Journal of Applied Analysis and Computation
Volume 1, Number 2, May 2011

Website:http://jaac-online.com/
pp. 155–171

RANDOM ATTRACTOR OF STOCHASTIC ZAKHAROV LATTICE SYSTEM

Yu Bai[a] and Shengfan Zhou[a,†]

Abstract In this paper, we consider a lattice system of stochastic Zakharov equation with white noise. We first show that the solutions of the system determine a continuous random dynamical system with random absorbing set. And then we prove the random asymptotic compactness on the random absorbing set. Finally, we obtain the existence of a random attractor for the system.

Keywords Stochastic Zakharov lattice system, continuous random dynamical system, random attractor.

MSC(2000) 35D, 35C.

1. Introduction

Stochastic lattice differential equations arise rapidly in a wide variety of applications where the spatial structure has a discrete character, and uncertainties or random influences are taken into account [1, 2, 3, 4, 5, 8, 10]. The global random attractor was first studied by Ruelle [7] to capture the essential dynamics with possibly extremely wide fluctuations. Later many researchers developed some general theories of random attractors.

Let n be a positive integer, and write L^2, l^2 as

$$L^2 = \{b = (b_j)_{j\in\mathbb{Z}^n} : j = (j_1, j_2, \ldots, j_n) \in \mathbb{Z}^n, b_j \in \mathbb{C}, \sum_{j\in\mathbb{Z}^n} |b_j|^2 < \infty\},$$

$$l^2 = \{b = (b_j)_{j\in\mathbb{Z}^n} : j = (j_1, j_2, \ldots, j_n) \in \mathbb{Z}^n, b_j \in \mathbb{R}, \sum_{j\in\mathbb{Z}^n} b_j^2 < \infty\}.$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with

$$\{\Omega = \omega \in C\left(\mathbb{R}, l^2 \times \mathbb{R}\right) : \omega\left(0\right) = 0\}.$$

$\mathcal{F}$ is the Borel $\sigma-$ algebra generated by the compact-open topology of Ω, $\mathbb{P}$ is the Wiener measure on $(\Omega, \mathcal{F})$. Define $(\theta_t)_{t\in\mathbb{R}}$ on Ω by

$$\theta_t\omega\left(\cdot\right) = \omega\left(\cdot + t\right) - \omega\left(t\right), t \in \mathbb{R}.$$

[†]the corresponding author. Email addresses: baiyu198686@163.com (Y.Bai), zhoushengfan@yahoo.com(S.Zhou)
[a]Department of Mathematics, Shanghai Normal University, Shanghai, 200234 China
[*]The authors were supported by National Natural Science Foundation of China (11071165), the National Ministry of Education of China (200802700002), and the Foundation of Shanghai Talented Persons (No. 2008049).

Then $\left(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}}\right)$ is an ergodic metric dynamical system (see [1]).

Consider the following stochastic Zakharov equations on infinite lattices:

$$\begin{cases} \frac{1}{\lambda} d\dot{x}_j + \alpha dx_j + (A(x+|z|^2))_j dt + \beta x_j dt = g_j dt + a_j dw_j^1, \\ i dz_j - ((Az)_j + x_j z_j - i\gamma z_j) dt = h_j dt + z_j \circ dw^2, \end{cases} \tag{1.1}$$

where $j \in \mathbb{Z}^n$, $z = (z_j)_{j\in\mathbb{Z}^n} \in L^2$, $x = (x_j)_{j\in\mathbb{Z}^n} \in l^2$, $a = (a_j)_{j\in\mathbb{Z}^n} \in l^2$, i is imaginary unit, $\alpha, \beta, \gamma, \lambda$ are positive constants, g_j and h_j are given , $(w_j^1, w^2)_{j\in\mathbb{Z}^n}$ are independent two-side Wiener process on probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\circ$ means the sense of Stratonovich in the random term. Define the coupled linear operator A in the following way: $A = \sum\limits_{m=1}^{n} A_m$, $A_m = B_m B'_m = B'_m B_m$, where B_m, B'_m and A_m are all linear operators acting on l^2 or L^2, $m = 1, 2, \ldots, n$. defined by: for any $b = (b_j)_{j\in\mathbb{Z}^n} \in l^2$ or L^2,

$$\begin{aligned} (A_m b)_j &= 2b_{(j_1,j_2,\ldots,j_m,\ldots,j_n)} - b_{(j_1,j_2,\ldots,j_{m+1},\ldots,j_n)} - b_{(j_1,j_2,\ldots,j_{m-1},\ldots,j_n)}, \\ (B_m b)_j &= b_{(j_1,j_2,\ldots,j_{m+1},\ldots,j_n)} - b_{(j_1,j_2,\ldots,j_m,\ldots,j_n)}, \\ (B'_m b)_j &= b_{(j_1,j_2,\ldots,j_{m-1},\ldots,j_n)} - b_{(j_1,j_2,\ldots,j_m,\ldots,j_n)}. \end{aligned}$$

Zakharov equation (in the form of partial differential equation) was introduced firstly in 1972, which successfully explained the density depression problems in the laser targeting practice from the international physics community and has received great attention. Recently, several authors have studied the asymptotic behavior for the determined lattice system arising from spatial discretizations of Zakharov equations without noises. Of those, Zhou, Zhao and Yin etc. have studied the existence, upper bound of Kolmogorov ε-entropy and upper semi-continuity of the global attractor, uniform attractor and kernel sections for autonomous and non-autonomous dissipative Zakharov lattice equations, see [9, 11, 12] and the references therein. Zhou & Zhao [10] and Han, Shen & Zhou [3] gave some sufficient conditions for the existence of a random attractor of stochastic locally coupled lattice dynamical system on the space of infinite sequence. Until now, as we know, there is no any result on the long-term asymptotic behavior of stochastic Zakharov lattice equations with white noises. In this paper, we consider the existence of a random attractor for stochastic Zakharov lattice system by using the method in [3, 10]. Firstly, we prove the solutions of the system determine a continuous random dynamical system with a random absorbing set, and we then prove that this random dynamical system is random asymptotical compact on the random absorbing set. Finally, we obtain the existence of random attractor.

2. Continuous Random Dynamical System

Define the inner product and norm on spaces l^2 and L^2 as : for any $x = (x_j)_{j\in\mathbb{Z}^n}$, $y = (y_j)_{j\in\mathbb{Z}^n} \in l^2$ or L^2 :

$$(x, y) = \sum_{j\in\mathbb{Z}^n} x_j \bar{y}_j, \quad (x, y)_{\lambda\beta} = \sum_{m=1}^{n} \lambda(B_m x, B_m y) + \lambda\beta(x, y),$$

$$\|x\|^2 = (x, x) = \sum_{j\in\mathbb{Z}^n} |x_j|^2, \quad \|x\|^2_{\lambda\beta} = (x, x)_{\lambda\beta} = \lambda \sum_{m=1}^{n} \|B_m x\|^2 + \lambda\beta\|x\|^2.$$

where $\bar{y}_j$ and y_j are conjugate ,and then the norms $\|\cdot\|$ and $\|\cdot\|_{\lambda\beta}$ are equivalent to each other.

Let $l^2_{\lambda\beta}$ be a Hilbert space with inner product $(\cdot,\cdot)_{\lambda\beta}$ and let $D = l^2_{\lambda\beta}\times l^2\times L^2$ be a Hilbert space with the inner product and norm given by

$$\left\{\begin{array}{l} \left(\varphi^{(1)},\varphi^{(2)}\right)_D = (x^{(1)},x^{(2)})_{\lambda\beta} + (y^{(1)},y^{(2)}) + (z^{(1)},z^{(2)}) \\ = \sum\limits_{j\in\mathbb{Z}^n} (\sum\limits_{m=1}^{n} \lambda(B_m x^{(1)})_j (B_m x^{(2)})_j + \lambda\beta x_j^{(1)} x_j^{(2)} + y_j^{(1)} y_j^{(2)} + z_j^{(1)} \bar{z}_j^{(2)}), \\ \|\varphi\|_D^2 = (\varphi,\varphi)_D\,, \end{array}\right. \tag{2.1}$$

where $\bar{z}_j^{(2)}$ and $z_j^{(2)}$ are conjugate, $\varphi^{(l)} = (x^{(l)},y^{(l)},z^{(l)}) = ((x_j^{(l)}),(y_j^{(l)}),(z_j^{(l)}))_{j\in\mathbb{Z}^n}\in D,\quad l=1,2.$

Let $\left\{e^j\right\}_{j\in\mathbb{Z}^n}$ is the orthonormal basis on l^2 and $w\left(t\right) = \sum\limits_{j\in\mathbb{Z}^n} a_j w_j^1\left(t\right) e^j$.

We consider the following Ornstein-Uhlenbeck equation

$$\left\{\begin{array}{l} dX + Xdt = dw(t), \\ dY + Ydt = dw^2(t), \end{array}\right. \tag{2.2}$$

where $w(t)(\omega) = \omega(t)$, then the stationary solutions of (2.2) are $X\left(\theta_t\omega\right)$ and $Y\left(\theta_t\omega\right)$, respectively, with properties:

$$\left\{\begin{array}{l} \lim\limits_{t\to\infty} \frac{|X(\theta_t\omega)|}{|t|} = \lim\limits_{t\to\infty} \frac{Y(\theta_t\omega)}{|t|} = 0, \\ \lim\limits_{t\to\infty} \frac{\int_0^t X(\theta_s\omega)ds}{t} = \lim\limits_{t\to\infty} \frac{\int_0^t Y(\theta_s\omega)ds}{t} = 0, \end{array}\right. \tag{2.3}$$

$X\left(\omega\right)$, $Y\left(\omega\right)$ are tempered and $X\left(\theta_t\omega\right)$ and $Y\left(\theta_t\omega\right)$ are continuous with respect to t for a.e. $\omega\in\Omega$.

