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GLOBAL DYNAMICS OF AN AUTOCATALYTIC REACTION-DIFFUSION SYSTEM WITH FUNCTIONAL RESPONSE

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Abstract In this work the existence, regularity, and finite fractal dimensionality of a global attractor in the L^2 phase space for the weak solution semiflow of an autocatalytic reaction-diffusion system with a functional response of the Holling type II are proved.

Keywords Reaction-diffusion system, global dynamics, autocatalysis, functional response, global attractor.

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1. Introduction

In recent years the asymptotic dynamics of autocatalytic reaction-diffusion systems typically represented by the Brusselator system, Gray-Scott equations and Oregonator system have been studied by the author in [16–18, 21] and the existence of global attractors of finite fractal dimensions and the existence of exponential attractors have been proved for these systems. The results have also been generalized to the reversible cubically autocatalytic reaction-diffusion systems in [19,20,22] and to the coupled two-cell or two-compartment models in [23, 24] with a variety of effective new methods developed.

In this paper, the following diffusive and autocatalytic model with a functional response of Holling type II will be considered,

$$\begin{aligned} \frac{\partial[U]}{\partial t} &= D_1\Delta[U] + k_1[U][V] - \frac{k_2[U]}{1+k_3[U]}, \\ \frac{\partial[V]}{\partial t} &= D_2\Delta[V] + k_0[A] - k_1[U][V], \end{aligned} \tag{1.1}$$

where Δ is the Laplace operator, $[U]$ and $[V]$ are concentrations of the reactants U and V as the unknown functions of time $t \geq 0$ and spatial variable $x \in \Omega$, respectively, $[A]$ is a constantly held concentration of a substance A, and the coefficients k_0, k_1, k_2 and k_3 are positive reaction rates. Assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$ with $n \leq 3$.

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Rescaling the coefficients and reassigning the variables and parameters, it is seen that the above reaction-diffusion system can be writiten as

$$\frac{\partial u}{\partial t} = d_1 \Delta u + uv - \frac{bu}{1+\mu u}, \tag{1.2}$$

$$\frac{\partial v}{\partial t} = d_2 \Delta v + a - uv, \tag{1.3}$$

where d_1, d_2, a, b and μ are positive constants, with the Dirichlet boundary condition assumed in this work,

$$u(t,x) = v(t,x) = 0, \quad t \geq 0, \ x \in \partial\Omega, \tag{1.4}$$

and an initial condition

$$u(0,x) = u_0(x), \ v(0,x) = v_0(x), \quad x \in \Omega. \tag{1.5}$$

This model features a quadratic autocatalysis and a functional response of the Holling type II, which is a saturation law with non-vanished initial slope and different from the sigmoidal functional response of the Holling type III. Mathematical analysis and numerical simulations have been reported recently [7,8,14,15] on steady state bifurcations and the effect of parameters on spatiotemporal pattern formation for the models with this kind of saturation law.

Besides, the similar Lengyel-Epstein system modeling the diffusive CIMA reaction, which is a significant reaction in physical chemistry, has ben studied in [4, 6] on the Turing patterns and the related global bifurcation.

This work focuses on the global weak solutions of the problem (1.2)–(1.5) and the existence and properties of an attractor for the solution semiflow in the L^2 and H^1 phase spaces. The results demonstrate that global dynamics of solutions of this significant reaction-diffusion system are determined by the finite-dimensional dynamics on a compact and invariant set in the asymptotically attracting sense.

For this reaction-diffusion system the vector version of the *asymptotically dissipative condition*,

$$\lim_{|s|\to\infty} f(s) \cdot s \leq C,$$

where $C \geq 0$ is some constant, usually assumed in the theory of dissipative infinite dimensional dynamical systems is *not* satisfied by the nonlinear terms of the equations, see (1.9) later. This is an obstacle for showing the absorbing property and the asymptotically compact property of the semiflow of the weak solutions.

We start with the formulation of an abstract evolutionary equation associated with the initial-boundary value problem (1.2)–(1.5) of this reaction-diffusion system. Define the product Hilbert spaces as follows,

$$H = [L^2(\Omega)]^2, \quad E = [H_0^1(\Omega)]^2, \quad \text{and} \quad \Pi = [(H_0^1(\Omega) \cap H^2(\Omega))]^2.$$

The norm and inner-product of H or the component space $L^2(\Omega)$ will be denoted by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$, respectively. The norm of $L^p(\Omega)$ or $[L^p(\Omega)]^2$ will be denoted by $\|\cdot\|_{L^p}$ if $p \neq 2$. By the Poincaré inequality and the homogeneous Dirichlet boundary condition (1.4), there is a constant $\gamma > 0$ such that

$$\|\nabla \varphi\|^2 \geq \gamma \|\varphi\|^2, \quad \text{for } \varphi \in H_0^1(\Omega) \text{ or } E, \tag{1.6}$$

and we shall take $\|\nabla \varphi\|$ to be the (equivalent) norm $\|\varphi\|_E$ of the space E or the norm $\|\varphi\|_{H_0^1}$ of the component space $H_0^1(\Omega)$. We use $|\cdot|$ to denote an absolute value or a vector norm in a Euclidean space.

Define

$$\begin{aligned} H_+ &= \{\varphi(\cdot) = (\varphi_1, \varphi_2) \in H : \varphi_i(x) \geq 0,\, x \in \Omega,\, i = 1, 2\}, \\ E_+ &= \{\varphi(\cdot) = (\varphi_1, \varphi_2) \in E : \varphi_i(x) \geq 0,\, x \in \Omega,\, i = 1, 2\}. \end{aligned} \tag{1.7}$$

Then H_+ and E_+ are the half-spaces in H and E, respectively,

It can be checked easily that, by the Lumer-Phillips theorem and the analytic semigroup generation theorem [11], the linear differential operator

$$A = \begin{pmatrix} d_1\Delta & 0 \\ 0 & d_2\Delta \end{pmatrix} : D(A)(= \Pi) \longrightarrow H \tag{1.8}$$

is the generator of an analytic C_0-semigroup on the Hilbert space H, which will be denoted by $\{e^{At}, t \geq 0\}$. By the fact that $H_0^1(\Omega) \hookrightarrow L^6(\Omega) \hookrightarrow L^4(\Omega)$ is a chain of continuous embeddings for $n \leq 3$ and using the Hölder inequality,

$$\|uv\| \leq \|u\|_{L^4}\|v\|_{L^4}, \quad \text{for } u, v \in L^4(\Omega),$$

one can verify that the nonlinear mapping

$$f(g) = \begin{pmatrix} uv - \frac{bu}{1+\mu u} \\ a - uv \end{pmatrix} : E_+ \longrightarrow H, \tag{1.9}$$

where $g = (u, v)$, is well defined on E and the mapping f is locally Lipschitz continuous. Thus the initial-boundary value problem (1.2)–(1.5) is formulated into an initial value problem of the abstract evolutionary equation,

$$\begin{aligned} &\frac{dg}{dt} = Ag + f(g), \quad t > 0, \\ &g(0) = g_0 = \operatorname{col}(u_0, v_0). \end{aligned} \tag{1.10}$$

where $g(t) = \operatorname{col}(u(t, \cdot), v(t, \cdot))$, simply written as $(u(t, \cdot), v(t, \cdot))$. We shall accordingly write $g_0 = (u_0, v_0)$.

The following proposition will be used in proving the existence of a weak solution to this initial value problem. Its proof is seen in [2, Theorem II.1.4].

Proposition 1.1. *Consider the Banach space*

$$W(0, \tau) = \left\{\zeta(\cdot) : \zeta \in L^2(0, \tau; E) \text{ and } \partial_t\zeta \in L^2(0, \tau; E^*)\right\} \tag{1.11}$$

with the norm

$$\|\zeta\|_W = \|\zeta\|_{L^2(0,\tau;E)} + \|\partial_t\zeta\|_{L^2(0,\tau;E^*)}.$$

Then the following statements hold:

(a) *The embedding $W(0, \tau) \hookrightarrow L^2(0, \tau; H)$ is compact.*

(b) *If $\zeta \in W(0, \tau)$, then it coincides with a function in $C([0, \tau]; H)$ for a.e. $t \in [0, \tau]$.*

(c) *If* $\zeta, \xi \in W(0,\tau)$*, then the function* $t \to \langle \zeta(t), \xi(t) \rangle_H$ *is absolutely continuous on* $[0,\tau]$ *and*

$$\frac{d}{dt}\langle \zeta(t), \xi(t) \rangle = \left(\frac{d\zeta}{dt}, \xi(t) \right) + \left(\zeta(t), \frac{d\xi}{dt} \right), \ a.e.\, t \in [0,\tau],$$

where $(\cdot,\cdot)$ *is the* (E^*, E) *dual product.*

By conducting *a priori* estimates on the Galerkin approximate solutions of the initial value problem (1.10) and through extracting the weak and weak* convergent subsequences in the appropriate spaces, we can prove the local and then global existence and uniqueness of the weak solution $g(t)$ of (1.10) in the next section, and we can prove the continuous dependence of the solutions on the initial data and the regularity properties satisfied by the weak solution. Therefore, the weak solutions of (1.10) for all initial data in H form a semiflow in the phase space H.

We refer to [11, 13] and many references therein for the concepts and basic facts in the theory of infinite dimensional dynamical systems.

