\mathcal{X})$ with W generated by $1, x^2, y^2$.

Below we present the main results of this article in $\mathbb{C}^2$ and in the next section in $\mathbb{C}^{n+1}$. In both cases we provide upper bounds for the number of invariant n–dimensional spheres concentric or not that a polynomial vector field in $\mathbb{C}^2$ or $\mathbb{C}^{n+1}$ can possess. Moreover we show that these bounds are in general not reached. We also determine the exact upper bound for the number of invariant circles concentric or not that polynomial vector fields of degree $(2,2)$ in $\mathbb{C}^2$ can have.

Theorem 1.1. *Assume that a polynomial vector field $\mathcal{X}$ in $\mathbb{C}^2$ of degree $\mathbf{m} = (m_1, m_2)$ with $m_1 \geq m_2$ has a finite number of invariant circles. Then the following statements hold.*

(a) The number of invariant circles of $\mathcal{X}$ taking into account its multiplicity is at most $4m_1 + m_2 - 2$.

(b) The number of concentric invariant circles of $\mathcal{X}$ taking into account its multiplicity is at most $m_1 + [(m_2+1)/2]$. Here $[x]$ denotes the integer part of $x \in \mathbb{R}$.

Corollary 1.1. Assume that a polynomial vector field $\mathcal{X}$ in $\mathbb{C}^2$ of degree $\mathbf{m} = (m, m)$ has a finite number of invariant circles. Then the following statements holds.

(a) The number of invariant circles of $\mathcal{X}$ taking into account its multiplicity is at most $5m - 2$.

(b) The number of concentric invariant circles of $\mathcal{X}$ taking into account its multiplicity is at most $[(3m+1)/2]$.

Proposition 1.2. The bounds given by Theorem 1.1 are not reached.

Corollary 1.2. Assume that a polynomial vector field $\mathcal{X}$ in $\mathbb{C}^2$ of degree $(2,2)$ has a finite number of invariant circles. Then the following statements holds.

(a) The number of invariant circles of $\mathcal{X}$ taking into account its multiplicity is at most 2.

(b) The number of concentric invariant circles of $\mathcal{X}$ taking into account its multiplicity is at most 1.

Now we generalize the previous results to invariant n–dimensional spheres of polynomial vector fields in $\mathbb{C}^{n+1}$.

The *multiplicity of an invariant n–dimensional sphere*

$$f = (x_1 - a_1)^2 + \ldots + (x_{n+1} - a_{n+1})^2 - r^2 = 0$$

is the greatest positive integer k such that f^k divides the polynomial $\varepsilon_W(\mathcal{X})$ with W generated by $\{1, x_1^2, x_1, \ldots, x_{n+1}^2, x_{n+1}\}$.

The *multiplicity of an invariant n–dimensional sphere with center at the origin*

$$f = x_1^2 + \ldots + x_{n+1}^2 - r^2 = 0$$

is the greatest positive integer k such that f^k divides the polynomial $\varepsilon_W(\mathcal{X})$ with W generated by $\{1, x_1^2, \ldots, x_{n+1}^2\}$.

Theorem 1.2. *Assume that a polynomial vector field $\mathcal{X}$ in $\mathbb{C}^{n+1}$ with $n \geq 1$ of degree $\mathbf{m} = (m_1, \ldots, m_{n+1})$ with $m_1 \geq \cdots \geq m_{n+1}$ has finitely many invariant n–dimensional spheres. Then the following statements hold.*

(a) The number of invariant n–dimensional spheres of $\mathcal{X}$ taking into account its multiplicity is at most

$$\left[\frac{1}{2}\left(\left(\sum_{k=1}^{n+1} 2m_k\right) + \binom{2(n+1)}{2}(m_1 - 1) + (n+1)\right)\right].$$

(b) The number of concentric invariant n–dimensional spheres of $\mathcal{X}$ taking into account its multiplicity is at most

$$\left[\frac{1}{2}\left(\left(\sum_{k=1}^{n+1} m_k\right) + \binom{n+1}{2}(m_1 - 1) + (n+1)\right)\right].$$

Corollary 1.3. Assume that a polynomial vector field $\mathcal{X}$ in $\mathbb{C}^{n+1}$ with $n \geq 1$ of degree $\mathbf{m} = (m, \ldots, m)$ has finitely many invariant n–dimensional spheres. Then the following statements hold.