Set

$$\left\{\begin{array}{l} y = \dot{x} + \rho x - \lambda X\left(\theta_t\omega\right) \quad (\rho = \frac{\alpha\beta\lambda}{\lambda\alpha^2+4\beta} > 0), \\ z^* = e^{iY(\theta_t\omega)}z \quad (|z^*| = |z|)\,. \end{array}\right. \tag{2.4}$$

Then

$$\left\{\begin{array}{l} \dot{x} + \rho x - y = \lambda X\left(\theta_t\omega\right), \\ \dot{y} + (\alpha\lambda-\rho)\,y - \left(\alpha\lambda\rho - \beta\lambda - \rho^2\right)x + A\lambda x = \left(\lambda + \rho\lambda - \alpha\lambda^2\right)X\left(\theta_t\omega\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -A\lambda\left|z\right|^2 + \lambda g, \\ \dot{z}^* + iAz^* + \gamma z^* = -iz^*Y\left(\theta_t\omega\right) - ixz^* - ihe^{iY(\theta_t\omega)}. \end{array}\right. \tag{2.5}$$

Write

$$\varphi = \begin{pmatrix} x \\ y \\ z^* \end{pmatrix},$$

$$C = \begin{pmatrix} \rho I & -I & 0 \\ \left(\beta\lambda + \rho^2 - \alpha\lambda\rho\right) I + A\lambda & (\alpha\lambda - \rho)\,I & 0 \\ 0 & 0 & iA + \gamma I \end{pmatrix}, \tag{2.6}$$

$$\psi\left(\theta_t\omega,\varphi\right) = \begin{pmatrix} \lambda X\left(\theta_t\omega\right) \\ \left(\lambda + \rho\lambda - \alpha\lambda^2\right)X\left(\theta_t\omega\right) - \lambda A\left|z^*\right|^2 + \lambda g \\ -iz^*Y\left(\theta_t\omega\right) - ixz^* - ihe^{iY(\theta_t\omega)} \end{pmatrix}. \tag{2.7}$$

Then system (1.1) can be written as

$$\dot{\varphi} + C\varphi = \psi\left(\theta_t\omega,\varphi\right). \tag{2.8}$$

Lemma 2.1. *For a. e. $\omega \in \Omega$, the mapping $\psi(\theta_t\omega, \varphi) : \mathbb{R}^+ \times D \to D$ is continuous with respect to t and φ, and for a. e. $\omega \in \Omega$, $t \in [0,T]$, $T > 0$, $\psi(\theta_t\omega, \varphi)$ is locally Lipschitz in φ.*

Proof. Let $\mathcal{B}$ is a bounded set in D. For $\varphi^{(1)}$, $\varphi^{(2)} \in \mathfrak{B}$, for a. e. $\omega \in \Omega$, $t \in [0,T]$,

$$\begin{aligned}
& \|\psi(\theta_t\omega, \varphi^{(1)}) - \psi(\theta_t\omega, \varphi^{(2)})\|_D^2 \\
= \ & \|(0, -\lambda A(|z^{*(1)}|^2 - |z^{*(2)}|^2), -iY(\theta_t\omega)(z^{*(1)} - z^{*(2)}) \\
& -i(x^{(1)}z^{*(1)} - x^{(2)}z^{*(2)}))^\top\|_D^2 \\
\leq \ & \lambda^2\|A\|_D^2\|z^{*(1)} - z^{*(2)}\|_D^2\|z^{*(1)} + z^{*(2)}\|_D^2 + \|x^{(1)}z^{*(1)} - x^{(2)}z^{*(2)}\|_D^2 \\
& +\|Y(\theta_t\omega)(z^{*(1)} - z^{*(2)})\|_D^2 \\
\leq \ & 64n\lambda^2 L(\mathfrak{B})\|z^{*(1)} - z^{*(2)}\|_D^2 + \|z^{*(1)}(x^{(1)} - x^{(2)}) + x^{(2)}(z^{*(1)} - z^{*(2)})\|_D^2 \\
& +Y^2(\theta_t\omega)\|z^{*(1)} - z^{*(2)}\|_D^2 \\
\leq \ & ((64n\lambda^2 + 1 + \frac{1}{\lambda\beta})L(\mathfrak{B}) + Y^2(\theta_t\omega))\|\varphi^{(1)} - \varphi^{(2)}\|_D^2 \\
\leq \ & M\|\varphi^{(1)} - \varphi^{(2)}\|_D^2,
\end{aligned}$$

where $M = ((64n\lambda^2 + 1 + \frac{1}{\lambda\beta})L(\mathfrak{B}) + \max_{t\in[0,T]} Y^2(\theta_t\omega))$, $L(\mathfrak{B}) = \sup_{\varphi\in\mathfrak{B}} \|\varphi\|_D^2$ are positive constants. So the operator ψ is locally Lipschitz with respect to φ in D. The proof is completed. □

Theorem 2.1. *For a. e. $\omega \in \Omega$ and $\varphi_0 \in D$, there exists a unique solution $\varphi(\cdot, \omega, \varphi_0) \in C^1([0, +\infty), D)$ to system (2.8) with initial data $\varphi(0, \omega, \varphi_0) = \varphi_0$. Furthermore, $\varphi(t, \omega, \varphi_0)$ is continuous with respect to t and φ_0, and measurable in ω.*

Proof. According to the existence and uniqueness of solutions of differential equations [6], for $\varphi_0 \in D$, there exists a unique local solution $\varphi(\cdot, \omega, \varphi_0) \in C^1([0, T_{\max}), D)$ of system (2.8) with $\varphi(0, \omega, \varphi_0) = \varphi_0$. Then we prove $T_{\max} = +\infty$. We take the inner product between z^* and the third equation of (2.5) and take the real part, we have

$$\frac{1}{2}\frac{d}{dt}\|z^*\|^2 + \gamma\|z^*\|^2 = Im(he^{iY(\theta_t\omega)}, z^*),$$

where

$$Im(he^{iY(\theta_t\omega)}, z^*) \leq \frac{1}{\gamma}\|h\|^2 + \frac{\gamma}{4}\|z^*\|^2.$$

Thus,

$$\frac{d}{dt}\|z^*\|^2 + 2\gamma\|z^*\|^2 \leq \frac{2}{\gamma}\|h\|^2 + \frac{\gamma}{2}\|z^*\|^2.$$

By Gronwall's inequality, we get

$$\|z^*\|^2 \leq \|z^*(0)\|^2 e^{-\frac{3}{2}\gamma t} + \frac{4}{3\gamma^2}\|h\|^2. \tag{2.9}$$

Taking inner product $(\cdot,\cdot)_D$ of equation (2.8) with φ, and taking the real part, we have

$$\operatorname{Re}(\dot{\varphi},\varphi)_D+\operatorname{Re}(C\varphi,\varphi)_D=\operatorname{Re}(\psi(\theta_t\omega,\varphi),\varphi)_D. \tag{2.10}$$

In fact

$$\operatorname{Re}(\dot{\varphi},\varphi)_D=\frac{1}{2}\frac{d}{dt}\|\varphi\|_D^2. \tag{2.11}$$

$$\operatorname{Re}(C\varphi,\varphi)_D=\sum_{m=1}^{n}\lambda\rho(\|B_mx\|^2+\beta\|x\|^2)+\rho(\rho-\alpha\lambda)(x,y)+(\alpha\lambda-\rho)\|y\|^2+\gamma\|z^*\|^2.$$

By $\rho=\frac{\alpha\beta\lambda}{\lambda\alpha^2+4\beta}>0$, we know $\rho-\alpha\lambda<0$. Choosing $\sigma=\frac{\lambda\alpha\beta}{\alpha\sqrt{\alpha^2\lambda^2+4\lambda\beta}+\alpha^2\lambda+4\beta}<\rho$, which satisfies

$$4\lambda(\rho-\sigma)\left(\frac{\alpha\lambda}{2}-\rho-\sigma\right)=\frac{\alpha^2\rho^2\lambda^2}{\beta}.$$

Then

$$\begin{aligned}
&\operatorname{Re}(C\varphi,\varphi)_D-\sigma(\|x\|_{\lambda\beta}^2+\|y\|^2)-\frac{\alpha\lambda}{2}\|y\|^2-\gamma\|z^*\|^2\\
=\;&\lambda(\rho-\sigma)(\sum_{m=1}^{n}\|B_mx\|^2+\beta\|x\|^2)+\rho(\rho-\alpha\lambda)(x,y)+(\frac{\alpha\lambda}{2}-\rho-\sigma)\|y\|^2\\
\geq\;&\lambda(\rho-\sigma)(\sum_{m=1}^{n}\|B_mx\|^2+\beta\|x\|^2)-\frac{\rho\alpha\lambda}{\sqrt{\beta}}(\sum_{m=1}^{n}\|B_mx\|^2+\beta\|x\|^2)^{\frac{1}{2}}\|y\|\\
&+(\frac{\alpha\lambda}{2}-\rho-\sigma)\|y\|^2.
\end{aligned}$$