Definition 1.1. Let $\{S(t)\}_{t\geq 0}$ be a semiflow on a Banach space $\mathcal{X}$. A bounded subset B_0 of $\mathcal{X}$ is called an *absorbing set* in $\mathcal{X}$ if, for any bounded subset $B \subset \mathcal{X}$, there is a finite time $t_0 \geq 0$ depending on B such that $S(t)B \subset B_0$ for all $t \geq t_0$.

Definition 1.2. A semiflow $\{S(t)\}_{t\geq 0}$ on a Banach space $\mathcal{X}$ is called *asymptotically compact* if for any bounded sequences $\{x_n\}$ in $\mathcal{X}$ and $\{t_n\} \subset (0,\infty)$ with $t_n \to \infty$, there exist subsequences $\{x_{n_k}\}$ of $\{u_n\}$ and $\{t_{n_k}\}$ of $\{t_n\}$, such that $\lim_{k\to\infty} S(t_{n_k})x_{n_k}$ exists in $\mathcal{X}$.

Definition 1.3. Let $\{S(t)\}_{t\geq 0}$ be a semiflow on a Banach space $\mathcal{X}$. A subset $\mathscr{A}$ of $\mathcal{X}$ is called a *global attractor* for this semiflow, if the following conditions are satisfied:

(i) $\mathscr{A}$ is a nonempty, compact and invariant subset of $\mathcal{X}$ in the sense that

$$S(t)\mathscr{A} = \mathscr{A} \quad \text{for any } \ t \geq 0.$$

(ii) $\mathscr{A}$ attracts any bounded set B of $\mathcal{X}$ in terms of the Hausdorff distance, i.e.

$$\operatorname{dist}_{\mathcal{X}}(S(t)B, \mathscr{A}) = \sup_{x\in B} \inf_{y\in\mathscr{A}} \|S(t)x - y\|_{\mathcal{X}} \to 0, \ \text{ as } t \to \infty.$$

The following proposition states concisely the basic result on the existence of a global attractor for a semiflow, cf. [11, 13].

Proposition 1.2. *Let* $\{S(t)\}_{t\geq 0}$ *be a semiflow on a Banach space or an invariant region* $\mathcal{X}$ *in it. If the following conditions are satisfied*:

(i) $\{S(t)\}_{t\geq 0}$ *has a bounded absorbing set* B_0 *in* $\mathcal{X}$*, and*

(ii) $\{S(t)\}_{t\geq 0}$ *is asymptotically compact in* $\mathcal{X}$*,*

then there exists a global attractor $\mathscr{A}$ *in* $\mathcal{X}$ *for this semiflow, which is given by*

$$\mathscr{A} = \omega(B_0) \overset{\text{def}}{=} \bigcap_{\tau\geq 0} Cl_{\mathcal{X}} \bigcup_{t\geq\tau} (S(t)B_0).$$

In Section 2 we prove the local existence and uniqueness of the weak solutions of the evolutionary equation (1.10) and in Section 3 we shall prove the global existence of the weak solutions and the absorbing property of this solution semiflow. In

Section 4 we shall prove the asymptotic compactness of this solutions semiflow and show the existence of a global attractor in the space H for this semiflow. In Section 5 we show that the global attractor has a finite Hausdorff dimension and a finite fractal dimension. In Section 6 we prove further regularity and attraction properties of the global attractor.

2. The local existence of weak solutions

In this paper, we shall write $u(t,x), v(t,x)$ simply as $u(t), v(t)$ or even as u, v. Similarly for other functions of (t,x).

The local existence and uniqueness of the solution to a system of multi-component reaction-diffusion equations such as the IVP (1.10) with certain regularity requirement is not a trivial issue. There are usually two different approaches to get a solution in a Sobolev space. One approach is the mild solutions provided by the "variation-of-constant formula" involving the linear semigroup $\{e^{At}\}_{t\geq 0}$, but the parabolic theory of mild solutions requires that $g_0 \in E$ instead of $g_0 \in H$ assumed in this work. The other approach is the weak solutions through the Galerkin approximations and the Lions-Magenes type of compactness treatment, cf. [2, 5].

Definition 2.1. A function $g(t,x), (t,x) \in [0,\tau] \times \Omega$, is called a weak solution to the initial value problem (IVP) of the parabolic evolutionary equation (1.10), if the following two conditions are satisfied:

(i) $\frac{d}{dt}(g,\phi) = (Ag,\phi) + (f(g),\phi)$ is satisfied for a.e. $t \in [0,\tau]$ and for any $\phi \in E$, where $(\cdot,\cdot)$ stands for the (E^*, E) dual product.

(ii) $g \in L^2(0,\tau;E) \cap C_w([0,\tau];H)$, where $C_w([0,\tau];H)$ denotes the space of weakly continuous functions on $[0,\tau]$ valued in H, such that $g(0) = g_0$.

Lemma 2.1. *For any given initial datum $g_0 \in H_+$, there exists a unique, local, weak solution $g(t) = (u(t), v(t))$, $t \in [0,\tau]$ for some $\tau > 0$, of the evolutionary equation* (1.10) *such that $g(0) = g_0$, which satisfies*

$$g \in C([0,\tau];H) \cap C^1((0,\tau);H) \cap L^2(0,\tau;E). \tag{2.1}$$

Proof. Using the orthonormal basis of eigenfunctions $\{e_j(x)\}_{j=1}^{\infty}$ of the Laplace operator with the homogeneous Dirichlet boundary condition:

$$\Delta e_j + \lambda_j e_j = 0 \text{ in } \Omega, \quad e_j|_{\partial\Omega} = 0, \quad j = 1, 2, \cdots, n, \cdots,$$

we consider the solution

$$g_m(t,x) = \sum_{j=1}^{m} q_j^m(t) e_j(x), \quad t \in [0,\tau],\ x \in \Omega, \tag{2.2}$$

of the approximate system

$$\begin{aligned} \frac{\partial g_m}{\partial t} &= A g_m + P_m f(g_m), \quad t > 0, \\ g_m(0) &= P_m\, g_0 \in H_m, \end{aligned} \tag{2.3}$$

where each $q_j^m(t)$ for $j = 1, \cdots, m$ is a three-dimensional vector function of t only, corresponding to the three unknowns u, v, and w, and $P_m : H \to H_m =$

Span$\{e_1, \cdots, e_m\}$ is the orthogonal projection. Note that for each given integer $m \geq 1$, (2.3) can be written as an IVP of a system of ODEs, whose unknown is a $3m$-dimensional vector function of all the coefficient functions of time t in the expansion of $g_m(t,x)$, namely,

$$q^m(t) = \mathrm{col}\,(q^m_{ju}(t), q^m_{jv}(t);\ j = 1, \cdots, m).$$

The IVP of this ODE system (2.3) can be written as

$$\begin{aligned} \frac{dq^m}{dt} &= \Lambda_m q^m(t) + f_m(q^m(t)),\ t > 0, \\ q^m(0) &= \mathrm{col}(P_m u_{0j},\ P_m v_{0j};\ j = 1, \cdots, m). \end{aligned} \tag{2.4}$$

Note that Λ_m is a matrix and f_m is a $2m$-dimensional vector of a quadratic polynomial and a continuous rational function of $2m$-variables, which is certainly a locally Lipschitz continuous vector function in $\mathbb{R}^{2m}$. Thus the solution of the initial value problem (2.4) exists uniquely on a time interval $[0, \tau_m]$, for some $\tau_m > 0$. Substituting all the components of this solution $q^m(t)$ into (2.2), we obtain a unique local solution $g_m(t,x)$ of the initial value problem (2.3), for any $m \geq 1$.

By the multiplier method we can conduct *a priori* estimates based on

$$\frac{1}{2}\|g_m(t)\|^2_{H_m} + \langle \boldsymbol{d}\nabla g_m(t), \nabla g_m(t)\rangle_{H_m} = \langle P_m f(g_m(t)), g_m(t)\rangle_{H_m},\ t \in [0, \tau_m],$$

where $\boldsymbol{d} = \mathrm{diag}(d_1, d_2, d_3)$ is a diagonal matrix. These estimates are similar to what we shall present in Lemma 3.1 in the next section. Note that $\|g_m(0)\| = \|P_m g_0\| \leq \|g_0\|$ for all $m \geq 1$. It follows that (as implied by the proof of Lemma 3.1) we can make $\tau_m = \tau$ for some $\tau > 0$ and for $m \geq 1$ such that

$$\{g_m\}_{m=1}^\infty \text{ is a bounded sequence in } L^2(0,\tau;E) \cap L^\infty(0,\tau;H),$$

and, since $A : E \to E^*$ is a bounded linear operator,

$$\{Ag_m\}_{m=1}^\infty \text{ is a bounded sequence in } L^2(0,\tau;E^*), \quad \text{where } E^* = [H^{-1}(\Omega)]^3.$$

Since $f : E_+ \to H$ is continuous,

$$\{P_m f(g_m)\}_{m=1}^\infty \text{ is a bounded sequence in } L^2(0,\tau;H) \subset L^2(0,\tau;E^*). \tag{2.5}$$