(a) The number of invariant n–dimensional spheres of $\mathcal{X}$ taking into account its multiplicity is at most

$$\left[\frac{1}{2}(n+1)(3m + 2n(m-1))\right].$$

(b) The number of invariant n–dimensional spheres of $\mathcal{X}$ taking into account its multiplicity is at most

$$\left[\frac{n+1}{2}\left(m + \frac{n(m-1)}{2} + 1\right)\right].$$

All the results in $\mathbb{C}^2$ are proved in section 2, and the results in $\mathbb{C}^{n+1}$ are shown in section 3.

2. Invariants circles

First we shall show that any invariant algebraic hypersurface is a factor of a convenient extactic algebraic hypersurface.

Proof of Proposition 1.1. Let $f = 0$ be an invariant algebraic hypersurface $\mathcal{X}$ such that $f \in W$. As was observed, the choice of the basis $v_1, \ldots, v_l$ of W plays no role in the definition of extactic algebraic hypersurface, therefore we can take $v_1 = f$ in (1.1).

Using induction we shall see that $\mathcal{X}^k(f) = K_k f$ being K_k a polynomial for $k = 1, 2, \ldots$ In fact, if $k = 1$ we have $\mathcal{X}(f) = K_1 f$ where K_1 is the cofactor of the invariant algebraic hypersurface $f = 0$. Suppose that $\mathcal{X}^k(f) = K_k f$ being K_k a polynomial, and see the case $k+1$. We have

$$\begin{aligned} \mathcal{X}^{k+1}(f) &= \mathcal{X}(\mathcal{X}^k(f)) = \mathcal{X}(K_k f) = \mathcal{X}(K_k) f + K_k \mathcal{X}(f) \\ &= (\mathcal{X}(K_k) + K_k K_1) f = K_{k+1} f. \end{aligned}$$

Thus f appears in all terms in the first column of the determinant $\varepsilon_W(\mathcal{X})$, and therefore f is a factor of the polynomial $\varepsilon_W(\mathcal{X})$. $\square$

Proof of Theorem 1.1. In order to prove the statement (a) we take the subspace W generated by the polynomials $\{1, x^2, x, y^2, y\}$. Then if $f = 0$ is a circle we have that $f \in W$.

By Proposition 1.1 if $f = 0$ is an invariant circle of $\mathcal{X}$, then f is a factor of the polynomial

$$\varepsilon_W(\mathcal{X}) = \det \begin{pmatrix} 1 & x^2 & x & y^2 & y \\ 0 & \mathcal{X}(x^2) & \mathcal{X}(x) & \mathcal{X}(y^2) & \mathcal{X}(y) \\ 0 & \mathcal{X}^2(x^2) & \mathcal{X}^2(x) & \mathcal{X}^2(y^2) & \mathcal{X}^2(y) \\ 0 & \mathcal{X}^3(x^2) & \mathcal{X}^3(x) & \mathcal{X}^3(y^2) & \mathcal{X}^3(y) \\ 0 & \mathcal{X}^4(x^2) & \mathcal{X}^4(x) & \mathcal{X}^4(y^2) & \mathcal{X}^4(y) \end{pmatrix} = \det(a_{ij}). \qquad (2.1)$$

Note that the degree of the polynomial a_{ij} of the previous matrix verifies that $\deg(a_{ij}) = \deg(a_{i-1\,j}) + m_1 - 1$ if $i = 2, 3, 4, 5$ and $j = 2, 3, 4, 5$. In fact, every time that we apply the polynomial vector field $\mathcal{X}$ to a polynomial, the degree of the resultant polynomial is determined by the first term that multiplies P_1 (polynomial of higher degree) and decreases by 1 for the derivation due to the application of the vector field. For instance, since the degree of $\mathcal{X}(x)$ is m_1 the degree of $\mathcal{X}^2(x)$ is $2m_1 - 1$.