Therefore

$$\operatorname{Re}(C\varphi,\varphi)_D\geq\sigma(\|x\|_{\lambda\beta}^2+\|y\|^2)+\frac{\alpha\lambda}{2}\|y\|^2+\gamma\|z^*\|^2, \tag{2.12}$$

$$\begin{aligned}
\operatorname{Re}(\psi(\theta_t\omega),\varphi)_D\;=\;&(\lambda X(\theta_t\omega),x)_{\lambda\beta}+\lambda(g,y)+\left(\lambda+\rho\lambda-\alpha\lambda^2\right)(X(\theta_t\omega),y)\\
&-\lambda(A(|z^*|^2),y)+\operatorname{Im}(he^{iY(\theta_t\omega)}+z^*Y(\theta_t\omega),z^*).
\end{aligned}$$

By Young's inequality:

$$\begin{cases}
(\lambda X(\theta_t\omega),x)_{\lambda\beta}\leq\frac{\sigma}{2}\|x\|_{\lambda\beta}^2+\frac{\lambda^2}{2\sigma}\|X(\theta_t\omega)\|_{\lambda\beta}^2\\
\qquad\qquad\qquad\quad\;\leq\frac{\sigma}{2}\|x\|_{\lambda\beta}^2+\frac{\lambda^2(4n\lambda+\lambda\beta)}{2\sigma}\|X(\theta_t\omega)\|^2,\\
\lambda(g,y)\leq\frac{\alpha\lambda}{8}\|y\|^2+\frac{2\lambda}{\alpha}\|g\|^2,\\
(X(\theta_t\omega),y)\leq\frac{\alpha}{8}\|y\|^2+\frac{2}{\alpha}\|X(\theta_t\omega)\|^2,\\
\lambda(A(|z^*|^2),y)\leq\frac{\alpha\lambda}{8}\|y\|^2+\frac{2\lambda}{\alpha}\|A(|z^*|^2)\|^2,\\
\operatorname{Im}(he^{iY(\theta_t\omega)}+z^*Y(\theta_t\omega),z^*)\leq\frac{1}{\gamma}\|h\|^2+\frac{\gamma}{4}\|z^*\|^2+\|z^*\|^2Y(\theta_t\omega).
\end{cases} \tag{2.13}$$

Put (2.9), (2.11) - (2.13) into (2.10), we get

$$\begin{aligned}
&\frac{d}{dt}\|\varphi\|_D^2+\sigma\|x\|_{\lambda\beta}^2+(2\sigma+\frac{\alpha\lambda}{4}-\frac{\alpha\lambda(\rho-\alpha\lambda)}{4})\|y\|^2+(\frac{3\gamma}{2}-2Y(\theta_t\omega))\|z^*\|^2\\
\leq\;&(\frac{\lambda^3(4n+\beta)}{\sigma}+\frac{4\left(\lambda+\rho\lambda-\alpha\lambda^2\right)}{\alpha})\|X(\theta_t\omega)\|^2+\frac{4\lambda}{\alpha}\|g\|^2+\frac{1024n\lambda}{9\alpha\gamma^4}\|h\|^4\\
&+\frac{2}{\gamma}\|h\|^2+\frac{64n\lambda}{\alpha}\|z^*(0)\|^4e^{-3\gamma t}+\frac{512n\lambda}{3\alpha\gamma^2}\|z^*(0)\|^2\|h\|^2e^{-\frac{3}{2}\gamma t}.
\end{aligned}$$

Write

$$\begin{cases} \delta = \min\{\sigma, 2\sigma + \frac{\alpha\lambda}{4} - \frac{\alpha\lambda(\rho-\alpha\lambda)}{4}, \frac{3\gamma}{2}\} > 0, \\ G = \max\{\frac{\lambda^3(4n+\beta)}{\sigma} + \frac{4\left(\lambda+\rho\lambda-\alpha\lambda^2\right)}{\alpha}, \frac{4\lambda}{\alpha}, \frac{2}{\gamma}, \frac{1024n\lambda}{9\alpha\gamma^4}\}, \\ Q = \frac{64n\lambda}{\alpha} \left\|z^*(0)\right\|^4 + \frac{512n\lambda}{3\alpha\gamma^2} \left\|z^*(0)\right\|^2 \left\|h\right\|^2, \\ \eta\left(\theta_t\omega\right) = \left\|X\left(\theta_t\omega\right)\right\|^2 + \left\|g\right\|^2 + \left\|h\right\|^2 + \left\|h\right\|^4. \end{cases} \tag{2.14}$$

Thus

$$\frac{d}{dt}\left\|\varphi\right\|_D^2 + \left(\delta - 2Y\left(\theta_t\omega\right)\right)\left\|\varphi\right\|_D \leq G\eta\left(\theta_t\omega\right) + Qe^{-\frac{3}{2}\gamma t}. \tag{2.15}$$

By Gronwall's inequality, we have

$$\begin{aligned} \left\|\varphi\right\|_D^2 &\leq \left\|\varphi_0\right\|_D^2 e^{-(\delta-2Y(\theta_t\omega))t} + \int_0^t e^{-(\delta-2Y(\theta_t\omega))(t-s)}(G\eta\left(\theta_s\omega\right) + Qe^{-\frac{3}{2}\gamma s})ds \\ &\leq e^{-\delta t+2\int_0^t Y(\theta_s\omega)ds}\left\|\varphi_0\right\|_D^2 + G\int_0^t e^{-\delta(t-s)+2\int_s^t Y(\theta_\tau\omega)d\tau}\eta\left(\theta_s\omega\right)ds \\ &\quad + \int_0^t e^{-\delta(t-s)-\frac{3\gamma}{2}s+2\int_s^t Y(\theta_\tau\omega)d\tau}Qds \qquad (2.16) \\ &\leq e^{-\delta t+2\int_0^t Y(\theta_s\omega)ds}\left\|\varphi_0\right\|_D^2 \\ &\quad +Ge^{-\delta t+2\int_0^t Y(\theta_\tau\omega)d\tau}\int_0^t e^{\delta s-2\int_0^s Y(\theta_\tau\omega)d\tau}\eta\left(\theta_s\omega\right)ds \\ &\quad +e^{-\delta t+2\int_0^t Y(\theta_\tau\omega)d\tau}\int_0^t e^{\left(\delta-\frac{3}{2}\gamma\right)s-2\int_0^s Y(\theta_\tau\omega)d\tau}Qds \\ &\leq e^{-\delta t+2\int_0^t Y(\theta_\tau\omega)d\tau}(\left\|\varphi_0\right\|_D^2 + G\int_0^t e^{\delta s-2\int_0^s Y(\theta_\tau\omega)d\tau}\eta\left(\theta_s\omega\right)ds \\ &\quad + \int_0^t e^{\left(\delta-\frac{3}{2}\gamma\right)s-2\int_0^s Y(\theta_\tau\omega)d\tau}Qds). \qquad (2.17) \end{aligned}$$

Hence, for $t \in [0, T_{\max})$, $\varphi(t) = (x(t), y(t), z^*(t))^\top$ is bounded, which implies that $T_{\max} = +\infty$.The proof is completed. □

Theorem 2.2. *The solution of system (2.8) $\varphi : \mathbb{R}^+ \times \Omega \times D \to D$ is a continuous random dynamical system.*

Let $\Psi = (x, y, z)^\top \in D$, then $\Psi(t, \omega, \Psi_0) = P^{-1}(\theta_t\omega)(\varphi(t, \omega, P(\omega)\Psi_0 - \eta(\omega)) + \eta(\theta_t\omega))$, where $P\left(\theta_t\omega\right) = \begin{pmatrix} I & 0 & 0 \\ \rho I & I & 0 \\ 0 & 0 & e^{iY(\theta_t\omega)}I \end{pmatrix}$ and $\eta\left(\theta_t\omega\right) = \left(0, \lambda X\left(\theta_t\omega\right), 0\right)^\top$. Obviously, $P(\theta_t\omega)$ is reversible .

Corollary 2.1. The solution $\Psi : \mathbb{R}^+ \times \Omega \times D \to D$ generated by system (1.1) is a continuous random dynamical system .

3. Random attractor

The existence of random absorbing set and random attractor of random dynamical system φ will be discussed in this section.