Therefore, by taking subsequences (which we will always relabel as the same as the original one), there exist limit functions

$$g(t,\cdot) \in L^2(0,\tau;E) \cap L^\infty(0,\tau;H) \quad \text{and} \quad \Phi(t,\cdot) \in L^2(0,\tau;H) \tag{2.6}$$

such that

$$\begin{aligned} g_m &\longrightarrow g \ \text{ weakly in } L^2(0,\tau;E), \\ g_m &\longrightarrow g \ \text{ weak* in } L^\infty(0,\tau;H), \\ Ag_m &\longrightarrow Ag \ \text{ weakly in } L^2(0,\tau;E^*), \end{aligned} \tag{2.7}$$

and

$$P_m f(g_m) \longrightarrow \Phi \ \text{ weakly in } L^2(0,\tau;H), \tag{2.8}$$

as $m \to \infty$. To estimate the (distributional) time derivative sequence $\{\partial_t g_m\}_{m=1}^{\infty}$, we take the supremum of the (E^*, E) dual product of the equation (2.3) with any $\eta \in E$ and use the fact that $(h, \eta) = \langle h, \eta \rangle$ for any $h \in H$ to obtain

$$\|\partial_t g_m(t)\|_{E^*} \le C \left(\|A g_m(t)\|_{E^*} + \|P_m f(g_m(t))\|_H \right), \quad t \in [0, \tau],\, m \ge 1,$$

where C is a uniform constant for all $m \ge 1$. Thus by the boundedness in (2.7) and (2.8) it holds that

$$\{\partial_t g_m\}_{m=1}^{\infty} \text{ is a bounded sequence in } L^2(0, \tau; E^*),$$

and, by further extracting of a subsequence if necessary, it follows from the uniqueness of distributional time derivative that

$$\partial_t g_m \longrightarrow \partial_t g \text{ weakly in } L^2(0, \tau; E^*), \text{ as } m \to \infty. \tag{2.9}$$

In order to show that the limit function g is a weak solution to the IVP (1.10), we need to show $\Phi = f(g)$. By Proposition 1.1, item (a), the boundedness of $\{g_m\}$ in $L^2(0, \tau; E)$ and $\{\partial_t g_m\}$ in $L^2(0, \tau; E^*)$ implies that (in the sense of subsequence extraction)

$$g_m \longrightarrow g \text{ strongly in } L^2(0, \tau; H), \text{ as } m \to \infty. \tag{2.10}$$

Consequently, there exists a subsequence such that

$$g_m(t, x) \longrightarrow g(t, x) \text{ for a.e. } (t, x) \in [0, \tau] \times \Omega, \text{ as } m \to \infty. \tag{2.11}$$

Due to the continuity of the mapping f, we have

$$f(g_m(t, x)) \longrightarrow f(g(t, x)) \text{ for a.e. } (t, x) \in [0, \tau] \times \Omega, \text{ as } m \to \infty. \tag{2.12}$$

According to [5, Lemma I.1.3] or [2, Lemma II.1.2] and by the triangle inequality in terms of the H-norm, the two facts (2.5) and (2.12) guarantee that

$$P_m f(g_m) \longrightarrow f(g) \text{ weakly in } L^2(0, \tau; H), \text{ as } m \to \infty. \tag{2.13}$$

By the uniqueness, (2.8) and (2.13) imply that $\Phi = f(g)$ in $L^2(0, \tau; H)$.

With (2.7), (2.9) and (2.13), by taking limit of the integral of the weak version of (2.3) with any given $\phi \in L^2(0, \tau; E)$,

$$\int_0^\tau \left(\frac{\partial g_m}{\partial t}, \phi \right) dt = \int_0^\tau \left[(A g_m, \phi) + (P_m f(g_m), \phi) \right] dt, \text{ as } m \to \infty,$$

we obtain

$$\int_0^\tau \left(\frac{\partial g}{\partial t}, \phi \right) dt = \int_0^\tau \left[(A g, \phi) + (f(g), \phi) \right] dt, \text{ for any } \phi \in L^2(0, \tau; E). \tag{2.14}$$

Let $\phi \in E$ be any constant function. Then we can use the property of Lebesgue points for each integral in (2.14) to obtain

$$\frac{d}{dt}(g, \phi) = (A g, \phi) + (f(g), \phi), \text{ for a.e. } t \in [0, \tau] \text{ and for any } \phi \in E. \tag{2.15}$$

Next, $\partial_t g \in L^2(0, \tau; E^*)$ shown in (2.9) implies that $g \in C_w([0, \tau]; E^*)$. Since the embedding $H \hookrightarrow E^*$ is continuous, this $g \in C_w([0, \tau]; E^*)$ and $g \in L^\infty(0, \tau; H)$ shown in (2.7) imply that

$$g \in C_w([0, \tau]; H).$$

due to [2, Theorem II.1.7 and Remark II.1.2]. Now we show that $g(0) = g_0$. By Proposition 1.1, item (c), for any $\phi \in C^1([0,\tau];E)$ with $\phi(\tau) = 0$, we have

$$\int_0^\tau (-g, \partial_t \phi)\, dt = \int_0^\tau [(Ag, \phi) + (f(g), \phi)]\, dt + \langle g(0), \phi(0) \rangle,$$

and

$$\int_0^\tau (-g_m, \partial_t \phi)\, dt = \int_0^\tau [(Ag_m, \phi) + (P_m f(g_m), \phi)]\, dt + \langle P_m g_0, \phi(0) \rangle.$$

Take the limit of the last equality as $m \to \infty$. Since $P_m g_0 \to g_0$ in H, we obtain $\langle g(0), \phi(0) \rangle = \langle g_0, \phi(0) \rangle$ for any $\phi(0) \in E$. The denseness of E in H implies that $g(0) = g_0$ in H. By Definition 2.1, we conclude that the limit function g is a weak solution to the initial value problem (1.10).

The uniqueness of weak solution can be shown by estimating the difference of any two possible weak solutions with the same initial value g_0 through the weak version of the evolutionary equation (or the variation-of-constant formula) and the Gronwall inequality.

By Proposition 1.1, item (b), and the fact that the weak solution $g \in W(0,\tau)$, the space defined in (1.11), we see that $g \in C([0,\tau];H)$, which also infers the continuous dependence of the weak solution $g(t) = g(t; g_0)$ on g_0 for any $t \in [0,\tau]$.

Moreover, since $g \in L^2(0,\tau;E)$, for any $t \in (0,\tau)$ there exists an earlier time $t_0 \in (0,t)$ such that $g(t_0) \in E$. Then the weak solution coincides with the strong solution expressed by the mild solution on $[t_0,\tau]$, cf. [2, 11], which turns out to be continuously differentiable in time at t strongly in H, cf. [11, Theorem 48.5]. Thus we have shown $g \in C^1((0,\tau);H)$ and the weak solution g satisfies the properties specified in (2.1). □

3. Absorbing property

In this section, we shall prove the global existence of the weak solutions and investigate the absorbing properties of the solution semiflow.

The following proposition [2, Theorem II.4.2] provides the necessary and sufficient conditions for a semilinear parabolic system

$$\frac{\partial \xi}{\partial t} = A_0\, \Delta \xi + \psi(\xi) + \theta(x), \quad t > 0, \ \ x \in \Omega, \tag{3.1}$$

on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ with the homogeneous Dirichlet (or Neumann) boundary condition to have the positive cone $\mathbb{R}^N_+$ as an invariant region [12]. Here A_0 is an $N \times N$ symmetric and positive definite matrix, $\xi(t,x) = \mathrm{col}\,(\xi_1, \xi_2, \cdots \xi_N), \theta(x) = \mathrm{col}\,(\theta_1, \theta_2, \cdots, \theta_N) \in [L^2(\Omega)]^N$ is a given vector function, and $\psi = \mathrm{col}\,(\psi_1, \psi_2, \cdots, \psi_N) : [H_0^1(\Omega)]^N (\text{or}\, [H^1(\Omega)]^N) \to [L^2(\Omega)]^N$ is locally Lipschitz continuous.

Proposition 3.1. *The positive cone $\mathbb{R}^N_+ = \{\xi \in \mathbb{R}^N : \xi_i \geq 0,\, i = 1, \cdots, N\}$ is an invariant region for* (3.1) *if and only if the following two conditions are satisfied:*

(i) *A_0 is a diagonal matrix; and*

(ii) *for every $i = 1, 2, \cdots, N$, it holds that*

$$\psi_i(\xi_1, \cdots, \xi_{i-1}, 0, \xi_{i+1}, \cdots, \xi_N) + \theta_i(x) \geq 0, \tag{3.2}$$

for any $x \in \Omega$ and $\xi_j \geq 0, j \neq i$.

Lemma 3.1. *For any initial datum* $g_0 = (u_0, v_0) \in H_+$, *there exists a unique, global, weak solution* $g(t) = (u(t), v(t))$, $t \in [0, \infty)$, *of the initial value problem of the evolutionary equation* (1.10) *and it becomes a strong solution on the time interval* $(0, \infty)$. *Moreover, the mapping* $g : (t, g_0) \longmapsto g(t; g_0)$ *is jointly continuous on* $[0, \infty) \times H_+$.