Since $m_1 \geq m_2$ and from the definition of determinant, it follows that the degree of the polynomial $\varepsilon_W(\mathcal{X})$ is

$$(4m_1 - 2) + (3m_1 - 2) + (m_1 + m_2) + m_2 = 8m_1 + 2m_2 - 4.$$

Note that the previous degree corresponds to the degree of the polynomial $\mathcal{X}^4(x^2)$ $\mathcal{X}^3(x)\mathcal{X}^2(y^2)\mathcal{X}(y)$, which corresponds to a permutation of four elements with maximal degree in the expansion of the determinant (2.1).

Since the polynomial $\varepsilon_W(\mathcal{X})$ can have at most its degree divided by 2 factors of the form $f(x, y) = (x-a)^2 + (y-b)^2 - r^2$, by Proposition 1.1 it follows statement (a) of the theorem.

Now consider a set of concentric circles. Doing an appropriate change of coordinates we can consider that the equations of these circles are the form $x^2 + y^2 - r^2 = 0$. Thus all these circles can be written in the form $f = 0$ with $f \in W$, where W is the $\mathbb{C}$-vector subspace of $\mathbb{C}[x, y]$ generated by $1, x^2, y^2$. Therefore by Proposition 1.1 if $f = 0$ is one of these circles, then f is a factor

$$\varepsilon_W(\mathcal{X}) = \det \begin{pmatrix} 1 & x^2 & y^2 \\ 0 & \mathcal{X}(x^2) & \mathcal{X}(y^2) \\ 0 & \mathcal{X}^2(x^2) & \mathcal{X}^2(y^2) \end{pmatrix}.$$

So the degree of the polynomial $\varepsilon_W(\mathcal{X})$ is $2m_1+m_2+1$ and therefore the statement (b) of the theorem holds. □

Proof of Corollary 1.1. The proof is obtained from Theorem 1.1 by a simple calculation. □

Now we show that the bounds given by Theorem 1.1 are not reached. For this we need the following lemma which is proved in [2]. Recall that an algebraic curve $f = f(x, y) = 0$ is non–singular if there are points at which f and its first derivatives f_x and f_y are all zero.

Lemma 2.1. *Assume that a polynomial differential system has a non–singular invariant algebraic curve $f = 0$. If $(f_x, f_y) = 1$, then the polynomial differential system can be written into the form*

$$\dot{x} = Af + Cf_y, \quad \dot{y} = Bf - Cf_x, \tag{2.2}$$

where A, B and C are suitable polynomials.

Proof of Proposition 1.2. We shall prove that the maximum number of invariant circles that a quadratic vector field can have is less than $4m_1 + m_2 - 2$. We also will prove that the maximum number of invariant concentric circles of a quadratic vector field is less than $m_1 + [(m_2 + 1)/2]$. Consequently the two upper bounds of Theorem 1.1 are not reached for quadratic vector fields.

Suppose that a quadratic vector field $\mathcal{X}$ has an invariant circle, doing an affine change of coordinates, if necessary, the center of the circle can be translated to the origin and its radius can be taken equals 1, i.e. it can be written as $x^2+y^2-1 = 0$. Therefore, Lemma 2.1 states that the quadratic differential system associated the field $\mathcal{X}$ can be written in the form (2.2) with $A, B \in \mathbb{R}$ and $C = (ax + by + c)/2$, this is

$$\begin{aligned} \dot{x} &= A(x^2 + y^2 - 1) + y(ax + by + c), \\ \dot{y} &= B(x^2 + y^2 - 1) - x(ax + by + c), \end{aligned} \tag{2.3}$$

which has the circle $x^2 + y^2 - 1 = 0$ invariant.