Definition 3.1. (Tempered random set) A random bounded set $K(\omega) \subset D$, is tempered in $(\theta_t)_{t\in\mathbb{R}}$, if for a.e. $\omega \in \Omega$,

$$\lim_{t\to\infty} e^{-\nu t} \sup \{\|b\|_D, b \in K(\theta_{-t}\omega)\} = 0, \quad \forall \nu > 0.$$

Theorem 3.1. *There exists a bounded, closed, tempered random absorbing set $\omega \to \Re(\omega)$ of random dynamical system φ, that is to say , for any tempered set $\omega \to K(\omega)$ in D, there exists $T_K(\omega) > 0$, such that for $t \geq T_K(\omega)$, $\varphi(t, \theta_{-t}\omega, K(\theta_{-t}\omega)) \subset \Re(\omega)$.*

Proof. Assume $\omega \to K(\omega)$ was a tempered random set in D. For any $\hat{\varphi}(\omega) \in K(\omega) \subset D$, it follows from (2.16) that

$$\begin{aligned} \|\varphi(t,\omega,\hat{\varphi}(\omega))\|_D^2 \quad \leq \quad & e^{-\delta t+2\int_0^t Y(\theta_s\omega)ds} \|\hat{\varphi}(\omega)\|_D^2 \\ & + G\int_0^t e^{-\delta(t-s)+2\int_s^t Y(\theta_\tau\omega)d\tau}\eta(\theta_s\omega)\,ds \\ & + \int_0^t e^{-\delta(t-s)-\frac{3}{2}\gamma s+2\int_s^t Y(\theta_\tau\omega)d\tau}Qds. \end{aligned}$$

Making the pullback mapping $\theta_{-t} : \omega \to \theta_{-t}\omega$ and integral transformation, we obtain

$$\begin{aligned} & \|\varphi(t,\theta_{-t}\omega,\hat{\varphi}(\theta_{-t}\omega))\|_D^2 \\ \leq \quad & e^{-\delta t+2\int_0^t Y(\theta_{s-t}\omega)ds} \|\hat{\varphi}(\theta_{-t}\omega)\|_D^2 + G\int_0^t e^{-\delta(t-s)+2\int_s^t Y(\theta_{\tau-t}\omega)d\tau}\eta(\theta_{s-t}\omega)\,ds \\ & + \int_0^t e^{-\delta(t-s)-\frac{3}{2}\gamma s+2\int_s^t Y(\theta_{\tau-t}\omega)d\tau}Qds \\ \leq \quad & e^{-\delta t+2\int_{-t}^0 Y(\theta_s\omega)ds} \|\hat{\varphi}(\theta_{-t}\omega)\|_D^2 + G\int_{-t}^0 e^{\delta s+2\int_{s+t}^t Y(\theta_{\tau-t}\omega)d\tau}\eta(\theta_s\omega)\,ds \\ & + e^{-\frac{3}{2}\gamma t}\int_{-t}^0 e^{\left(\delta-\frac{3}{2}\gamma\right)s+2\int_s^0 Y(\theta_\tau\omega)d\tau}Qds \\ \leq \quad & e^{-\delta t+2\int_{-t}^0 Y(\theta_s\omega)ds} \|\hat{\varphi}(\theta_{-t}\omega)\|_D^2 + G\int_{-\infty}^0 e^{\delta s+2\int_s^0 Y(\theta_\tau\omega)d\tau}\eta(\theta_s\omega)\,ds \\ & + e^{-\frac{3}{2}\gamma t}\int_{-t}^0 e^{\left(\delta-\frac{3}{2}\gamma\right)s+2\int_s^0 Y(\theta_\tau\omega)d\tau}Qds. \end{aligned}$$

Recall $Y(\omega)$'s properties and $\hat{\varphi}(\omega) \in K(\omega)$, there exists $T_1(\omega) > 0$, such that for $t > T_1(\omega)$,

$$e^{-\delta t+2\int_{-t}^0 Y(\theta_s\omega)ds} \|\hat{\varphi}(\theta_{-t}\omega)\|_D^2 < \frac{1}{2}.$$

Furthermore, because $\left(\delta - \frac{3}{2}\gamma\right) s > 0$ $(s \in (-t, 0))$, there surely exist $T_2(\omega) > 0$, such that for $t > T_2(\omega)$,

$$e^{-\frac{3}{2}\gamma t}\int_{-t}^0 e^{\left(\delta-\frac{3}{2}\gamma\right)s+2\int_s^0 Y(\theta_\tau\omega)d\tau}Qds < \frac{1}{2}.$$

Denote

$$r_0(\omega) = G\int_{-\infty}^0 e^{\delta s+2\int_s^0 Y(\theta_\tau\omega)d\tau}\eta(\theta_s\omega)\,ds,$$

then for any $\nu > 0$,

$$\begin{aligned} e^{-\nu t} r_0(\theta_{-t}\omega) &= e^{-\nu t} G \int_{-\infty}^{0} e^{\delta s + 2\int_s^0 Y(\theta_{\tau - t}\omega) d\tau} \eta(\theta_{s-t}\omega)\, ds \\ &= e^{-\nu t} G \int_{-\infty}^{-t} e^{\delta(s+t) + 2\int_s^{-t} Y(\theta_t \omega) d\tau} \eta(\theta_s \omega)\, ds. \end{aligned}$$

Then

$$\lim_{t\to\infty} e^{-\nu t} r_0(\theta_{-t}\omega) = 0. \tag{3.1}$$

So $\omega \to r_0(\omega)$ is tempered random variables. Write $r(\omega) = r_0(\omega) + 1$, then $r(\omega)$ is also tempered random variables. Denote a bounded closed ball $\Re(\omega) = O_D(0, r^{\frac{1}{2}}(\omega))$ with radius $r^{\frac{1}{2}}(\omega)$ and centered at origin, which indicate that $\Re(\omega)$ is a bounded, closed, tempered random set. For any tempered random set $\omega \to K(\omega)$ in D, choose $T_K(\omega) = \max\{T_1(\omega), T_2(\omega)\}$, then for $t > T_K(\omega)$,

$$\|\varphi(t, \theta_{-t}\omega, \hat{\varphi}(\theta_{-t}\omega))\|_D^2 < r(\omega) \quad (\forall \hat{\varphi}(\theta_{-t}\omega) \in K(\omega)).$$

Therefore, $\omega \to \Re(\omega)$ is the random absorbing set of random dynamical system φ. The proof is completed . □

Now let us consider the random asymptotic compactness of φ on the absorbing set $\omega \to \Re(\omega)$.

Theorem 3.2. *Let $\Re(\omega)$ be the absorbing set as in Theorem 3.1. If $g \in l^2$, $h \in L^2$, then for any $\varepsilon > 0$, a. e. $\omega \in \Omega$, there is $T(\varepsilon, \omega, \Re) > 0$, $J(\varepsilon, \omega) > 0$, such that the solution $(x_j(t), y_j(t), z_j^*(t))^\top$ of equation (2.8) with initial data $\hat{\varphi} \in \Re(\omega)$, satisfies*

$$\sum_{|j_0| > J(\varepsilon,\omega)} \|\varphi_j(t, \theta_{-t}\omega, \hat{\varphi}(\theta_{-t}\omega))\|_D^2 < \varepsilon, \quad \forall t \ge T(\varepsilon, \omega, \Re(\omega)),$$

where $|j_0| = \max_{1\le k\le n}\{|j_k|\}$, $|j(m)| = \max\{|j_0|, |j_m| + 1\}$.

Proof. Choose a smooth function $f \in C(\mathbb{R}^+, \mathbb{R})$ as follows:

$$\begin{cases} f(a) = 0, & 0 \le a \le 1, \\ f(a) = 1, & a \ge 2, \\ 0 \le f(a) \le 1, & 1 \le a \le 2, \\ |f'(a)| \le f_0, \quad f_0 > 0, & \forall a \in \mathbb{R}^+. \end{cases} \tag{3.2}$$

Let J be a positive integer and $\phi = (p, q, v)^\top$,

$$(\phi_j)_{j\in\mathbb{Z}^n} = ((p_j), (q_j), (v_j))_{j\in\mathbb{Z}^n}^\top = (f(\frac{|j_0|}{J})x_j, f(\frac{|j_0|}{J})y_j, f(\frac{|j_0|}{J})z_j^*)_{j\in\mathbb{Z}^n}^\top.$$

Taking the inner product of system (2.8) with $\phi = (p, q, v)^\top$, and taking the real part, we have

$$\mathrm{Re}(\dot{\varphi}, \phi)_D + \mathrm{Re}(C\varphi, \phi)_D = \mathrm{Re}(\psi(\theta_t\omega), \phi)_D. \tag{3.3}$$

(i) Consider the first term of (3.3), we have

$$\mathrm{Re}(\dot{\varphi}, \phi)_D = \sum_{j\in\mathbb{Z}^n} (\lambda \sum_{m=1}^{n} (B_m \dot{x})_j (B_m p)_j) + \sum_{j\in\mathbb{Z}^n} (\lambda\beta(\dot{x}_j, p_j) + (\dot{y}_j, q_j) + (\dot{z}_j^*, v_j)). \tag{3.4}$$

where

$$
\begin{aligned}
(B_m\dot{x}, B_mp) \quad &= \quad \sum_{j\in\mathbb{Z}^n} f(\frac{|j_0|}{J})\,(B_m\dot{x})_j\,(B_mx)_j \\
&\quad + \sum_{j\in\mathbb{Z}^n} (f(\frac{|j(m)|}{J}) - f(\frac{|j_0|}{J}))(\dot{x}_{(j_1,\ldots,j_m+1,\ldots,j_n)} - \dot{x}_j) \\
&\quad \cdot x_{(j_1,\ldots,j_m+1,\ldots,j_n)}.
\end{aligned} \tag{3.5}
$$