Proof. By checking the conditions of Proposition 3.1 it is easy to see that the cone $\mathbb{R}^2_+$ is an invariant region for the reaction-diffusion problem (1.2)–(1.5). For any $g_0 = (u_0, v_0) \in H_+$, the local weak solution will also be confined in the invariant conic region H_+. By Lemma 2.1, there is a maximal interval of existence, denoted by $I_{max} = [0, \tau_{max}), \tau_{max} > 0$, for the corresponding weak solution $g(t) = g(t; g_0)$ such that for any $[0, \tau] \subset I_{max}$,

$$g \in C([0, \tau]; H_+) \cap C^1((0, \tau); H_+) \cap L^2(0, \tau; E_+). \tag{3.3}$$

Taking the inner-product $\langle (1.3), v(t, \cdot) \rangle$ (meaning for each term of (1.3)), we get

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 + d_2\|\nabla v\|^2 = \int_\Omega av\,dx - \int_\Omega uv^2\,dx \le \frac{a^2|\Omega|}{2d_2\gamma} + \frac{d_2\gamma}{2}\int_\Omega v^2\,dx.$$

Hence, by the Poincaré inequality (1.6),

$$\frac{d}{dt}\|v\|^2 + d_2\gamma\|v\|^2 \le \frac{d}{dt}\|v\|^2 + d_2\|\nabla v\|^2 \le \frac{a^2|\Omega|}{d_2\gamma}. \tag{3.4}$$

By the Gronwall inequality, we obtain

$$\|v(t)\|^2 \le e^{-d_2\gamma t}\|v_0\|^2 + \frac{a^2|\Omega|}{(d_2\gamma)^2}, \quad \text{for } t \in [0, T_{max}). \tag{3.5}$$

In order to deal with the u-component, we can add up (1.2) and (1.3) and get the following equation satisfied by $y(t, x) = u(t, x) + v(t, x)$,

$$\frac{\partial y}{\partial t} = d_1\Delta y + a + (d_2 - d_1)\Delta v - \frac{bu}{1 + \mu u}. \tag{3.6}$$

Taking the inner-product $\langle (3.6), y(t, \cdot) \rangle$ we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|y\|^2 + d_1\|\nabla y\|^2 &= \int_\Omega ay\,dx + (d_2 - d_1)\int_\Omega y\Delta v\,dx - \int_\Omega \frac{buy}{1 + \mu u}\,dx \\ &\le \frac{a^2|\Omega|}{d_1\gamma} + \frac{d_1\gamma}{4}\|y\|^2 + \frac{|d_1 - d_2|^2}{d_1}\|\nabla v\|^2 + \frac{d_1}{4}\|\nabla y\|^2 \\ &\quad + \frac{1}{d_1\gamma}\int_\Omega \left(\frac{bu}{1 + \mu u}\right)^2 dx + \frac{d_2\gamma}{4}\|y\|^2,\end{aligned}$$

where it is easy to find that $\psi(s) = \frac{bs}{1+\mu s}$ satisfies $\psi'(s) > 0$ for $s \ge 0$, and

$$\sup_{s\ge 0} |\psi(s)|^2 = \lim_{s\to\infty} |\psi(s)|^2 = \frac{b^2}{\mu^2}.$$

Then we can deduce that, for $t \in [0, T_{max})$,

$$\begin{aligned}\frac{d}{dt}\|y\|^2 + \frac{1}{2}d_1\gamma\|y\|^2 &\le \frac{d}{dt}\|y\|^2 + \frac{1}{2}d_1\|\nabla y\|^2 \\ &\le \left(2a^2 + \frac{2b^2}{\mu^2}\right)\frac{|\Omega|}{d_1\gamma} + \frac{2|d_1 - d_2|^2}{d_1}\|\nabla v\|^2.\end{aligned} \tag{3.7}$$

Integration of the above inequality yields the estimate

$$\|y(t)\|^2 \le \|u_0 + v_0\|^2 + \frac{t|\Omega|}{d_1\gamma}\left(2a^2 + \frac{2b^2}{\mu^2}\right) + \frac{2|d_1 - d_2|^2}{d_1}\int_0^t \|\nabla v(s)\|^2\, ds,$$

for $t \in [0, T_{max})$. From (3.4) we see that

$$d_2 \int_0^t \|\nabla v(s)\|^2\, ds \le \|v_0\|^2 + \frac{a^2|\Omega|}{d_2\gamma} t.$$

Thus we obtain

$$\begin{aligned} \|y(t)\|^2 &\le \|u_0 + v_0\|^2 + \frac{2|d_1 - d_2|}{d_1 d_2}\|v_0\|^2 \\ &\quad + t|\Omega|\left[\frac{2}{d_1\gamma}\left(a^2 + \frac{b^2}{\mu^2}\right) + \frac{2a^2|d_1 - d_2|}{d_1 d_2^2 \gamma}\right], \quad t \in [0, T_{max}). \end{aligned} \tag{3.8}$$

Since $u(t) = y(t) - v(t)$, (3.5) and (3.8) imply that

$$\begin{aligned} \|u(t)\|^2 &\le \|u_0 + v_0\|^2 + \left(1 + \frac{2|d_1 - d_2|}{d_1 d_2}\right)\|v_0\|^2 + \frac{a^2|\Omega|}{(d_2\gamma)^2} \\ &\quad + t|\Omega|\left[\frac{2}{d_1\gamma}\left(a^2 + \frac{b^2}{\mu^2}\right) + \frac{2a^2|d_1 - d_2|}{d_1 d_2^2 \gamma}\right], \quad t \in [0, T_{max}). \end{aligned} \tag{3.9}$$

Therefore, for any $g_0 \in H_+$, the weak solution $g(t) = (u(t,\cdot), v(t,\cdot))$ of the initial value problem (1.10) will never blow up at any finite time so that $T_{max} = +\infty$. This proves the global existence and uniqueness of a weak solution of (1.10) for any initial data.

According to (3.3), for any $g_0 \in H_+$ and any $t > 0$, since the weak solution $g(\cdot) \in L^2(0, t; E_+)$, there is a time $t_0 \in (0, t)$ such that $g(t) \in E_+$. Then the variation-of-constant formula of the mild solution shows that this $g(t), t \in [t_0, \infty)$, turns out to be a strong solution, cf. [11, 13]. As a result, any weak solution with $g_0 \in H_+$ becomes a strong solution on the time interval $(0, \infty)$.

The strong continuity of the weak solution $g(t; g_0)$ on g_0 can be shown by making estimates on the equation satisfied by the difference of any two trajectories and a further argument of the joint continuity with respect to (t, g_0). Here the details are omitted. □

The family of all the global weak solutions $\{g(t\cdot; g_0) : g_0 \in H_+\}$ defines a semiflow on H_+, i.e.

$$S(t) : g_0 \longmapsto g(t; g_0), \quad g_0 \in H_+,\ t \ge 0, \tag{3.10}$$

which will be briefly called the *solution semiflow* of (1.10).

Lemma 3.2. *There exists an absorbing set B_0 in H_+ for the solution semiflow $\{S(t)\}_{t\ge 0}$ of* (1.10) *defined by* (3.10)*,*

$$B_0 = \left\{g \in H_+ : \|g\|^2 \le K_0\right\}, \tag{3.11}$$

where K_0 is a positive constant independent of initial data.

Proof. From (3.5) and Lemma 3.1 we see that for any initial status $g_0 \in H_+$,

$$\limsup_{t\to\infty} \|v(t)\|^2 < \rho_1 = \frac{a^2|\Omega|}{(d_2\gamma)^2} + 1. \tag{3.12}$$

From (3.7) we have the inequality

$$\begin{aligned}\frac{d}{dt}\left(e^{d_1\gamma t/2}\|y(t)\|^2\right) \le{}& \frac{2|d_1-d_2|^2}{d_1} e^{d_1\gamma t/2}\|\nabla v(t)\|^2 \\ &+ \left(a^2 + \frac{b^2}{\mu^2}\right)\frac{2|\Omega|}{d_1\gamma} e^{d_1\gamma t/2}.\end{aligned} \tag{3.13}$$

Integrating (3.13) over $[0,t]$, we obtain

$$\begin{aligned}\|y(t)\|^2 \le{}& e^{-d_1\gamma t/2}\|u_0+v_0\|^2 + \left(a^2 + \frac{b^2}{\mu^2}\right)\frac{4|\Omega|}{(d_1\gamma)^2} \\ &+ \frac{2|d_1-d_2|^2}{d_1}\int_0^t e^{-d_1\gamma(t-\tau)/2}\|\nabla v(\tau)\|^2\, d\tau.\end{aligned} \tag{3.14}$$

To treat the integral term in (3.14), we multiply the second inequality of (3.4) by $e^{d_1\gamma t/2}$ and then integrate it to get

$$\int_0^t \left(e^{d_1\gamma\tau/2}\frac{d}{d\tau}\|v(\tau)\|^2\right) d\tau + d_2\int_0^t e^{d_1\gamma\tau/2}\|\nabla v(\tau)\|^2\, d\tau \le \frac{2a^2|\Omega|}{d_1 d_2\gamma^2} e^{d_1\gamma t/2},$$

which implies that

$$\begin{aligned} d_2\int_0^t e^{d_1\gamma\tau/2}\|\nabla v(\tau)\|^2\, d\tau \le{}& \frac{2a^2|\Omega|}{d_1 d_2\gamma^2} e^{d_1\gamma t/2} - \int_0^t \left(e^{d_1\gamma\tau/2}\frac{d}{d\tau}\|v(\tau)\|^2\right) d\tau \\ ={}& \frac{2a^2|\Omega|}{d_1 d_2\gamma^2} e^{d_1\gamma t/2} - \left(e^{d_1\gamma t/2}\|v(t)\|^2 - \|v_0\|^2\right) + \frac{1}{2}d_1\gamma\int_0^t e^{d_1\gamma\tau/2}\|v(\tau)\|^2\, d\tau \\ \le{}& \frac{2a^2|\Omega|}{d_1 d_2\gamma^2} e^{d_1\gamma t/2} + \|v_0\|^2 + \frac{1}{2}d_1\gamma\int_0^t e^{d_1\gamma\tau/2}\left[e^{-d_2\gamma\tau}\|v_0\|^2 + \frac{a^2|\Omega|}{(d_2\gamma)^2}\right] d\tau \\ ={}& \left(\frac{2}{d_1} + \frac{1}{d_2}\right)\frac{a^2|\Omega|}{d_2\gamma^2} e^{d_1\gamma t/2} + \left(1 + \frac{1}{2}d_1\gamma\,\theta(t)\right)\|v_0\|^2, \quad t > 0,\end{aligned} \tag{3.15}$$

where (3.5) is used and

$$\theta(t) = \int_0^t e^{(d_1-2d_2)\gamma\,\tau/2}\, d\tau = \begin{cases} \frac{2}{|d_1-2d_2|\gamma}\, e^{(d_1-2d_2)\gamma t/2}, & \text{if } d_1 > 2d_2, \\ t, & \text{if } d_1 = 2d_2, \\ \frac{2}{|d_1-2d_2|\gamma}, & \text{if } d_1 < 2d_2. \end{cases} \tag{3.16}$$