Without loss of generality we can assume that $B = 0$. Indeed if $B = 0$ we have what we want. Suppose that $B \neq 0$. Then changing the variables (x, y) for the variables

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

with $\cos\theta = A/\sqrt{A^2 + B^2}$ and $\sin\theta = -B/\sqrt{A^2 + B^2}$, the quadratic system in the new variables becomes

$$\dot{X} = \overline{A}(X^2 + Y^2 - 1) + Y(\overline{a}X + \overline{b}Y + \overline{c}), \quad \dot{Y} = -X(\bar{a}X + \bar{b}Y + \bar{c}),$$

where

$$\bar{A} = \sqrt{A^2 + B^2}, \;\; \overline{a} = \frac{aA + bB}{\sqrt{A^2 + B^2}}, \;\; \overline{b} = \frac{Ab - aB}{\sqrt{A^2 + B^2}}, \;\; \overline{c} = c.$$

Note that renaming the coefficients of this quadratic system we obtain the quadratic system (2.3) with $B = 0$. So we can work with the quadratic system

$$\dot{x} = A(x^2 + y^2 - 1) + y(ax + by + c), \quad \dot{y} = -x(ax + by + c).$$

Consider the circle with center (p,q) and radius r, i.e. $f(x,y)=(x-p)^2+(y-q)^2-r^2=0$. If $f=0$ is an invariant circle of $\mathcal{X}$, then it verifies $\mathcal{X}f=Kf$, where $K=K(x,y)=Dx+Ey+F$. Therefore, we obtain

$$\begin{aligned}&[A(x^2+y^2-1)+y(ax+by+c)]2(x-p)-x(ax+by+c)\\&2(y-q)=(Dx+Ey+F)[(x-p)^2+(y-q)^2-r^2].\end{aligned}\tag{2.4}$$

We will analyze the two cases that interest us, solutions where we get concentric invariant circles and, in the general case solutions which give rise to any invariant circle that the quadratic vector field $\mathcal{X}$ can have.

In the case of concentric invariants circles we have $p=q=0$. Then equation (2.4) becomes

$$\begin{aligned}&2Ax^3+2Axy^2-2Ax=\\&Dx^3+Dxy^2-Dr^2x+Ex^2y+Ey^3-Er^2y+Fx^2+Fy^2-Fr^2.\end{aligned}$$

Thus $D=2A$, $2A=Dr^2$, $E=0$ y $F=0$. If $A\neq 0$, $D=Dr^2$ so that $r=1$.

If $A=0$ then also $D=0$, and the resulting system associated with the vector field $\mathcal{X}$ is $\dot{x}=y(c+ax+by)$, $\dot{y}=-x(c+ax+by)$, which has all the circles $f(x,y)=x^2+y^2-r^2=0$ with arbitrary $r>0$ invariant. Therefore this vector field has infinitely many invariant circles, and these fields are not considered in Theorem 1.1.

In short a quadratic vector field can have at most one concentric invariant circle.

Nontrivial real solutions different from the unit circle that we obtain by solving equation (2.4) are

(*i*) $a=0$, $b=-D$, $c=\frac{1}{2}D(r^2-2)$, $A=\frac{D}{2}$, $E=0$, $F=0$, $q=-1$, $p=0$. This is, we obtain the invariant circle $f(x,y)=x^2+(y+1)^2-r^2=0$ for the system

$$\begin{aligned}\dot{x}&=\frac{1}{2}D(x^2+y^2-1)+y\left(-Dy+\frac{1}{2}D(r^2-2)\right),\\\dot{y}&=-x\left(-Dy+\frac{1}{2}D(r^2-2)\right).\end{aligned}\tag{2.5}$$

(*ii*) $a=0$, $b=-D$, $c=\frac{1}{2}D(2-r^2)$, $A=\frac{D}{2}$, $E=0$, $F=0$, $q=1$, $p=0$. This is, we obtain the invariant circle $f(x,y)=x^2+(y-1)^2-r^2=0$ for the system

$$\begin{aligned}\dot{x}&=\frac{1}{2}D(x^2+y^2-1)+y\left(-Dy+\frac{1}{2}D(2-r^2)\right),\\\dot{y}&=-x\left(-Dy+\frac{1}{2}D(2-r^2)\right).\end{aligned}\tag{2.6}$$