It is easy to check that

$$
\begin{aligned}
&|\sum_{j\in\mathbb{Z}^n} (f(\frac{|j\,(m)|}{J}) - f(\frac{|j_0|}{J}))\left(\dot{x}_{(j_1,\ldots,j_m+1,\ldots,j_n)} - \dot{x}_j\right) x_{(j_1,\ldots,j_m+1,\ldots,j_n)}| \\
= \quad &|\sum_{j\in\mathbb{Z}^n} f'(\frac{|\hat{j}|}{J})\frac{1}{J}\left(\dot{x}_{(j_1,\ldots,j_m+1,\ldots,j_n)} - \dot{x}_j\right) x_{(j_1,\ldots,j_m+1,\ldots,j_n)}| \\
\leq \quad &\frac{f_0 r\,(\omega)}{\lambda\beta J}\left(2\rho^2 + 1 + 2\lambda\beta\right) + \frac{2\lambda^2 f_0}{J}\sum_{|j_0|>J} |X_j\,(\theta_t\omega)|^2, \hat{j} \in [j_0, j(m)].
\end{aligned} \tag{3.6}
$$

and

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\sum_{j\in\mathbb{Z}^n} f(\frac{|j_0|}{J})\,\|\varphi_j\|_D^2 \quad &= \quad \sum_{j\in\mathbb{Z}^n} f(\frac{|j_0|}{J})(\lambda\sum_{m=1}^{n}(B_m\dot{x})_j\,(B_mx)_j) + \lambda\beta\,(\dot{x}, p) \\
&\quad + (\dot{y}, q) + (\dot{z}^*, v)\,.
\end{aligned} \tag{3.7}
$$

Synthesize (3.4)-(3.7), we obtain that

$$
\begin{aligned}
\mathrm{Re}\,(\dot{\varphi}, \phi)_D \quad &= \quad \sum_{j\in\mathbb{Z}^n} f(\frac{|j_0|}{J})(\lambda\sum_{m=1}^{n}(B_m\dot{x})_j\,(B_mx)_j) + \lambda\beta\,(\dot{x}, p) + (\dot{y}, q) + (\dot{z}^*, v) \\
&\quad + \lambda\sum_{j\in\mathbb{Z}^n}\sum_{m=1}^{n}(f(\frac{|j\,(m)|}{J}) - f(\frac{|j_0|}{J})\left(\dot{x}_{(j_1,\ldots,j_m+1,\ldots,j_n)} - \dot{x}_j\right) \\
&\quad \cdot x_{(j_1,\ldots,j_m+1,\ldots,j_n)} \\
&\geq \quad \frac{1}{2}\frac{d}{dt}\sum_{j\in\mathbb{Z}^n} f(\frac{|j_0|}{J})\,\|\varphi_j\|_D^2 - \frac{2n\lambda^3 f_0}{J}\sum_{|j_0|>J}|X_j\,(\theta_t\omega)|^2 \\
&\quad - \frac{nf_0 r\,(\omega)}{\beta J}\left(2\rho^2 + 1 + 2\lambda\beta\right).
\end{aligned} \tag{3.8}
$$

(ii) Estimate the second term of (3.3) :

$$
\begin{aligned}
\mathrm{Re}\,(C\varphi, \phi)_D \quad &= \quad \lambda\rho\sum_{m=1}^{n}(B_mx, B_mp) - \lambda\sum_{m=1}^{n}(B_my, B_mp) + \lambda\sum_{m=1}^{n}(B_mx, B_mq) \\
&\quad + \lambda\beta\rho\,(x, p) + \left(\rho^2 - \alpha\lambda\rho\right)(x, q) + (\alpha\lambda - \rho)\,(y, q) + \gamma\,(z^*, v) \\
&\quad + \mathrm{Re}\,(iAz^*, v)\,.
\end{aligned} \tag{3.9}
$$

In fact

$$
\left\{
\begin{array}{l}
(x, p) = \sum_{j\in\mathbb{Z}^n} f(\frac{|j_0|}{J})x_j^2, \quad (y, p) = \sum_{j\in\mathbb{Z}^n} f(\frac{|j_0|}{J})y_jx_j = (x, q), \\
(y, q) = \sum_{j\in\mathbb{Z}^n} f(\frac{|j_0|}{J})y_j^2, \quad (z^*, v) = \sum_{j\in\mathbb{Z}^n} f(\frac{|j_0|}{J})|z_j^*|^2.
\end{array}
\right. \tag{3.10}
$$

$$
\begin{aligned}
&(B_m x, B_m p)\\
= &\sum_{j\in\mathbb{Z}^n}(B_m x)_j(f(\frac{|j\,(m)|}{J})x_{(j_1,\ldots,j_m+1,\ldots,j_n)}-f(\frac{|j_0|}{J})x_j)\\
= &\sum_{j\in\mathbb{Z}^n}(B_m x)_j(f(\frac{|j_0|}{J})\,(B_m x)_j+(f(\frac{|j\,(m)|}{J})-f(\frac{|j_0|}{J}))x_{(j_1,\ldots,j_m+1,\ldots,j_n)})\\
= &\sum_{j\in\mathbb{Z}^n}f(\frac{|j_0|}{J})|(B_m x)_j|^2\\
&+\sum_{j\in\mathbb{Z}^n}(f(\frac{|j\,(m)|}{J})-f(\frac{|j_0|}{J}))(x_{(j_1,\ldots,j_m+1,\ldots,j_n)}-x_j)x_{(j_1,\ldots,j_m+1,\ldots,j_n)}\\
\geq &\sum_{j\in\mathbb{Z}^n}f(\frac{|j_0|}{J})|(B_m x)_j|^2-\frac{3f_0 r\,(\omega)}{\lambda\beta J}. \qquad (3.11)
\end{aligned}
$$

$$
\begin{aligned}
&(B_m x, B_m q)-(B_m y, B_m p)\\
= &\sum_{j\in\mathbb{Z}^n}(x_{(j_1,\ldots,j_m+1,\ldots,j_n)}-x_j)(f(\frac{|j\,(m)|}{J})y_{(j_1,\ldots,j_m+1,\ldots,j_n)}-f(\frac{|j_0|}{J})y_j)\\
&-\sum_{j\in\mathbb{Z}^n}(y_{(j_1,\ldots,j_m+1,\ldots,j_n)}-y_j)(f(\frac{|j\,(m)|}{J})x_{(j_1,\ldots,j_m+1,\ldots,j_n)}-f(\frac{|j_0|}{J})x_j)\\
= &\sum_{j\in\mathbb{Z}^n}(f(\frac{|j\,(m)|}{J})-f(\frac{|j_0|}{J}))(y_j x_{(j_1,\ldots,j_m+1,\ldots,j_n)}-x_j y_{(j_1,\ldots,j_m+1,\ldots,j_n)})\\
\geq &-\frac{f_0 r(\omega)(1+2\lambda\beta)}{\lambda\beta J}. \qquad (3.12)
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{Re}\,(iAz^*, v)=-\mathrm{Im}\,(Az^*, v)=-\mathrm{Im}\sum_{m=1}^{n}(B_m z^*, B_m v)\\
= &-\mathrm{Im}\sum_{m=1}^{n}\sum_{j\in\mathbb{Z}^n}(z^*_{(j_1,\ldots,j_m+1,\ldots,j_n)}-z^*_j)(f(\frac{|j\,(m)|}{J})z^*_{(j_1,\ldots,j_m+1,\ldots,j_n)}-f(\frac{|j_0|}{J})z^*_j)\\
\geq &-\mathrm{Im}\sum_{m=1}^{n}\sum_{j\in\mathbb{Z}^n}(f(\frac{|j\,(m)|}{J})-f(\frac{|j_0|}{J}))|z^*_{(j_1,\ldots,j_m+1,\ldots,j_n)}||z^*_j|\\
\geq &-\frac{nf_0 r(\omega)}{J}. \qquad (3.13)
\end{aligned}
$$

Substituting (3.10)-(3.13) into (3.9), we get:

$$
\begin{aligned}
\mathrm{Re}\,(C\varphi,\phi)_D \quad\geq\quad &\sum_{j\in\mathbb{Z}^n}f(\frac{|j_0|}{J})(\lambda\rho\sum_{m=1}^{n}|(B_m x)_j|^2+\lambda\beta\rho\,|x_j|^2+(\rho^2-\alpha\lambda\rho)x_j y_j\\
&+(\alpha\lambda-\rho)|y_j|^2+\gamma|z^*_j|^2)-\frac{nf_0 r\,(\omega)}{J}(\frac{3\rho+1}{\beta}+2\lambda+1). \qquad (3.14)
\end{aligned}
$$