Substituting (3.15) and (3.16) into (3.14), we find that

$$\begin{aligned}\|y(t)\|^2 \le{}& e^{-d_1\gamma t/2}\|u_0+v_0\|^2 + \left(a^2 + \frac{b^2}{\mu^2}\right)\frac{4|\Omega|}{(d_1\gamma)^2} \\ &+ \frac{2|d_1-d_2|^2}{d_1 d_2} e^{-d_1\gamma t/2}\left[\left(\frac{2}{d_1} + \frac{1}{d_2}\right)\frac{a^2|\Omega|}{d_2\gamma^2} e^{d_1\gamma t/2} + \left(1 + \frac{1}{2}d_1\gamma\,\theta(t)\right)\|v_0\|^2\right],\end{aligned} \tag{3.17}$$

where, as seen from (3.16),

$$0 \le e^{-d_1\gamma t/2}\left(1+\frac{1}{2}d_1\gamma\,\theta(t)\right) \to 0, \quad \text{as } t\to\infty.$$

From (3.17) and the above limit it follows that

$$\limsup_{t\to\infty} \|y(t)\|^2 < \rho_2, \tag{3.18}$$

with

$$\rho_2 = \left(a^2+\frac{b^2}{\mu^2}\right)\frac{4|\Omega|}{(d_1\gamma)^2} + \frac{2a^2|d_1-d_2|^2}{d_1 d_2^2\gamma^2}\left(\frac{2}{d_1}+\frac{1}{d_2}\right)|\Omega| + 1.$$

Finally, from (3.12) and (3.18) we can conclude that, for any initial data $g_0 = (u_0, v_0) \in H_+$, the weak solution $g(t; g_0)$ satisfies

$$\begin{aligned}\limsup_{t\to\infty} \|g(t)\|^2 &= \limsup_{t\to\infty} (\|u(t)\|^2 + \|v(t)\|^2) \\ &= \limsup_{t\to\infty} (\|y(t)-v(t)\|^2 + \|v(t)\|^2) < K_0 = 3\rho_1 + 2\rho_2.\end{aligned}$$

With this positive constant K_0, the set B_0 in (3.11) is an absorbing set for the solution semiflow of (1.10). The proof is completed. □

4. Asymptotic compactness and global attractor

In this section we shall prove that the solution semiflow $\{S(t)\}_{t\ge 0}$ of (1.10) is asymptotically compact in H_+ and has a global attractor. The following proposition is about the uniform Gronwall inequality, which is an instrumental tool in the analysis of asymptotic compactness, cf. [11, 13].

Proposition 4.1. *Let β, ζ, and h be nonnegative functions in $L^1_{loc}[0,\infty;\mathbb{R})$. Assume that β is absolutely continuous on $(0,\infty)$ and the following differential inequality is satisfied,*

$$\frac{d\beta}{dt} \le \zeta\beta + h, \quad \text{for } t > 0. \tag{4.1}$$

If there is a finite time $t_1 > 0$ and some $r > 0$ such that

$$\int_t^{t+r} \zeta(\tau)\,d\tau \le A, \quad \int_t^{t+r} \beta(\tau)\,d\tau \le B, \quad \text{and} \quad \int_t^{t+r} \beta(\tau)\,d\tau \le C, \tag{4.2}$$

for any $t > t_1$, where A, B, and C are some positive constants, then

$$\beta(t) \le \left(\frac{B}{r} + C\right)e^A, \quad \text{for any } t > t_1 + r.$$

Lemma 4.1. *There is a constants $M > 0$ such that*

$$\int_t^{t+1} \|g(\tau)\|^2\,d\tau = \int_t^{t+1} (\|u(\tau)\|^2 + \|v(\tau)\|^2)\,d\tau \le M, \tag{4.3}$$

for any $t > T_0$, $g_0 = (u_0, v_0) \in B_0$, where $T_0 = T_0(B_0)$ is a positive constant depending only on the absorbing ball B_0.

Proof. Note that the absorbing set B_0 shown in (3.11) absorbs itself. Thus there is a finite time $T_0 = T_0(B_0) > 0$ such that

$$S(t)B_0 \subset B_0, \quad \text{for any } t > T_0.$$

From the second inequality in (3.4) we see that

$$\begin{aligned}\int_t^{t+1} \|\nabla v(\tau)\|^2 \, d\tau &\le \frac{1}{d_2}\left(\|v(t)\|^2 + \frac{a^2|\Omega|}{d_2\gamma}\right) \\ &\le M_1 = \frac{1}{d_2}\left(K_0 + \frac{a^2|\Omega|}{d_2\gamma}\right), \quad \text{for } t > T_0,\ g_0 \in B_0.\end{aligned} \tag{4.4}$$

From the second inequality in (3.7) and (4.4) we see that

$$\begin{aligned}&\int_t^{t+1} \|\nabla y(\tau)\|^2 \, d\tau \\ &\le \frac{2}{d_1}\left[\|y(t)\|^2 + \left(a^2 + \frac{b^2}{\mu^2}\right)\frac{2|\Omega|}{d_1\gamma} + \frac{2|d_1 - d_2|}{d_1}\int_t^{t+1} \|\nabla v(\tau)\|^2 \, d\tau\right] \\ &\le M_2 = \frac{2}{d_1}\left[K_0 + \left(a^2 + \frac{b^2}{\mu^2}\right)\frac{2|\Omega|}{d_1\gamma} + \frac{2|d_1 - d_2|}{d_1 d_2}\left(K_0 + \frac{a^2|\Omega|}{d_2\gamma}\right)\right],\end{aligned} \tag{4.5}$$

for $t > T_0$, $g_0 \in B_0$. Since $u(t) = y(t) - v(t)$, (4.3) follows from (4.4) and (4.5) with the constant M given by

$$M = 3M_1 + 2M_2.$$

The proof is completed. □

According to Lemma 3.1, any weak solution $g(t; g_0)$, $t \ge 0$, becomes a strong solution on $(0, \infty)$ and we have

$$S(t)g_0 \in E_+ = [H_0^1(\Omega)]^2 \subset [L^6(\Omega)]^2 \subset [L^4(\Omega)]^2 \quad \text{for } t > 0.$$

There exists a constant $\eta > 0$ such that the following embedding inequality holds,

$$\|\nabla \varphi\|^2 \ge \eta \|\varphi\|_{L^4}^2, \quad \text{for } \varphi \in H_0^1(\Omega) \text{ or } E. \tag{4.6}$$

Lemma 4.2. *The solution semiflow $\{S(t)\}_{t \ge 0}$ of (1.10) is asymptotically compact in the invariant cone H_+ of the phase space H.*

Proof. Taking the L^2 inner-product $\langle (1.2), -\Delta u\rangle$, by the homogeneous Dirichlet boundary conditions, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + d_1\|\Delta u\|^2 &= -\int_\Omega uv\Delta u \, dx + \int_\Omega \frac{bu}{1 + \mu u}\Delta u \, dx \\ &\le \left(\frac{d_1}{2} + \frac{d_1}{2}\right)\|\Delta u\|^2 + \frac{1}{2d_1}\int_\Omega u^2 v^2 \, dx + \frac{b^2|\Omega|}{2d_1\mu^2}.\end{aligned}$$

By the Cauchy inequality and (4.6), it follows that, for $t > 0$,

$$\begin{aligned}\frac{d}{dt}\|\nabla u\|^2 &\le \frac{1}{d_1}\int_\Omega u^2 v^2 \, dx + \frac{b^2|\Omega|}{d_1\mu^2} \le \frac{1}{d_1}\|u^2\|\|v^2\| + \frac{b^2|\Omega|}{d_1\mu^2} \\ &= \frac{1}{d_1}\|u^2\|_{L^4}^2\|v^2\|_{L^4}^2 + \frac{b^2|\Omega|}{d_1\mu^2} \le \frac{1}{d_1\eta^2}\|\nabla u\|^2\|\nabla v\|^2 + \frac{b^2|\Omega|}{d_1\mu^2}.\end{aligned} \tag{4.7}$$

The above inequality (4.7) can be wriiten as

$$\frac{d}{dt}\beta_u \le \zeta_1 \beta_u + h_1, \quad t > 0, \tag{4.8}$$

where

$$\beta_u(t) = \|\nabla u(t)\|^2, \quad \zeta_1(t) = \frac{1}{d_1 \eta^2}\|\nabla v(t)\|^2, \quad \text{and} \quad h_1 = \frac{b^2|\Omega|}{d_1 \mu^2}.$$

By Lemma 4.1 and applying the uniform Gronwall inequality in Proposition 4.1 to (4.8), we obtain

$$\|\nabla u(t)\|^2 \le \left(M + \frac{b^2|\Omega|}{d_1\mu^2}\right)\exp\left(\frac{M}{d_1\eta^2}\right), \quad \text{for } t > T_0 + 1,\ g_0 \in B_0, \tag{4.9}$$

where M and T_0 are the constants shown in Lemma 4.1.