(*iii*) $a=0$, $b=-D$, $c=\dfrac{D(1+q^2-r^2)}{2q}$, $A=\dfrac{D}{2}$, $E=0$, $F=0$, $p=0$ y $q\neq 0$. This is, we obtain the invariant circle $f(x,y)=x^2+(y-q)^2-r^2=0$ for the system

$$\begin{aligned}\dot{x}&=\frac{1}{2}D(x^2+y^2-1)+y\left(\frac{D(1+q^2-r^2)}{2q}-Dy\right),\\\dot{y}&=-x\left(-Dy+\frac{D(1+q^2-r^2)}{2q}\right).\end{aligned}\tag{2.7}$$

Thus, systems (2.5), (2.6) and (2.7) possess two invariant circles. Therefore we have proved that the maximum number of invariant circles for a quadratic vector field is two. □

Proof of Corollary 1.2. The proof follows immediately from the proof of Proposition 1.2. □

3. Invariant n–dimensional spheres

In this section we shall prove Theorem 1.2 and Corollary 1.3 which contain our results for polynomial vector fields in $\mathbb{C}^{n+1}$.

Proof of Theorem 1.2. Let W be the vectorial subspace of $\mathbb{C}[x_1,\ldots,x_{n+1}]$ generated by $1, x_1^2, x_1,\ x_2^2, x_2, \ldots, x_{n+1}^2, x_{n+1}$. Then if $f=0$ is a n–dimensional sphere, $f \in W$.

From Proposition 1.1 if $f=0$ is an invariant n–dimensional sphere of $\mathcal{X}$, then f is a factor of the polynomial $\varepsilon_W(\mathcal{X})$ given by

$$\det\begin{pmatrix} 1 & x_1^2 & x_1 & \cdots & x_{n+1}^2 & x_{n+1} \\ 0 & \mathcal{X}(x_1^2) & \mathcal{X}(x_1) & \cdots & \mathcal{X}(x_{n+1}^2) & \mathcal{X}(x_{n+1}) \\ 0 & \mathcal{X}^2(x_1^2) & \mathcal{X}^2(x_1) & \cdots & \mathcal{X}^2(x_{n+1}^2) & \mathcal{X}^2(x_{n+1}) \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & \mathcal{X}^{2(n+1)}(x_1^2) & \mathcal{X}^{2(n+1)}(x_1) & \cdots & \mathcal{X}^{2(n+1)}(x_{n+1}^2) & \mathcal{X}^{2(n+1)}(x_{n+1}) \end{pmatrix}. \tag{3.1}$$

Note that for $k=1,\ldots,n+1$, the degree of the polynomials $\mathcal{X}(x_k)$, $\mathcal{X}^2(x_k)$, $\mathcal{X}^3(x_k)$, …, $\mathcal{X}^{2(n+1)}(x_k)$ are m_k, m_1+m_k-1, $2(m_1-1)+m_k$, …, $(2(n+1)-1)(m_1-1)+m_k$ respectively, and the degree of the polynomials $\mathcal{X}(x_k^2)$, $\mathcal{X}^2(x_k^2)$, $\mathcal{X}^3(x_k^2)$, …, $\mathcal{X}^{2(n+1)}(x_k^2)$ are m_k+1, m_1+m_k, $2(m_1-1)+m_k+1$,…, $(2(n+1)-1)(m_1-1)+m_k+1$. In general we have that $\mathcal{X}^d(x_k)=(d-1)(m_1-1)+m_k$ and $\mathcal{X}^d(x_k^2)=(d-1)(m_1-1)+m_k+1$ for $d=1,\ldots,2(n+1)$.