Since

$$\begin{aligned}
&\mathrm{Re}(C\varphi,\phi)_D-\sum_{j\in\mathbb{Z}^n}f(\frac{|j_0|}{J})(\sigma(\lambda\sum_{m=1}^{n}|(B_mx)_j|^2+\lambda\beta|x_j|^2+|y_j|^2)\\
&+\frac{\lambda\alpha}{2}|y_j|^2+\gamma|z_j^*|^2)\\
\geq\ &\sum_{j\in\mathbb{Z}^n}f(\frac{|j_0|}{J})((\rho-\sigma)(\lambda\sum_{m=1}^{n}|(B_mx)_j|^2+\lambda\beta\,|x_j|^2)+(\frac{\alpha\lambda}{2}-\rho-\sigma)|y_j|^2\\
&+\rho(\rho-\alpha\lambda)x_jy_j)-\frac{nf_0r(\omega)}{J}(\frac{3\rho+1}{\beta}+2\lambda+1)\\
\geq\ &\sum_{j\in\mathbb{Z}^n}f(\frac{|j_0|}{J})((\rho-\sigma)\lambda\beta|x_j|^2+(\frac{\alpha\lambda}{2}-\rho-\sigma)|y_j|^2-\rho\alpha\lambda x_jy_j)\\
&-\frac{nf_0r(\omega)}{J}(\frac{3\rho+1}{\beta}+2\lambda+1)\\
\geq\ &-\frac{nf_0r(\omega)}{J}(\frac{3\rho+1}{\beta}+2\lambda+1),
\end{aligned}$$

$$\begin{aligned}
\mathrm{Re}(C\varphi,\phi)_D\ \geq\ &\sum_{j\in\mathbb{Z}^n}f(\frac{|j_0|}{J})(\sigma(\lambda\sum_{m=1}^{n}|(B_mx)_j|^2+\lambda\beta|x_j|^2+|y_j|^2)+\frac{\alpha\lambda}{2}|y_j|^2\\
&+\gamma|z_j^*|^2)-\frac{nf_0r(\omega)}{J}(\frac{3\rho+1}{\beta}+2\lambda+1).
\end{aligned}\tag{3.15}$$

(iii) Estimate the third term of (3.3):

Taking inner product between v and the third equation of (2.5) and take the real part, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{|j_0|>J}f(\frac{|j_0|}{J})|z_j^*|^2-\mathrm{Im}\,(Az^*,v)+\gamma\sum_{|j_0|>J}f(\frac{|j_0|}{J})|z_j^*|^2\\
=\ &Im(he^{iY(\theta_t\omega)},v)\leq\frac{\gamma}{4}\sum_{|j_0|>J}f(\frac{|j_0|}{J})|z_j^*|^2+\frac{1}{\gamma}\sum_{|j_0|>J}f(\frac{|j_0|}{J})|h_j|^2.
\end{aligned}$$

by (3.13),

$$\frac{d}{dt}\sum_{|j_0|>J}f(\frac{|j_0|}{J})\,|z_j^*|^2+\frac{3}{2}\gamma\sum_{|j_0|>J}f(\frac{|j_0|}{J})|z_j^*|^2\leq\frac{2}{\gamma}\sum_{|j_0|>J}f(\frac{|j_0|}{J})|h_j|^2+\frac{2nf_0r(\omega)}{J}.$$

By Gronwall's inequality, we find

$$\begin{aligned}
\sum_{|j_0|>J}f(\frac{|j_0|}{J})|z_j^*|^2\ \leq\ &\sum_{|j_0|>J}f(\frac{|j_0|}{J})|z_j^*(0)|^2e^{-\frac{3}{2}\gamma t}\\
&+\int_0^t e^{-\frac{3}{2}\gamma(t-s)}(\frac{2}{\gamma}\sum_{|j_0|>J}f(\frac{|j_0|}{J})|h_j|^2+\frac{2nf_0r(\omega)}{J})ds\\
\leq\ &r(\omega)e^{-\frac{3}{2}\gamma t}+\frac{4}{3\gamma^2}\sum_{|j_0|>J}f(\frac{|j_0|}{J})\,|h_j|^2+\frac{4nf_0r(\omega)}{3\gamma J}.
\end{aligned}\tag{3.16}$$

In fact

$$\begin{aligned}\mathrm{Re}(\psi\left(\theta_t\omega\right),\phi)_D &= \sum_{m=1}^{n}(\lambda^2 B_m X(\theta_t\omega), B_m p) + (\lambda^2\beta X(\theta_t\omega), p) - \left(\lambda A|z^*|^2, q\right)\\ &+((\lambda+\rho\lambda-\alpha\lambda^2)X(\theta_t\omega), q) + (\lambda g, q)\\ &+\mathrm{Im}(he^{iY(\theta_t\omega)} + z^*Y(\theta_t\omega), v), \end{aligned}\tag{3.17}$$

where

$$\begin{aligned}&(\lambda^2 B_m X(\theta_t\omega), B_m p)\\ =& \sum_{j\in\mathbb{Z}^n}\lambda^2(X_{(j_1,\ldots,j_m+1,\ldots,j_n)}(\theta_t\omega) - X_j(\theta_t\omega))(f(\frac{|j(m)|}{J})x_{(j_1,\ldots,j_m+1,\ldots,j_n)}\\ &-f(\frac{|j_0|}{J})x_j)\\ \leq& \sum_{j\in\mathbb{Z}^n}\lambda^2(X_{(j_1,\ldots,j_m+1,\ldots,j_n)}(\theta_t\omega) - X_j(\theta_t\omega))f(\frac{|j(m)|}{J})x_{(j_1,\ldots,j_m+1,\ldots,j_n)}\\ &+\sum_{j\in\mathbb{Z}^n}\lambda^2(X_{(j_1,\ldots,j_m+1,\ldots,j_n)}(\theta_t\omega) - X_j(\theta_t\omega))f(\frac{|j_0|}{J})x_j\\ \leq& \frac{\sigma\lambda\beta}{4n}\sum_{j\in\mathbb{Z}^n}f(\frac{|j_0|}{J})x_j^2 + \frac{16n\lambda^4}{\sigma\lambda\beta}\sum_{|j_0|>J}\|X(\theta_t\omega)\|^2, \end{aligned}\tag{3.18}$$

$$\begin{aligned}(\lambda^2\beta X(\theta_t\omega), p) &= \sum_{j\in\mathbb{Z}^n}f(\frac{|j_0|}{J})(x_j, \lambda^2\beta X_j(\theta_t\omega))\\ &\leq \frac{\sigma\lambda\beta}{4}\sum_{j\in\mathbb{Z}^n}f(\frac{|j_0|}{J})x_j^2 + \frac{\lambda^4\beta^2}{\sigma\lambda\beta}\sum_{|j_0|>J}\|X_j\left(\theta_t\omega\right)\|^2, \end{aligned}\tag{3.19}$$

$$\begin{aligned}(\lambda A|z^*|^2, q) &= \sum_{|j_0|>J}f(\frac{|j_0|}{J})(y_j, \lambda A|z^*|_j^2)\\ &\leq \frac{3\lambda^2}{2\sigma+\alpha\lambda}\sum_{|j_0|>J}f(\frac{|j_0|}{J})(A(|z^*|^2))_j^2 + \frac{2\sigma+\alpha\lambda}{12}\sum_{j\in\mathbb{Z}^n}f(\frac{|j_0|}{J})y_j^2\\ &\leq \frac{48n\lambda^2}{2\sigma+\alpha\lambda}\sum_{|j_0|>J}f(\frac{|j_0|}{J})|z_j^*|^4 + \frac{2\sigma+\alpha\lambda}{12}\sum_{j\in\mathbb{Z}^n}f(\frac{|j_0|}{J})y_j^2\\ &\leq \frac{48n\lambda^2}{2\sigma+\alpha\lambda}r(\omega)\sum_{|j_0|>J}f(\frac{|j_0|}{J})|z_j^*|^2 + \frac{2\sigma+\alpha\lambda}{12}\sum_{j\in z^n}f(\frac{|j_0|}{J})y_j^2\\ &\leq \frac{48n\lambda^2}{2\sigma+\alpha\lambda}r^2(\omega)e^{-\frac{3}{2}\gamma t} + \frac{64n\lambda^2}{(2\sigma+\alpha\lambda)\gamma^2}r(\omega)\sum_{|j_0|>J}f(\frac{|j_0|}{J})|h_j|^2\\ &\quad+\frac{2\sigma+\alpha\lambda}{12}\sum_{j\in\mathbb{Z}^n}f(\frac{|j_0|}{J})y_j^2 + \frac{64n^2\lambda^2 f_0}{(2\sigma+\alpha\lambda)\gamma J}r^2(\omega), \end{aligned}\tag{3.20}$$

$$
\begin{aligned}
(\lambda g, q) &= \sum_{j\in\mathbb{Z}^n} f(\frac{|j_0|}{J})(\lambda g_j, y_j) \\
&\le \frac{3\lambda^2}{2\sigma+\alpha\lambda}\sum_{|j_0|>J} f(\frac{|j_0|}{J})|g_j|^2 + \frac{2\sigma+\alpha\lambda}{12}\sum_{j\in\mathbb{Z}^n} f(\frac{|j_0|}{J})y_j^2, \qquad (3.21)
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{Im}(he^{iY(\theta_t\omega)} + z^* Y(\theta_t\omega), v) = \mathrm{Im}\sum_{j\in\mathbb{Z}^n} f(\frac{|j_0|}{J}) z_j^*(he^{iY(\theta_t\omega)} + z_j^* Y(\theta_t\omega)) \\
\le\ & \frac{\gamma}{4}\sum_{|j_0|>J} f(\frac{|j_0|}{J})|z_j^*|^2 + \frac{1}{\gamma}\sum_{|j_0|>J} f(\frac{|j_0|}{J})|h_j|^2 + Y(\theta_t\omega)|z_j^*|^2. \qquad (3.22)
\end{aligned}
$$

