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EXISTENCE OF GENERALIZED TRAVELING WAVES IN TIME RECURRENT AND SPACE PERIODIC MONOSTABLE EQUATIONS*

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Abstract This paper is concerned with the extension of the concepts and theories of traveling wave solutions of time and space periodic monostable equations to time recurrent and space periodic ones. It first introduces the concept of generalized traveling wave solutions of time recurrent and space periodic monostable equations, which extends the concept of periodic traveling wave solutions of time and space periodic monostable equations to time recurrent and space periodic ones. It then proves that in the direction of any unit vector ξ, there is $c^*(\xi)$ such that for any $c > c^*(\xi)$, a generalized traveling wave solution in the direction of ξ with averaged propagation speed c exists. It also proves that if the time recurrent and space periodic monostable equation is indeed time periodic, then $c^*(\xi)$ is the minimal wave speed in the direction of ξ and the generalized traveling wave solution in the direction of ξ with averaged speed $c > c^*(\xi)$ is a periodic traveling wave solution with speed c, which recovers the existing results on the existence of periodic traveling wave solutions in the direction of ξ with speed greater than the minimal speed in that direction.

Keywords Monostable equation, generalized traveling wave solution, average propagating speed, spreading speed, sub-solution, super-solution, comparison principle, compact flow, recurrent function.

MSC(2000) 35B15, 35B35, 35B40, 35K55, 35K57, 92D25.

1. Introduction

The current paper is devoted to the study of traveling wave solutions of reaction diffusion equations of the form,

$$\frac{\partial u}{\partial t} = \Delta u + \Sigma_{i=1}^N a_i(t,x)\frac{\partial u}{\partial x_i} + uf(t,x,u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $a_i(t,x)$ $(i = 1, 2, \cdots, N)$ and $f(t,x,u)$ are recurrent and unique ergodic in t and periodic in x, and $f(t,x,u)$ is monostable in u. More precisely, $a_i(t,x)$ $(i = 1, 2, \cdots, N)$ and $f(t,x,u)$ satisfy the following three assumptions.

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(H1) *($\{a_i(t,x)\}_{i=1}^N, f(t,x,u), \frac{\partial f}{\partial u}(t,x,u)$) is recurrent and unique ergodic in t (see Definition 2.2 for detail), is periodic in x_j with period p_j ($j = 1, 2, \cdots, N$), and is globally Hölder continuous in t, x.*

(H2) *There are $\beta_0 > 0$ and $P_0 > 0$ such that $f(t,u) \leq -\beta_0$ for $t \in \mathbb{R}$ and $u \geq P_0$ and $\frac{\partial f}{\partial u}(t,u) \leq -\beta_0$ for $t \in \mathbb{R}$ and $u \geq 0$.*

(H3) *The principal Lyapunov exponent of the linearization of* (1.1) *at* 0 *is positive (see section 2.2 for the definition of principal Lyapunov exponent and basic properties).*

Observe that under the assumptions (H1)-(H3), the trivial solution $u \equiv 0$ of (1.1) is linearly unstable and (1.1) has a unique positive solution $u = u^+(t,x)$ which is recurrent and unique ergodic in t, periodic in x_i with period p_i, and is globally stable with respect to positive space periodic perturbations (see Proposition 3.1).

Equation (1.1) satisfying (H1)-(H3) is hence called a monostable equation. Here is a typical example of monostable equations,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u), \qquad x \in \mathbb{R}, \tag{1.2}$$

which was introduced in the pioneering papers of Fisher [9] and Kolmogorov, Petrowsky, Piscunov [23] for the evolutionary take-over of a habitat by a fitter genotype, where u is the frequency of one of two forms of a gene. Monostable equations are then also called Fisher's or KPP type equations in literature. They are used to model many other systems in biology and ecology (see [1], [2], [5]).