Then take the L^2 inner-product $\langle (1.3), -\Delta v\rangle$ to get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla v\|^2 + d_2\|\Delta v\|^2 &= -\int_\Omega a\Delta v\, dx + \int_\Omega uv\Delta u\, dx \\ &\le \left(\frac{d_2}{2} + \frac{d_2}{2}\right)\|\Delta v\|^2 + \frac{a^2|\Omega|}{2d_2} + \frac{1}{2d_2}\int_\Omega u^2v^2\, dx.\end{aligned}$$

Again by the Cauchy inequality and (4.6), it follows that, for $t > 0$,

$$\begin{aligned}\frac{d}{dt}\|\nabla v\|^2 &\le \frac{1}{d_2}\int_\Omega u^2v^2\, dx + \frac{a^2|\Omega|}{d_2} \le \frac{1}{d_2}\|u^2\|\|v^2\| + \frac{a^2|\Omega|}{d_2} \\ &= \frac{1}{d_2}\|u^2\|_{L^4}^2\|v^2\|_{L^4}^2 + \frac{a^2|\Omega|}{d_2} \le \frac{1}{d_2\eta^2}\|\nabla u\|^2\|\nabla v\|^2 + \frac{a^2|\Omega|}{d_2}.\end{aligned} \tag{4.10}$$

The above inequality (4.10) can be wriiten as

$$\frac{d}{dt}\beta_v \le \zeta_2\, \beta_v + h_2, \quad t > 0, \tag{4.11}$$

where

$$\beta_v(t) = \|\nabla v(t)\|^2, \quad \zeta_2(t) = \frac{1}{d_2 \eta^2}\|\nabla u(t)\|^2, \quad \text{and} \quad h_2 = \frac{a^2|\Omega|}{d_2}.$$

By Lemma 4.1 and applying the uniform Gronwall inequality in Proposition 4.1 to (4.11), we obtain

$$\|\nabla v(t)\|^2 \le \left(M + \frac{a^2|\Omega|}{d_2}\right)\exp\left(\frac{M}{d_2\eta^2}\right), \quad \text{for } t > T_0 + 1,\ g_0 \in B_0, \tag{4.12}$$

where M and T_0 are the constants shown in Lemma 4.1.

The inequalities (4.9) and (4.12) yield the following estimate of boundedness,

$$\|g(t)\|_E^2 = \|\nabla u(t)\|^2 + \|\nabla v(t)\|^2 \le M^*, \quad \text{for } t > T_0 + 1,\ g_0 \in B_0, \tag{4.13}$$

where

$$M^* = \left(M + \frac{b^2|\Omega|}{d_1\mu^2}\right)\exp\left(\frac{M}{d_1\eta^2}\right) + \left(M + \frac{a^2|\Omega|}{d_2}\right)\exp\left(\frac{M}{d_2\eta^2}\right).$$

The boundedness shown in (4.13) combined with the absorbing property shown in Lemma 3.2 confirms that, for any given bounded set $B \subset H_+$, there exists a finite time $T(B) > 0$ such that $\{S(t)B : t > T(B)\} \subset B_0$ and

$$\{S(t)B : t > T(B) + T_0 + 1\} \text{ is a bounded set in } E,$$

which in turn is a precompact set in H_+ due to that E is compactly embedded in H. Therefore, by Definition 1.2 the solution semiflow $\{S(t)\}_{t\geq 0}$ of (1.10) is asymptotically compact in H_+. □

Finally we can prove the main result on the existence of a global attractor for this semiflow $\{S(t)\}_{t\geq 0}$.

Theorem 4.1. *For any positive parameters d_1, d_2, a, b and μ in the reaction-diffusion system* (1.2)–(1.3) *with the Dirichlet boundary condition* (1.4)*, there exists a global attractor $\mathscr{A}$ in H_+ for the solution semiflow $\{S(t)\}_{t\geq 0}$ generated by* (1.10)*.*

Proof. Lemma 3.2 and Lemma 4.2 demonstrate that the two conditions in Proposition 1.2 are satisfied by this solution semiflow $\{S(t)\}_{t\geq 0}$ generated by (1.10), where we let $\mathfrak{X} = H_+$. Therefore, there exists a global attractor $\mathscr{A}$ in H_+ for this solution semiflow. □

Moreover, we can show that the global attractor $\mathscr{A}$ is actually a bounded set in the more regular space $[L^\infty(\Omega)]^2$.

Theorem 4.2. *The global attractor $\mathscr{A}$ of the solution semiflow of* (1.10) *is a bounded subset in $[L^\infty(\Omega)]^2$.*

Proof. By the (L^p, L^∞) regularity of the analytic C_0-semigroup $\{e^{At}\}_{t\geq 0}$, cf. [11, Theorem 38.10], one has $e^{At} : [L^p(\Omega)]^2 \to [L^\infty(\Omega)]^2$ for $t > 0$, and there is a constant $C(p) > 0$ such that

$$\|e^{At}\|_{\mathcal{L}(L^p, L^\infty)} \leq C(p)\, t^{-\frac{n}{2p}}, \quad t > 0, \quad \text{where } n = \dim \Omega. \tag{4.14}$$

Note that there is a constant steady state $\hat{g} = (\hat{u}, \hat{v}) \in \mathscr{A}$, where

$$\hat{u} = \frac{a}{b - a\mu}, \quad \hat{v} = b - a\mu,$$

and without loss of generality assuming that $b \neq a\mu$. Thus $f(\hat{g}) = 0$. By the variation-of-constant formula satisfied by the mild solutions, certainly valid for the strong solutions associated with any $g \in \mathscr{A}$ $(\subset E_+)$, we have, for $t \geq 0$,

$$\begin{aligned} \|S(t)g\|_{L^\infty} &\leq \|e^{At}\|_{\mathcal{L}(L^2, L^\infty)} \|g\| + \int_0^t \|e^{A(t-\sigma)}\|_{\mathcal{L}(L^2, L^\infty)} \|f(S(\sigma)g) - f(\hat{g})\|\, d\sigma \\ &\leq C(2) t^{-\frac{3}{4}} \|g\| + \int_0^t C(2)(t-\sigma)^{-\frac{3}{4}} L(\sqrt{M^*}) \|S(\sigma)g - \hat{g}\|_E\, d\sigma, \end{aligned} \tag{4.15}$$

where the space dimension $n \leq 3$, $C(2)$ is specified in (4.14) with $p = 2$, M^* is the constant in (4.13) and $L(\sqrt{M^*})$ is the Lipschitz constant of the nonlinear map f on the closed bounded ball in E centered at the origin and with radius $\sqrt{M^*}$. By the invariance of the global attractor $\mathscr{A}$, we have

$$\begin{aligned} \{S(t)\mathscr{A} : t \geq 0\} &= \mathscr{A} \subset B_0 = \{g \in H_+ : \|g\|^2 \leq K_0\} \subset H_+, \\ \{S(t)\mathscr{A} : t \geq 0\} &= \mathscr{A} \subset B_1 = \{g \in E_+ : \|g\|_E^2 \leq M^*\} \subset E_+. \end{aligned} \tag{4.16}$$

Then from (4.15) we get

$$\begin{aligned}\|S(t)g\|_{L^\infty} &\le C(2)\sqrt{K_0}t^{-\frac{3}{4}} + \int_0^t C(2)L(\sqrt{M^*})2\sqrt{M^*}(t-\sigma)^{-\frac{3}{4}}\,d\sigma \\ &= C(2)[\sqrt{K_0}\,t^{-\frac{3}{4}} + 8L(\sqrt{M^*})\sqrt{M^*}\,t^{\frac{1}{4}}], \quad \text{for } t > 0.\end{aligned} \tag{4.17}$$

Specifically one can take $t = 1$ in (4.17) and use the invariance $S(t)\mathscr{A} = \mathscr{A}$ to obtain

$$\|g\|_{L^\infty} \le C(2)(\sqrt{K_0} + 8\sqrt{M^*}L(\sqrt{M^*})), \quad \text{for any } g \in \mathscr{A}.$$

Thus the global attractor $\mathscr{A}$ is a bounded subset in $[L^\infty(\Omega)]^2$. □

5. Finite dimensionality of the global attractor

Now consider the Hausdorff dimension and fractal dimension of the global attractor $\mathscr{A}$ of the solution semiflow $\{S(t)\}_{t\ge 0}$ of (1.10) in H_+. Let $q_m = \limsup_{t\to\infty} q_m(t)$, where, cf. [13],

$$q_m(t) = \sup_{g_0\in\mathscr{A}} \sup_{\substack{g_i\in H, \|g_i\|=1 \\ i=1,\cdots,m}} \left(\frac{1}{t}\int_0^t \mathrm{Tr}\,(A + f'(S(\tau)g_0)) \circ Q_m(\tau)\,d\tau\right), \tag{5.1}$$

in which $Q_m(t)$ stands for the orthogonal projection of space H on the subspace spanned by $G_1(t), \cdots, G_m(t)$, with $G_i(t) = L(S(t), g_0)g_i, i = 1, \cdots, m$. Here $f'(S(\tau)g_0)$ is the Fréchet derivative of the map f at $S(\tau)g_0$, and $L(S(t), g_0)$ is the Fréchet derivative of the map $S(t)$ at g_0, with t fixed. The definitions of Hausdorff dimension and fractal dimension can be seen in [13, Chapter 5] as well as the following proposition.