It follows from (3.3), (3.8), (3.15), (3.18)-(3.22) that

$$
\begin{aligned}
&\frac{d}{dt}\sum_{j\in\mathbb{Z}^n} f(\frac{|j_0|}{J})\|\varphi_j\|^2 + \sum_{j\in\mathbb{Z}^n} f(\frac{|j_0|}{J})((2\sigma\,\|x\|_{\lambda\beta}^2 - \sigma\,\|x\|_{\lambda\beta}^2) + (\sigma + \frac{\lambda\alpha}{2})\|y_j\|^2 \\
&+(\frac{3\gamma}{2} - 2Y\,(\theta_t\omega))|z_j^*|^2) \\
\le\ & (\frac{4n\lambda^3 f_0}{J} + \frac{2\lambda^3(16n^2+\beta^2)}{\sigma\beta} + \frac{6(\lambda+\rho\lambda-\alpha\lambda^2)^2}{2\sigma+\alpha\lambda})\sum_{|j_0|>J}\|X_j\,(\theta_t\omega)\|^2 \\
&+(\frac{6\lambda^2}{2\sigma+\alpha\lambda}\sum_{|j_0|>J} f(\frac{|j_0|}{J})|g_j|^2 + \frac{2}{\gamma}\sum_{|j_0|>J} f(\frac{|j_0|}{J})|h_j|^2) + \frac{128n^2\lambda^2 f_0}{(2\sigma+\alpha\lambda)\gamma J}r^2(\omega) \\
&+\frac{128n\lambda^2}{(2\sigma+\alpha\lambda)\,\gamma^2}r(\omega)\sum_{|j_0|>J} f(\frac{|j_0|}{J})|h_j|^2 + \frac{96n\lambda^2}{2\sigma+\alpha\lambda}r^2\,(\omega)\,e^{-\frac{3}{2}\gamma t} \\
&+\frac{1}{J}(\frac{2nf_0\left(2\rho^2+3\rho+2+4\lambda\beta+\beta\right)}{\beta})r\,(\omega)\,.
\end{aligned}
$$

Set

$$
\begin{aligned}
L_1 &= \frac{4n\lambda^3 f_0}{J} + \frac{2\lambda^3(16n^2+\beta^2)}{\sigma\beta} + \frac{6(\lambda+\rho\lambda-\alpha\lambda^2)^2}{2\sigma+\alpha\lambda}, \\
L_2 &= \max\{\frac{6\lambda^2}{2\sigma+\alpha\lambda}, \frac{2}{\gamma}\}, \\
L_3 &= \frac{128n\lambda^2}{(2\sigma+\alpha\lambda)\,\gamma^2}, \\
F(\omega) &= (\frac{2nf_0\left(2\rho^2+3\rho+2+4\lambda\beta+\beta\right)}{\beta})r(\omega) + \frac{128n^2\lambda^2 f_0}{(2\sigma+\alpha\lambda)\gamma}r^2(\omega).
\end{aligned}
$$

Then

$$
\begin{aligned}
&\frac{d}{dt}\sum_{j\in\mathbb{Z}^n} f(\frac{|j_0|}{J})\|\varphi_j\|_D^2 + (\delta - 2Y\,(\theta_t\omega))\sum_{j\in\mathbb{Z}^n} f(\frac{|j_0|}{J})\|\varphi_j\|_D^2 \\
\le\ & L_1\sum_{|j_0|>J}\|X_j(\theta_t\omega)\|^2 + L_2\sum_{|j_0|>J}(|h_j|^2 + g_j^2) + L_3 r\,(\omega)\sum_{|j_0|>J} f(\frac{|j_0|}{J})|h_j|^2 \\
&+\frac{F(\omega)}{J} + \frac{96n\lambda^2}{2\sigma+\alpha\lambda}r^2\,(\omega)\,e^{-\frac{3}{2}\gamma t}.
\end{aligned}
$$

By Gronwall's inequality for $t > T_{\Re}(\omega)$,

$$
\begin{aligned}
\sum_{j\in\mathbb{Z}^n} f(\frac{|j_0|}{J})\|\varphi_j\|_D^2 \leq\ & e^{-\delta(t-T_{\Re})+2\int_{T_{\Re}}^t Y(\theta_s\omega)ds}\sum_{j\in\mathbb{Z}^n} f(\frac{|j_0|}{J})\|\varphi_j(T_{\Re},\omega,\hat{\varphi}_0(\omega))\|_D^2 \\
&+L_1\int_{T_{\Re}}^t\sum_{|j_0|>J} e^{-\delta(t-\tau)+2\int_\tau^t Y(\theta_s\omega)ds}|X_j(\theta_\tau\omega)|^2 d\tau \\
&+L_2\int_{T_{\Re}}^t\sum_{|j_0|>J} e^{-\delta(t-\tau)+2\int_\tau^t Y(\theta_s\omega)ds}(g_j^2+|h_j|^2)d\tau \\
&+L_3\sum_{|j_0|>J}|h_j|^2\int_{T_{\Re}}^t\sum_{|j_0|>J} e^{-\delta(t-\tau)+2\int_\tau^t Y(\theta_s\omega)ds} r(\omega)\,d\tau \\
&+\int_{T_{\Re}}^t e^{-\delta(t-\tau)+2\int_\tau^t Y(\theta_s\omega)ds}\frac{F(\omega)}{J}d\tau \\
&+\frac{96n\lambda^2}{2\sigma+\alpha\lambda}\int_{T_{\Re}}^t e^{-\delta(t-\tau)+2\int_\tau^t Y(\theta_s\omega)ds} r^2(\omega)\,e^{-\frac{3}{2}\gamma\tau}d\tau. \quad (3.23)
\end{aligned}
$$

Making the pullback mapping $\theta_{-t}:\omega\rightarrow\theta_{-t}\omega$,

$$
\begin{aligned}
& e^{-\delta(t-T_{\Re})+2\int_{T_{\Re}}^t Y(\theta_{s-t}\omega)ds}\sum_{j\in\mathbb{Z}^n} f(\frac{|j_0|}{J})\|\varphi_j(T_{\Re},\theta_{-t}\omega,\hat{\varphi}_0(\theta_{-t}\omega))\|_D^2 \\
\leq\ & e^{-\delta(t-T_{\Re})+2\int_{T_{\Re}}^t Y(\theta_{s-t}\omega)ds}\|\varphi(T_{\Re},\theta_{-t}\omega,\hat{\varphi}_0(\theta_{-t}\omega))\|_D^2 \\
\leq\ & e^{-\delta(t-T_{\Re})+2\int_{T_{\Re}-t}^0 Y(\theta_s\omega)ds}\|\varphi(T_{\Re},\theta_{-t}\omega,\hat{\varphi}_0(\theta_{-t}\omega))\|_D^2 .
\end{aligned}
$$

Since $\|\varphi(T_{\Re},\theta_{-t}\omega,\hat{\varphi}_0(\theta_{-t}\omega))\|_D^2$ is bounded, for any $\varepsilon>0$, there exists $T_1(\varepsilon,\omega,\Re)$ $> T_{\Re}$, such that for $t > T_1(\varepsilon,\omega,\Re)$,

$$
e^{-\delta(t-T_{\Re})+2\int_{T_{\Re}}^t Y(\theta_{s-t}\omega)ds}\sum_{j\in\mathbb{Z}^n} f(\frac{|j_0|}{J})\|\varphi_j(T_{\Re},\theta_{-t}\omega,\hat{\varphi}_0(\theta_{-t}\omega))\|_D^2 \leq \frac{\varepsilon}{6}. \quad (3.24)
$$

$$
\begin{aligned}
& L_1\int_{T_{\Re}}^t\sum_{|j_0|>J} e^{-\delta(t-\tau)+2\int_\tau^t Y(\theta_{s-t}\omega)ds}|X_j(\theta_{\tau-t}\omega)|^2 d\tau \\
=\ & L_1\int_{T_{\Re}-t}^0\sum_{|j_0|>J} e^{\delta\tau+2\int_\tau^0 Y(\theta_s\omega)ds}|X_j(\theta_\tau\omega)|^2 d\tau \\
=\ & L_1\int_{-T^*}^0\sum_{|j_0|>J} e^{\delta\tau+2\int_\tau^0 Y(\theta_s\omega)ds}|X_j(\theta_\tau\omega)|^2 d\tau \\
& +L_1\int_{T_{\Re}-t}^{-T^*}\sum_{|j_0|>J} e^{\delta\tau+2\int_\tau^0 Y(\theta_s\omega)ds}|X_j(\theta_\tau\omega)|^2 d\tau,
\end{aligned}
$$

where $T^*>0$. Since $\omega\rightarrow X(\theta_t\omega)$ is tempered and $X(\theta_t\omega)$ is continuous with respect to t, there is a tempered variable $\xi(\omega)$ such that

$$
\|X(\theta_t\omega)\|^2 \leq \xi(\omega)\,e^{\frac{\delta}{2}|t|},\quad \forall t\in\mathbb{R}.
$$

By (2.3), taking sufficient large $T^0>0$ such that for any $\tau\le -(T^0+T_{\Re})$,

$$\frac{\delta}{2}+\frac{2}{\tau}\int_{\tau}^{0}Y\left(\theta_s\omega\right)ds\ge\frac{\delta}{4}.$$

Thus for $t>T^0+T_{\Re}$, we have

$$\begin{aligned}&L_1\int_{T_{\Re}-t}^{-T^0}\sum_{|j_0|>J}e^{\delta\tau+2\int_{\tau}^{0}Y(\theta_s\omega)ds}|X_j\left(\theta_\tau\omega\right)|^2d\tau\\ \le\ &L_1\int_{T_{\Re}-t}^{-T^0}\xi\left(\omega\right)e^{\frac{\delta}{4}\tau}d\tau\le\frac{4L_1\xi\left(\omega\right)}{\delta}e^{-\frac{\delta}{4}T^0}.\end{aligned}$$