One of the central problems about monostable equations is the traveling wave problem. This problem is well understood for the classical Fisher or KPP equation (1.2). For example, Fisher in [9] found traveling wave solutions $u(t,x) = \phi(x - ct)$, ($\phi(-\infty) = 1, \phi(\infty) = 0$) of all speeds $c \geq 2$ and showed that there are no such traveling wave solutions of slower speed. He conjectured that the take-over occurs at the asymptotic speed 2. This conjecture was proved in [23] by Kolmogorov, Petrowsky, and Piscunov, that is, they proved that for any nonnegative solution $u(t,x)$ of (1.2), if at time $t = 0$, u is 1 near $-\infty$ and 0 near ∞, then $\lim_{t\to\infty} u(t,ct)$ is 0 if $c > 2$ and 1 if $c < 2$. Put $c^* = 2$. c^* is of the following spatially spreading property: for any nonnegative solution $u(t,x)$ of (1.2), if at time $t = 0$, $u(0,x) \geq \sigma$ for some $\sigma > 0$ and $x \ll -1$ and $u(0,x) = 0$ for $x \gg 1$, then

$$\inf_{x \leq c' t} |u(t,x) - 1| \to 0, \ \forall c' < c^* \quad \text{and} \quad \sup_{x \geq c'' t} u(t,x) \to 0 \quad \forall c'' > c^* \text{ as } \quad t \to \infty.$$

In literature, c^* is hence called the *spreading speed* for (1.2). The results on traveling wave solutions of (1.2) have been well extended to general time and space independent monostable equations (see [1], [2], [6], [12], [22], [39], [45], etc.).

Due to the inhomogeneity of the underline media of biological models in nature, the investigation of the traveling wave problem for time and/or space dependent monostable equations is gaining more and more attention. A huge amount of research has been carried out toward the traveling wave solutions of various time and/or space dependent monostable equations. See, for example, [3], [4], [5], [10], [16], [24], [25], [26], [31], [33], [46], [47], and references therein for space and/or time periodic reaction diffusion equations of KPP type, see [7], [10], [13], [32], [34], [35], [36], [48], [49], and references therein for KPP models in random media, and see [25], [26], [27], [46], [47], and references therein for time discrete KPP models.

Recall that when (1.1) is time periodic in t with period T, it has been proved that for any $\xi \in \mathbb{R}^N$ with $\|\xi\| = 1$, there is a $c^*(\xi) \in \mathbb{R}$ such that for any $c \geq c^*(\xi)$, there is a traveling wave solution connecting u^+ and 0 and propagating in the direction of ξ with speed c, and there is no such traveling wave solution of slower speed. The minimal wave speed $c^*(\xi)$ is of some important spreading properties and is called the *spreading speed* in the direction of ξ (see [26], [31], [33], [47], and references therein). Moreover, the following variational principle for $c^*(\xi)$ holds,

$$c^*(\xi) = \inf_{\mu>0} \frac{\lambda^*(\mu,\xi)}{\mu}, \tag{1.3}$$

where $\lambda^*(\mu,\xi)$ is the principal eigenvalue (i.e. the eigenvalue with largest real part and a positive eigenfunction) of the following periodic parabolic eigenvalue problem,

$$\begin{cases} -\frac{\partial u}{\partial t} + \Delta u + \sum_{i=1}^N a_i^{\mu,\xi}(t,x)\frac{\partial u}{\partial x_i} + a_0^{\mu,\xi}(t,x)u = \lambda u, \quad x \in \mathbb{R}^N \\ u(t,\cdot + p_i\mathbf{e_i}) = u(t,\cdot),\ \nabla u(t,\cdot+p_i\mathbf{e_i}) = \nabla u(t,\cdot), \quad i = 1,2,\cdots,N \\ u(t+T,\cdot) = u(t,\cdot) \end{cases} \tag{1.4}$$

$a_i^{\mu,\xi}(t,x) = a_i(t,x) - 2\mu\xi_i$ $(i = 1,2,\cdots,N)$, $a_0^{\mu,\xi}(t,x) = a_0(t,x) - \mu\sum_{i=1}^N a_i(t,x)\xi_i$ $+\mu^2$, $a_0(t,x) = f(t,x,0)$, and $\mathbf{e_i} = (\delta_{i1},\delta_{i2},\cdots,\delta_{iN})$, $\delta_{ij} = 1$ if $i = j$ and 0 if $i \neq j$.