Proposition 5.1. *If there is an integer m such that $q_m < 0$, then the Hausdorff dimension $d_H(\mathscr{A})$ and the fractal dimension $d_F(\mathscr{A})$ of $\mathscr{A}$ satisfy*

$$d_H(\mathscr{A}) \le m, \quad \text{and} \quad d_F(\mathscr{A}) \le m \max_{1\le j\le m-1}\left(1 + \frac{(q_j)_+}{|q_m|}\right) \le 2m. \tag{5.2}$$

It is standard to show that for any given $t > 0$, $S(t)$ on H_+ is Fréchet differentiable and and its Fréchet derivative at g_0 is given by

$$L(S(t), g_0)Z_0 \overset{\text{def}}{=} Z(t) = (\mathcal{U}(t), \mathcal{V}(t)),$$

for any $Z_0 = (\mathcal{U}_0, \mathcal{V}_0) \in H$, where $(\mathcal{U}(t), \mathcal{V}(t))$ is the weak solution of the following initial-boundary value problem of the variational system associated with the trajectory $\{S(t)g_0 : t \ge 0\}$,

$$\begin{aligned}&\frac{\partial \mathcal{U}}{\partial t} = d_1\Delta\mathcal{U} + v(t)\mathcal{U} + u(t)\mathcal{V} - \frac{b}{(1+\mu u)^2}\mathcal{U}, \\ &\frac{\partial \mathcal{V}}{\partial t} = d_2\Delta\mathcal{V} - v(t)\mathcal{U} - u(t)\mathcal{V}, \\ &\qquad \mathcal{U}\mid_{\partial\Omega} = \mathcal{V}\mid_{\partial\Omega} = 0, \quad t > 0, \\ &\qquad \mathcal{U}(0) = \mathcal{U}_0, \quad \mathcal{V}(0) = \mathcal{V}_0.\end{aligned} \tag{5.3}$$

Here $(u(t), v(t)) = g(t) = S(t)g_0$ is the weak solution of (1.10) satisfying the initial condition $g(0) = g_0$. The initial-boundary value problem (5.3) can be written as

$$\begin{aligned} \frac{dZ}{dt} &= (A+f'(S(t)g_0))Z, \quad t > 0, \\ Z(0) &= Z_0. \end{aligned} \tag{5.4}$$

Theorem 5.1. *The global attractors $\mathscr{A}$ for the solution semiflow $\{S(t)\}_{t\geq 0}$ of* (1.10) *has a finite Hausdorff dimesion and a finite fractal dimension.*

Proof. By Proposition 5.1, we shall estimate $\mathrm{Tr}\,(A + f'(S(\tau)g_0)) \circ Q_m(\tau)$. At any given time $\tau > 0$. Let $\{\varphi_j(\tau) : j = 1, \cdots, m\}$ be an H-orthonormal basis for the subspace

$$Q_m(\tau)H = \mathrm{Span}\,\{Z_1(\tau), \cdots, Z_m(\tau)\},$$

where $Z_1(t), \cdots, Z_m(t)$ are the weak solutions of (5.4) with the respective initial data $Z_{1,0}, \cdots, Z_{m,0}$ and, without loss of generality, assuming that $Z_{1,0}, \cdots, Z_{m,0}$ are linearly independent in H.

Note that $Z_1(t), \cdots, Z_m(t)$ turn out to be strong solutions for $t > 0$. By the Gram-Schmidt orthogonalization, $\varphi_j(\tau) = (\varphi_j^1(\tau), \varphi_j^2(\tau)) \in E$ for $\tau > 0$, $j = 1, \cdots, m$, and $\varphi_j(\tau)$ are strongly measurable in τ. Set $d_0 = \min\{d_1, d_2\}$. Then

$$\begin{aligned} &\mathrm{Tr}\,(A + f'(S(\tau)g_0) \circ Q_m(\tau) \\ &= \sum_{j=1}^{m} (\langle A\varphi_j(\tau), \varphi_j(\tau)\rangle + \langle f'(S(\tau)g_0)\varphi_j(\tau), \varphi_j(\tau)\rangle) \\ &\leq -\, d_0 \sum_{j=1}^{m} \|\nabla\varphi_j(\tau)\|^2 + J_1 + J_2, \quad \tau > 0, \end{aligned} \tag{5.5}$$

where the two terms J_1 and J_2 are respectively given by

$$\begin{aligned} J_1 &= -\sum_{j=1}^{m} \int_\Omega \left(\frac{b}{(1+\mu u(\tau))^2} - v(\tau) \right) |\varphi_j^1(\tau)|^2\, dx + \sum_{j=1}^{m} \int_\Omega u(t)\varphi_j^1(\tau)\varphi_j^2(\tau)\, dx \\ &\leq \sum_{j=1}^{m} \int_\Omega \left(v(\tau)|\varphi_j^1(\tau)|^2 + u(t)\varphi_j^1(\tau)\varphi_j^2(\tau) \right) dx, \end{aligned}$$

and

$$\begin{aligned} J_2 &= -\sum_{j=1}^{m} \int_\Omega v(\tau)\varphi_j^1(\tau)\varphi_j^2(\tau)\, dx - \sum_{j=1}^{m} \int_\Omega u(\tau)|\varphi_j^2(\tau)|^2\, dx \\ &\leq -\sum_{j=1}^{m} \int_\Omega v(\tau)\varphi_j^1(\tau)\varphi_j^2(\tau)\, dx. \end{aligned}$$

By the generalized Hölder inequality and the invariance of the global attractor $\mathscr{A}$, we get

$$\begin{aligned} J_1 &\le \sum_{j=1}^{m} \left(\|v(\tau)\| \|\varphi_j^1(\tau)\|_{L^4}^2 + \|u(\tau)\| \|\varphi_j^1(\tau)\|_{L^4} \|\varphi_j^2(\tau)\|_{L^4} \right) \\ &\le \sum_{j=1}^{m} \|S(\tau)g_0\| \left(\|\varphi_j^1(\tau)\|_{L^4}^2 + \|\varphi_j^1(\tau)\|_{L^4} \|\varphi_j^2(\tau)\|_{L^4} \right) \le 2\sqrt{K_0} \sum_{j=1}^{m} \|\varphi_j(\tau)\|_{L^4}^2, \end{aligned} \tag{5.6}$$

for any $\tau > 0$ and any $g_0 \in \mathscr{A}$. Now we apply the Garliardo-Nirenberg interpolation inequality, cf. [11, Theorem B.3],

$$\|\varphi\|_{W^{k,p}} \le C \|\varphi\|_{W^{m,q}}^{\theta} \|\varphi\|_{L^r}^{1-\theta}, \quad \text{for } \varphi \in W^{m,q}(\Omega), \tag{5.7}$$

provided that $p, q, r \ge 1, 0 < \theta < 1$, and

$$k - \frac{n}{p} \le \theta \left(m - \frac{n}{q} \right) - (1-\theta) \frac{n}{r}, \quad \text{where } n = \dim \Omega.$$

Here let $W^{k,p}(\Omega) = L^4(\Omega), W^{m,q}(\Omega) = H_0^1(\Omega), L^r(\Omega) = L^2(\Omega)$, and $\theta = n/4 \le 3/4$. It follows from (5.7) that

$$\|\varphi_j(\tau)\|_{L^4} \le C \|\nabla \varphi_j(\tau)\|^{\frac{n}{4}} \|\varphi_j(\tau)\|^{1-\frac{n}{4}} = C \|\nabla \varphi_j(\tau)\|^{\frac{n}{4}}, \ 1 \le j \le m, \tag{5.8}$$

since $\|\varphi_j(\tau)\| = 1, 1 \le j \le m$, where C is a positive constant. Substituting (5.8) into (5.6) we obtain

$$J_1 \le 2\sqrt{K_0}\, C^2 \sum_{j=1}^{m} \|\nabla \varphi_j(\tau)\|^{\frac{n}{2}}.$$

Similarly we can get

$$J_2 \le \sqrt{K_0} \sum_{j=1}^{m} \|\varphi_j(\tau)\|_{L^4}^2 \le \sqrt{K_0}\, C^2 \sum_{j=1}^{m} \|\nabla \varphi_j(\tau)\|^{\frac{n}{2}}.$$

Substituting the above two inequalities into (5.5), we obtain

$$\begin{aligned} &\operatorname{Tr}(A + f'(S(\tau)g_0) \circ Q_m(\tau) \\ &\le -d_0 \sum_{j=1}^{m} \|\nabla \varphi_j(\tau)\|^2 + 3\sqrt{K_0}\, C^2 \sum_{j=1}^{m} \|\nabla \varphi_j(\tau)\|^{\frac{n}{2}}. \end{aligned} \tag{5.9}$$