Setting $T^*>\max\left\{\frac{4}{\delta}\ln\frac{48L_1\xi(\omega)}{\varepsilon\delta},T^0\right\}$, then

$$L_1\int_{T_{\Re}-t}^{-T^*}\sum_{|j_0|>J}e^{\delta\tau+2\int_{\tau}^{0}Y(\theta_s\omega)ds}|X_j\left(\theta_\tau\omega\right)|^2d\tau\le\frac{\varepsilon}{12}.$$

Let T^* be fixed, by the Lebesgue's dominated convergence theorem, there exists $J_1\left(\varepsilon,\omega\right)$, such that for $J>J_1\left(\varepsilon,\omega\right)$,

$$L_1\int_{-T^*}^{0}\sum_{|j_0|>J}e^{\delta\tau+2\int_{\tau}^{0}Y(\theta_s\omega)ds}|X_j\left(\theta_\tau\omega\right)|^2d\tau\le\frac{\varepsilon}{12}.$$

So

$$L_1\int_{T_{\Re}}^{t}\sum_{|j_0|>J}e^{-\delta(t-\tau)+2\int_{\tau}^{t}Y(\theta_{s-t}\omega)ds}\left|X_j\left(\theta_{\tau-t}\omega\right)\right|^2d\tau\le\frac{\varepsilon}{6}. \tag{3.25}$$

Since $g_j\in l^2$, $h_j\in L^2$, there exists $J_2(\varepsilon,\omega)$, such that for $J>J_2(\varepsilon,\omega)$,

$$\begin{aligned}&L_2\int_{T_{\Re}}^{t}\sum_{|j_0|>J}e^{-\delta(t-\tau)+2\int_{\tau}^{t}Y(\theta_{s-t}\omega)ds}({g_j}^2+|h_j|^2)d\tau\\ =\ &L_2\sum_{|j_0|>J}({g_j}^2+|h_j|^2)\int_{T_{\Re}}^{t}e^{-\delta(t-\tau)+2\int_{\tau-t}^{0}Y(\theta_s\omega)ds}d\tau\\ \le\ &\frac{\varepsilon}{6}.\end{aligned}\tag{3.26}$$

By $h_j\in L^2$, $r\left(\omega\right)$ is tempered random variables, therefore exists $J_3\left(\varepsilon,\omega\right)>0$, such that for $J>J_3\left(\varepsilon,\omega\right)$,

$$\begin{aligned}&L_3\sum_{|j_0|>J}|h_j|^2\int_{T_{\Re}}^{t}e^{-\delta(t-\tau)+2\int_{\tau}^{t}Y(\theta_{s-t}\omega)ds}r\left(\theta_{-\tau}\omega\right)d\tau\\ \le\ &L_3\sum_{|j_0|>J}|h_j|^2\int_{T_{\Re}}^{t}e^{-\delta(t-\tau)+2\int_{\tau-t}^{0}Y(\theta_s\omega)ds}r\left(\theta_{-\tau}\omega\right)d\tau\\ \le\ &\frac{\varepsilon}{6}.\end{aligned}\tag{3.27}$$

Since $F(\omega)$ is a tempered random variables, there exists $T_2(\varepsilon,\omega,\Re) > T_\Re$, $J_4(\varepsilon,\omega) > 0$, such that for $t > T_2(\varepsilon,\omega,\Re)$, $J > J_4(\varepsilon,\omega)$,

$$\begin{aligned} & \int_{T_\Re}^{t} e^{-\delta(t-\tau)+2\int_\tau^t Y(\theta_{s-t}\omega)ds} \frac{F(\theta_{-\tau}\omega)}{J} d\tau \\ \leq \; & \int_{T_\Re}^{t} e^{-\delta(t-\tau)+2\int_{\tau-t}^0 Y(\theta_s\omega)ds} \frac{F(\theta_{-\tau}\omega)}{J} d\tau \\ \leq \; & \frac{\varepsilon}{6}. \end{aligned} \tag{3.28}$$

Now that $\omega \to r(\omega)$ is tempered random variables, there exists $T_3(\varepsilon,\omega,\Re) > T_\Re$, such that for $t > T_3(\varepsilon,\omega,\Re)$,

$$\begin{aligned} & \frac{96n\lambda^2}{2\sigma+\alpha\lambda}\int_{T_\Re}^{t} e^{-\delta(t-\tau)+2\int_\tau^t Y(\theta_{s-t}\omega)ds} r^2(\theta_{-\tau}\omega) e^{-\frac{3}{2}\gamma\tau} d\tau \\ \leq \; & e^{-\delta t}\frac{96n\lambda^2}{2\sigma+\alpha\lambda}\int_{T_\Re}^{t} e^{\left(\delta-\frac{3}{2}\gamma\right)\tau+2\int_{\tau-t}^0 Y(\theta_s\omega)ds} r^2(\theta_{-\tau}\omega) d\tau \\ \leq \; & \frac{\varepsilon}{6}. \end{aligned} \tag{3.29}$$

Denote

$$J(\varepsilon,\omega) = \max\{J_1(\varepsilon,\omega), J_2(\varepsilon,\omega), J_3(\varepsilon,\omega), J_4(\varepsilon,\omega)\},$$

$$T(\varepsilon,\omega,\Re) = \max\{T_1(\varepsilon,\omega,\Re), T_2(\varepsilon,\omega,\Re), T_3(\varepsilon,\omega,\Re), T^* + T_\Re\}.$$

By (3.23)-(3.29), we have that for any $\varepsilon > 0$, each $\omega \in \Omega$, $t > T(\varepsilon,\omega,\Re)$, $\hat\varphi \in \Re(\omega)$,

$$\sum_{|j_0| > J(\varepsilon,\omega)} \|\varphi_j(t,\theta_{-t}\omega,\hat\varphi(\theta_{-t}\omega))\|_D^2 \leq \sum_{j\in\mathbb{Z}^n} f\left(\frac{|j_0|}{J}\right)\|\varphi_j(t,\theta_{-t}\omega,\hat\varphi(\theta_{-t}\omega))\|_D^2 \leq \varepsilon.$$

□

By Theorem 2.2, Theorem 3.1, Theorem 3.2 and Theorem 3.1 in [10], we obtain the following main result.

Theorem 3.3. *There exists a random attractor $\omega \to \varrho(\omega)$ to random dynamical system φ in D, and*

$$\varrho(\omega) = \bigcap_{\tau \geq T_K(\omega)} \overline{\bigcup_{t\geq\tau} \varphi(t,\theta_{-t}\omega,\Re(\theta_{-t}\omega))}, \qquad \omega \in \Omega,$$

where $\omega \to \Re(\omega)$ is the random absorbing set to φ.

According to Corollary 2.1 , Theorem 3.3 , we know

Corollary 3.1. The random dynamical system Ψ generated by equation (1.1) possesses a random attractor $\omega \mapsto \tilde\varrho(\omega)$, and $\tilde\varrho(\omega) = P^{-1}(\omega)(\varrho(\omega) + \eta(\omega))$, $\omega \in \Omega$.

References

[1] P.W. Bates, H. Lisei and K. Lu, *Attractors for stochastic lattice dynamical systems*, Stoch. Dyn., 6(2006), 1-21.

[2] T. Caraballo and K. Lu, *Attractors for stochastic lattice dynamical systems with a multiplicative noise*, Front. Math. China, DOI 10.1007/s11464-008-0028-7(2008).

[3] X. Han, W. Shen and S. Zhou, *Random attractors for stochastic lattice dynamical systems in weighted spaces*, J. Differential Equations, 250(2011), 1235-1266.

[4] J. Huang, *The random attractor of stochastic FitzHughCNagumo equations in an infinite lattice with white noise*, Phys. D, 233(2007), 83-94.

[5] Y. Lv and J.H. Sun, *Dynamical behavior for stochastic lattice systems*, Chaos, Solitons & Fractals, 27(2006), 1080-1090.

[6] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 2007.

[7] D. Ruelle, *Characteristic exponents for a viscous fluid subjected to time dependent forces*, Comm. Math. Phys., 93(1984), 285-300.

[8] X. Wang, S. Li and D. Xu, *Random attractors for second-order stochastic lattice dynamical systems*, Nonlinear Anal., 72(2010), 483-494.

[9] F.Q. Yin, S.F. Zhou, Z.G. OuYang and C.H. Xiao, *Attractor for Lattice System of Dissipative Zakharov Equation*, Acta Math. Sinica, 25:2(2009), 321-342.

[10] C.D. Zhao and S.F. Zhou, *Sufficient conditions for the existence of global random attractors for stochastic lattice dynamical systems and applications*, J. Math. Anal. Appl., 354(2009), 78-95.

[11] S. Zhou, J. Huan and X. Han, *Compact kernel sections for dissipative non-autonomous Zakharov equation on infinite lattices*, Commu. Pure Appl. Anal., 9:1(2010), 193-210.

[12] S. Zhou, C. Zhao and X. Liao, *Compact uniform attractor for dissipative non-autonomous lattice dynamical systems*, Commu. Pure Appl. Anal., 6:4(2007), 1087-1111.