However, there is little understanding of the traveling wave problem for general time dependent and space periodic monostable equations. The objective of the current paper is to investigate the extent to which the concepts and theories of traveling wave solutions of time and/or space periodic monostable stable equations may be generalized.

To this end, we first introduce the concept of generalized traveling wave solutions, which generalize the classical concept of traveling wave solutions. Roughly, a solution $u = u(t,x)$ of (1.1) is called a *generalized traveling wave solution of average propagating speed* c in the direction of $\xi \in S^{N-1}$ if it is an entire solution of (1.1) and

$$u(t,x) = \Phi(x - \zeta(t)\xi, t, \zeta(t)\xi) \tag{1.5}$$

for some $\zeta : \mathbb{R} \to \mathbb{R}$ and $\Phi : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^+$ satisfying that

$$\lim_{x\cdot\xi\to-\infty}(\Phi(x,t,z) - u^+(t,x+z)) = 0, \qquad \lim_{x\cdot\xi\to\infty}\Phi(x,t,z) = 0$$

uniformly in $t \in \mathbb{R}$, $z \in \mathbb{R}^N$,

$$\Phi(x,t,z-x) = \Phi(x^{'},t,z-x^{'}) \quad \forall x,x^{'} \in \mathbb{R}^N,\ x\cdot\xi = x^{'}\cdot\xi,$$

$$\Phi(x,t,z+p_i\mathbf{e_i}) = \Phi(x,t,z), \quad i = 1,2,\cdots,N,$$

and

$$\lim_{t\to\infty}\frac{\zeta(t+s)-\zeta(s)}{t} = c \quad \text{uniformly in} \quad s \in \mathbb{R}$$

(see Definition 4.1 for detail).

Note that if $u = \Phi(x - \zeta(t)\xi, t, \zeta(t)\xi)$ is a generalized traveling wave solution of (1.1) in the direction of ξ, then it is also of the following form,

$$u(t,x) = \Psi(x\cdot\xi - \zeta(t), t, x) \tag{1.6}$$

for some $\Psi : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N$ satisfying

$$\lim_{r\to-\infty}(\Psi(r,t,z) - u^+(t,z)) = 0, \quad \lim_{r\to\infty}\Psi(r,t,z) = 0$$

and

$$\Psi(r,t,z+p_i\mathbf{e_i}) = \Psi(r,t,z)$$

(see Remark 4.1).

To state the main results of the paper, for given $\xi \in S^{N-1}$ and $\mu > 0$, let $\lambda(\mu,\xi)$ be the principal Lyapunov exponent (see Definition 2.4) of

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + \sum_{i=1}^N a_i^{\mu,\xi}(t,x)\frac{\partial u}{\partial x_i} + a_0^{\mu,\xi}(t,x)u, \quad x \in \mathbb{R}^N \\ u(t,\cdot + p_i\mathbf{e_i}) = u(t,\cdot),\ \nabla u(t,\cdot+p_i\mathbf{e_i})\nabla u(t,\cdot), \quad i = 1,2,\cdots,N, \end{cases} \tag{1.7}$$

where, as in (1.4), $a_i^{\mu,\xi}(t,x) = a_i(t,x) - 2\mu\xi_i$ $(i = 1,2,\cdots,N)$, $a_0^{\mu,\xi}(t,x) = a_0(t,x) - \mu\sum_{i=1}^N a_i(t,x)\xi_i + \mu^2$, and $a_0(t,x) = f(t,x,0)$. Observe that principal Lyapunov exponent of (1.7) is the analogue of the principal eigenvalue of the periodic parabolic problem (1.4) (see section 2 for basic properties of principal Lyapunov exponents of time recurrent parabolic equations). When $a_i(t,x)$ and $a_0(t,x)$ are periodic in t with period T, $\lambda(\mu,\xi)$ equals the principal eigenvalue $\lambda^*(\mu,\xi)$ of (1.4).