By Young's inequality, for $n \le 3$, we have

$$3\sqrt{K_0}\, C^2 \sum_{j=1}^{m} \|\nabla \varphi_j(\tau)\|^{\frac{n}{2}} \le \frac{d_0}{2} \sum_{j=1}^{m} \|\nabla \varphi_j(\tau)\|^2 + K(n)m,$$

where $K(n)$ is a positive constant depending only on $n = \dim \Omega$ and the involved constants d_0, K_0, and C. Hence, for any $\tau > 0$ and any $g_0 \in \mathscr{A}$, the following inequality is valid,

$$\operatorname{Tr}(A + f'(S(\tau)g_0) \circ Q_m(\tau) \le -\frac{d_0}{2} \sum_{j=1}^{m} \|\nabla \varphi_j(\tau)\|^2 + K(n)m.$$

According to the generalized Sobolev-Lieb-Thirring inequality [13, Appendix, Corollary 4.1], since $\{\varphi_1(\tau), \cdots, \varphi_m(\tau)\}$ is an orthonormal set in H, there exists a constant $\Psi > 0$ only depending on the shape and dimension of Ω such that

$$\sum_{j=1}^{m} \|\nabla \varphi_j(\tau)\|^2 \geq \frac{\Psi\, m^{1+\frac{2}{n}}}{|\Omega|^{\frac{2}{n}}}.$$

Therefore, for any $\tau > 0$ and any $g_0 \in \mathscr{A}$,

$$\mathrm{Tr}\,(A + f'(S(\tau)g_0) \circ Q_m(\tau) \leq -\frac{d_0 \Psi}{2|\Omega|^{\frac{2}{n}}} m^{1+\frac{2}{n}} + K(n)m. \tag{5.10}$$

Then we conclude that for ant $t > 0$,

$$\begin{aligned} q_m(t) &= \sup_{g_0 \in \mathscr{A}} \sup_{\substack{g_i \in H, \|g_i\|=1 \\ i=1,\cdots,m}} \left(\frac{1}{t} \int_0^t \mathrm{Tr}\,(A + f'(S(\tau)g_0)) \circ Q_m(\tau)\, d\tau \right) \\ &\leq -\frac{d_0 \Psi}{2|\Omega|^{\frac{2}{n}}} m^{1+\frac{2}{n}} + K(n)m. \end{aligned} \tag{5.11}$$

Consequently,

$$q_m = \limsup_{t \to \infty} q_m(t) \leq -\frac{d_0 \Psi}{2|\Omega|^{\frac{2}{n}}} m^{1+\frac{2}{n}} + K(n)m < 0,$$

if the positive integer m satisfies the following condition,

$$m - 1 \leq |\Omega| \left(\frac{2K(n)}{d_0 \Psi} \right)^{n/2} < m. \tag{5.12}$$

According to Proposition 5.1, we have shown that the Hausdorff dimension and the fractal dimension of the global attractor $\mathscr{A}$ are finite with the upper bounds given by

$$d_H(\mathscr{A}) \leq m \quad \text{and} \quad d_F(\mathscr{A}) \leq 2m,$$

respectively, where m is the positive integer satisfying (5.12). □

6. Regularity and attraction in E_+

In this section we show that the global attractor $\mathscr{A}$ of the solution semiflow of (1.10) is an (H_+, E_+) global attractor. This concept was introduced in [1].

Definition 6.1. Let $\mathcal{X}$ be a Banach space or a closed invariant cone in a Banach space and $\{\Sigma(t)\}_{t \geq 0}$ be a semiflow on $\mathcal{X}$. Let $\mathcal{Y}$ be a compactly imbedded subspace or sub-cone of $\mathcal{X}$. A subset $\mathcal{A}$ of $\mathcal{Y}$ is called an $(\mathcal{X}, \mathcal{Y})$ global attractor for this semiflow if $\mathcal{A}$ has the following properties,

(i) $\mathcal{A}$ is a nonempty, compact, and invariant set in $\mathcal{Y}$.

(ii) $\mathcal{A}$ attracts any bounded set $B \subset \mathcal{X}$ with respect to the $\mathcal{Y}$-norm, namely, there is a time $\tau = \tau(B)$ such that $\Sigma(t)B \subset \mathcal{Y}$ for $t > \tau$ and $dist_{\mathcal{Y}}(\Sigma(t)B, \mathcal{A}) \to 0$, as $t \to \infty$.

Lemma 6.1. *Let $\{g_m\}$ be a sequence in E such that $\{g_m\}$ converges to $g_0 \in E$ weakly in E and $\{g_m\}$ converges to g_0 strongly in H, as $m \to \infty$. Then*

$$\lim_{m\to\infty} S(t)g_m = S(t)g_0 \text{ strongly in } E,$$

where the convergence is uniform with respect to t in any given compact interval $[t_0, t_1] \subset (0, \infty)$.

The proof of this lemma is seen in [20, Lemma 4.2].

Theorem 6.1. *The global attractor $\mathscr{A}$ in H_+ for the solution semiflow $\{S(t)\}_{t\geq 0}$ of* (1.10) *is indeed an (H_+, E_+) global attractor.*

Proof. By (4.13) in the proof of Lemma 4.2 and that B_0 is an absorbing set for the solution semiflow $\{S(t)\}_{t\geq 0}$ of (1.10) in H_+, we find that

$$B_1 = \{\varphi \in E_+ : \|\varphi\|_E = \|\nabla\varphi\|^2 \leq M^*\} \tag{6.1}$$

as in (4.16) is an absorbing set for the semiflow $\{S(t)\}_{t\geq 0}$ in E_+. Indeed, for any E-bounded subset $B \subset E_+$, B must also be bounded in H_+ so that there is a finite time $T^0(B) \geq 0$ such that $S(t)B \subset B_0$ for all $t > T^0$. Then (4.13) implies that

$$S(t)B \subset B_1, \quad \text{for any } t > T^0 + T_0 + 1, \tag{6.2}$$

where T_0 has been specified in the proof of Lemma 4.2.

Next we show that the solution semiflow $\{S(t)\}_{t\geq 0}$ of (1.10) is asymptotically compact with respect to the strong topology in E. For any time sequence $\{t_n\}, t_n \to \infty$, and any E-bounded sequence $\{g_n\} \subset E_+$, there exists a finite time $t_0 \geq 0$ such that $S(t)\{g_n\} \subset B_0$, for any $t > t_0$. Then for an arbitrarily given $T > t_0 + T_0 + 1$, there is an integer $n_0 \geq 1$ such that $t_n > 2T$ for all $n > n_0$.

By Lemma 4.2, it holds that

$$\{S(t_n - T)g_n\}_{n>n_0} \text{ is a bounded set in } E_+.$$

Since E is a Hilbert space, there is an increasing sequence of integers $\{n_j\}_{j=1}^{\infty}$, with $n_1 > n_0$, such that

$$\lim_{j\to\infty} S(t_{n_j} - T)g_{n_j} = g^* \text{ weakly in } E.$$

By the compact imbedding $E \hookrightarrow H$, there is a subsequence of $\{n_j\}$, which is relabeled as the same as $\{n_j\}$, such that

$$\lim_{j\to\infty} S(t_{n_j} - T)g_{n_j} = g^* \text{ strongly in } H_+,$$

because H_+ is a closed invariant cone of H. Moreover, the uniqueness of limit implies that $g^* \in E_+$. Then by Lemma 6.1, we have the following convergence with respect to the E-norm,

$$\lim_{j\to\infty} S(t_{n_j})g_{n_j} = \lim_{j\to\infty} S(T)S(t_{n_j} - T)g_{n_j} = S(T)g^* \text{ strongly in } E_+. \tag{6.3}$$

This proves that $\{S(t)\}_{t\geq 0}$ is asymptotically compact on E_+.

Therefore, by Proposition 1.2, there exists a global attractor $\mathscr{A}_E$ for this solution semiflow $\{S(t)\}_{t\geq 0}$ in the invariant cone E_+. Note that B_1 attracts the H-absorbing ball B_0 in the E-norm as demonstrated earlier in this proof, we see that this global attractor $\mathscr{A}_E$ is an (H_+, E_+) global attractor according to Definition 6.1. Then the invariance and the boundedness of $\mathscr{A}$ in H and of $\mathscr{A}_E$ in E imply that

$$\mathscr{A} \text{ attracts } \mathscr{A}_E \text{ in } H_+, \text{ so that } \mathscr{A}_E \subset \mathscr{A},$$
$$\mathscr{A}_E \text{ attracts } \mathscr{A} \text{ in } E_+, \text{ so that } \mathscr{A} \subset \mathscr{A}_E.$$

Therefore, $\mathscr{A} = \mathscr{A}_E$ and, as a consequence, the global attractor $\mathscr{A}$ in H_+ is itself an (H_+, E_+) global attractor for this semiflow $\{S(t)\}_{t\geq 0}$. $\square$

As a remark, if we add a condition that $u_\Omega(t) = \frac{1}{\Omega}\int_\Omega u(t,x)\,dx = 0$ and $v_\Omega(t) = \frac{1}{\Omega}\int_\Omega v(t,x)\,dx = 0$, $t \geq 0$, then the results shown in this paper are also valid for the homogeneous Neumann boundary condition.

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