Since $m_1 \geq \cdots \geq m_{n+1}$ and from the definition of the determinant, it follows that the degree of the polynomial $\varepsilon_W(\mathcal{X})$ is the degree of one of the polynomials of the determinant (3.1) which corresponds to a permutation of $2n+2$ of their elements with maximal degree and is given by the expression

$$\mathcal{X}^{2(n+1)}(x_1^2)\mathcal{X}^{2(n+1)-1}(x_1)\mathcal{X}^{2(n+1)-2}(x_2^2)\mathcal{X}^{2(n+1)-3}(x_2)\cdots \\ \mathcal{X}^4(x_n^2)\mathcal{X}^3(x_n)\mathcal{X}^2(x_{n+1}^2)\mathcal{X}(x_{n+1}).$$

Note that the degree of this polynomial is

$$\begin{aligned} &[(2(n+1)-1)(m_1-1)+m_1+1]+[(2(n+1)-2)(m_1-1)+m_1] \\ &+[(2(n+1)-3)(m_1-1)+m_2+1]+[(2(n+1)-4)(m_1-1)+m_2] \\ &+\cdots \\ &+[3(m_1-1)+m_n+1]+[2(m_1-1)+m_n] \\ &+[(m_1-1)+m_{n+1}+1]+m_{n+1} \\ =\ &\left(\sum_{k=1}^{n+1} 2m_k\right)+\binom{2(n+1)}{2}(m_1-1)+(n+1). \end{aligned}$$

Since the number of factors $(x_1-a_1)^2+(x_2-a_2)^2+\cdots+(x_{n+1}-a_{n+1})^2-r^2=0$ of the polynomial $\varepsilon_W(\mathcal{X})$ is at most its degree divided by 2, by Proposition 1.1 it follows the number of statement (a) of the theorem.

Now we consider a set of concentric invariant n–dimensional spheres. Doing an appropriate change of coordinates we can consider that the equations of these spheres have the form $x_1^2+x_2^2+\cdots+x_{n+1}^2-r^2=0$. Then all these spheres can be written in the form $f=0$ with $f\in W$, where W is the $\mathbb{C}$-vector subspace of $\mathbb{C}[x_1,x_2,\cdots,x_{n+1}]$ generated by $1,x_1^2,x_2^2,\ldots,x_{n+1}^2$. So, by Proposition 1.1 if $f=0$ is one of these spheres, then f is a factor of

$$\varepsilon_W(\mathcal{X})=\det\begin{pmatrix}1 & x_1^2 & x_2^2 & \cdots & x_{n+1}^2\\ 0 & \mathcal{X}(x_1^2) & \mathcal{X}(x_2^2) & \cdots & \mathcal{X}(x_{n+1}^2)\\ 0 & \mathcal{X}^2(x_1^2) & \mathcal{X}^2(x_2^2) & \cdots & \mathcal{X}^2(x_{n+1}^2)\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ 0 & \mathcal{X}^{n+1}(x_1^2) & \mathcal{X}^{n+1}(x_2^2) & \cdots & \mathcal{X}^{n+1}(x_{n+1}^2)\end{pmatrix}. \tag{3.2}$$

Since $m_1\geq\cdots\geq m_{n+1}$ and from the definition of determinant it follow that the degree of the polynomial $\varepsilon_W(\mathcal{X})$ is the degree of one of the polynomials of the determinant (3.2) corresponding to a permutation of $n+1$ of their elements with maximal degree, which is given by the expression

$$\mathcal{X}^{n+1}(x_1^2)\mathcal{X}^{n}(x_2^2)\mathcal{X}^{n-1}(x_3^2)\cdots\mathcal{X}^{2}(x_n^2)\mathcal{X}(x_{n+1}^2).$$

Thus the degree of the polynomial $\varepsilon_W(\mathcal{X})$ is

$$\begin{aligned}&(n(m_1-1)+m_1+1)+((n-1)(m_1-1)+m_2+1)\\ &+\cdots+(m_1+m_n)+(m_{n+1}+1)\\ =\ &\left(\sum_{k=1}^{n+1}m_k\right)+\binom{n+1}{2}(m_1-1)+(n+1),\end{aligned}$$

and therefore the number of concentric invariant spheres $\mathcal{X}$ taking into account their multiplicities is given by the number of statement (b) of theorem. □

Proof of Corollary 1.3. The proof is obtained by a simple calculation using Theorem 1.2. □

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