For given $\xi \in S^{N-1}$, let $\mu^*(\xi) > 0$ be such that

$$\frac{\lambda(\mu^*(\xi),\xi)}{\mu^*(\xi)} = \inf_{\mu>0}\frac{\lambda(\mu,\xi)}{\mu}$$

and

$$\frac{\lambda(\mu,\xi)}{\mu} > \frac{\lambda(\mu^*(\xi),\xi)}{\mu^*(\xi)} \quad \text{for} \quad 0 < \mu < \mu^*(\xi)$$

(the existence of $\mu^*(\xi)$ follows from Theorem 2.6). Let

$$c^*(\xi) = \frac{\lambda(\mu^*(\xi),\xi)}{\mu^*(\xi)}.$$

Among others, we prove (see Theorem 4.1 for detail)

(a) For any $c > c^*(\xi)$, (1.1) has a generalized traveling wave solution $u(t,x) = \Phi(x - \zeta(t)\xi, t, \zeta(t)\xi)$ in the direction of ξ with average propagating speed c.

(b) The generalized traveling wave solution $u(t,x) = \Phi(x - \zeta(t)\xi, t, \zeta(t)\xi)$ in (a) has uniform exponential decay rate μ at ∞, where $\mu \in (0,\mu^*(\xi))$ is such that $c = \frac{\lambda(\mu,\xi)}{\mu}$.

(c) If $a_i(t,x)$ and $f(t,x,u)$ are periodic in t with period T, then the generalized traveling wave solution $u(t,x) = \Phi(x-\zeta(t)\xi, t, \zeta(t)\xi)$ in (a) is periodic in t in the sense that $\Phi(x,t+T,z) = \Phi(x,t,z)$ and $\zeta^{'}(t+T) = \zeta^{'}(t)$. Moreover, it can be written as $u(t,x) = \tilde{\Phi}(x-ct,t,ct)$ for some function $\tilde{\Phi} : \mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^+$ with similar properties as $\Phi(\cdot,\cdot,\cdot)$.

As mentioned above, if $a_i(t,x)$ and $f(t,x,u)$ are periodic in T, it has been proved (see [31], [47]) that for any $c \geq c^*(\xi)$, (1.1) has a periodic traveling wave solution in the direction of ξ. The results of the current paper recover the existing results on periodic traveling wave solutions with speed $c > c^*(\xi)$. It should be pointed out that the approach used in this paper is different from the approaches in other papers.

When (1.1) is recurrent and unique ergodic in t but independent of x, it has been proved in [43] that for any $c \geq c^*(\xi)$, (1.1) has a generalized traveling wave solution in the direction of ξ with averaged speed c. Hence the results of the current paper also recover the existing results on generalized traveling wave solutions with average propagating speed $c > c^*(\xi)$.

Observe that when (1.1) is periodic in t or is independent of x, it has been proved that $c^*(\xi)$ is the spreading speed as well as the minimal average propagating speed of generalized traveling wave solutions of (1.1) in the direction of ξ (see [42], [43], [47]) (the reader is referred to [15], [42] for the studies of spreading speeds of time recurrent monostable equations). However it remains open whether in general $c^*(\xi)$ is the spreading speed of (1.1) in the direction of ξ (if exists) and is the minimal average propagating speed of generalized traveling wave solutions in the direction of ξ.

The rest of the paper is organized as follows. In section 2, we collect some fundamental properties of principal Lyapunov exponent of (1.7), which together with comparison principles for parabolic equations are the main tools for the proofs of the main results of the paper. We also recall the concepts of recurrent and unique ergodic functions and compact minimal flows in section 2. We prove the existence, uniqueness, and stability of space periodic and time recurrent and unique ergodic positive solutions of (1.1) and construct some important super- and sub-solutions of (1.1) in section 3. In section 4, we study the existence of generalized traveling wave solutions of (1.1) and prove the main results.

2. Preliminary

In this section, we first recall the concepts of compact minimal flows and recurrent and unique ergodic functions. We then collect some fundamental properties of principal Lyapunov exponent of (1.7).

2.1. Compact flows and recurrent functions

In this subsection, we recall the definitions of compact flows and recurrent functions and collect some basic properties.

Definition 2.1. Let Z be a compact metric space and $\mathcal{B}(Z)$ be the Borel σ-algebra of Z.

(1) $(Z,\mathbb{R}) := (Z,\{\sigma_t\}_{t\in\mathbb{R}})$ is called a *compact flow* if $\sigma_t\colon Z \to Z$ $(t \in \mathbb{R})$ satisfies: $[\,(t,z) \mapsto \sigma_t z\,]$ is jointly continuous in $(t,z) \in \mathbb{R}\times Z$, $\sigma_0 = \mathrm{id}$, and $\sigma_s\circ\sigma_t = \sigma_{s+t}$ for any $s,t \in \mathbb{R}$. We may write $z\cdot t$ or (z,t) for $\sigma_t z$.

(2) Assume that $(Z,\{\sigma_t\}_{t\in\mathbb{R}})$ is a compact flow. A probability measure μ on $(Z,\mathcal{B}(Z))$ is called an *invariant measure* for $(Z,\{\sigma_t\}_{t\in\mathbb{R}})$ if for any $E \in \mathcal{B}(Z)$ and any $t \in \mathbb{R}$, $\mu(\sigma_t(E)) = \mu(E)$. An invariant measure μ for $(Z,\{\sigma_t\}_{t\in\mathbb{R}})$ is said to be *ergodic* if for any $E \in \mathcal{B}(Z)$ satisfying $\mu(\sigma_t^{-1}(E) \triangle E) = 0$ for all $t \in \mathbb{R}$, $\mu(E) = 1$ or $\mu(E) = 0$.

(3) Assume that $(Z, \{\sigma_t\}_{t\in\mathbb{R}})$ is a compact flow. $(Z, \{\sigma_t\}_{t\in\mathbb{R}})$ is said to be *uniquely ergodic* if it has a unique invariant measure (in such case, the unique invariant measure is necessarily ergodic).

(4) We say that $(Z, \{\sigma_t\}_{t\in\mathbb{R}})$ is *minimal* or *recurrent* if for any $z \in Z$, the orbit $\{\,\sigma_t z | t \in \mathbb{R}\,\}$ is dense in Z.

Given $g \in C(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}^m)$, let

$$H(g) = cl\{g\cdot\tau|\tau\in\mathbb{R}\},$$

where $g\cdot\tau(t,x) = g(t+\tau, x)$ and the closure is taken under the compact open topology. $H(g)$ is usually called the *hull of* g. Let $(H(g), \{\sigma_t\}_{t\in\mathbb{R}})$ be *the translation flow* defined by $\sigma_t(\tilde{g}) = \tilde{g}(t+\cdot,\cdot)$ for $\tilde{g}\in H(g)$. If g is bounded and uniformly continuous on $\mathbb{R}\times E$ for any bounded subset $E\subset\mathbb{R}^n$, then $H(g)$ is compact and metrizable under the compact open topology and hence $(H(g), \{\sigma_t\}_{t\in\mathbb{R}})$ is a compact flow (see [40]).

Definition 2.2. (1) A function $g \in C(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}^m)$ is said to be *recurrent* in the first independent variable if it is bounded and uniformly continuous on $\mathbb{R}\times E$ for any bounded subset $E\subset\mathbb{R}^n$ and $(H(g), \{\sigma_t\}_{t\in\mathbb{R}})$ is minimal.

(2) A function $g \in C(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}^m)$ is said to be *unique ergodic* in the first independent variable if it is bounded and uniformly continuous on $\mathbb{R}\times E$ for any bounded subset $E\subset\mathbb{R}^n$ and $(H(g), \{\sigma_t\}_{t\in\mathbb{R}})$ is unique ergodic.

Remark 2.1. If $g \in C(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}^m)$ is periodic or almost periodic in the first independent variable and is bounded and uniformly continuous on $\mathbb{R}\times E$ for any bounded subset $E\subset\mathbb{R}^n$, then it is both recurrent and unique ergodic in the first variable. The reader is referred to [8] for the definition and basic properties of almost periodic functions.

Theorem 2.1. *If* $g \in C(\mathbb{R}\times\mathbb{R}^n, \mathbb{R})$ *is recurrent and unique ergodic in the first independent variable, then the limit* $\lim_{t-s\to\infty}\frac{1}{t-s}\int_s^t g(\tau,x)d\tau$ *exists for any* $x\in\mathbb{R}$.

Proof. It follows from the results contained in [21]. □

2.2. Spectral theory for linear recurrent parabolic equations with periodic boundary conditions

In this subsection, we collect some fundamental properties of the principal spectrum for time recurrent parabolic equations with periodic boundary conditions. The reader is referred to [42] for detail. The reader is also referred to [17], [18], [19], [20], [28], [29], [37], [42] for the studies of principal spectral theory for general time dependent parabolic equations with Dirichlet, or Neumann, or Robin boundary conditions.

We first consider a family of linear parabolic equations of the form,

$$u_t = \Delta u + \sum_{i=1}^{N} b_i(t,x)\frac{\partial u}{\partial x_i} + b_0(t,x)u, \quad x\in\mathbb{R}^N \tag{2.1}$$

complemented with the periodic boundary condition

$$u(t,\cdot+p_i\mathbf{e_i}) = u(t,\cdot),\ \nabla u(t,\cdot+p_i\mathbf{e_i}) = \nabla u(t,\cdot), \quad i=1,2,\cdots,N, \tag{2.2}$$

where $p_j > 0$ $(j = 1, 2, \cdots, N)$, $b := (b_i, b_0) := (\{b_i\}_{i=1}^N, b_0) \in Y$, and Y is a subset of $C(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^{N+1})$. To emphasis the dependence of (2.1) on b, we may write it as $(2.1)_b$.

We make the following standard assumption on Y.

(H-Y) *For any $b = (b_i, b_0) \in Y$, $b(t, x)$ is recurrent in t and periodic in x_j with period $p_j > 0$ $(j = 1, 2, \cdots, N)$ and are globally Hölder continuous in t, x. Moreover, Y is translation invariant in t (i.e. for any $b \in Y$ and $t \in \mathbb{R}$, $\sigma_t b := b \cdot t := b(t + \cdot, \cdot) \in Y$) and is connected and compact under open compact topology.*

In the following, we assume that Y satisfies (H-Y) and Y is equipped with the open compact topology. Then $(Y, (\sigma_t)_{t\in\mathbb{R}})$ is a compact flow. For a given Banach space X, $\|\cdot\|_X$ denotes the norm in X.

Let

$$X_L = \{u \in C(\mathbb{R}^N, \mathbb{R}) \,|\, u(x + p_i \mathbf{e_i}) = u(x),\, i = 1, 2, \cdots, N\} \tag{2.3}$$

equipped with uniform convergence topology. Let $\mathcal{L}(X_L, X_L)$ be the space of bounded linear operators from X_L to X_L. Let

$$X_L^+ = \{u \in X_L | u(x) \geq 0,\, x \in \mathbb{R}^N\} \tag{2.4}$$

and

$$\mathrm{Int}(X_L^+) := X_L^{++} = \{u \in X_L^+ | u(x) > 0,\, x \in \mathbb{R}^N\}. \tag{2.5}$$

For $u_1, u_2 \in X_L$, we write

$$u_1 \leq u_2 \quad (u_1 \geq u_2) \quad \text{if} \quad u_2 - u_1 \in X_L^+ \quad (u_1 - u_2 \in X_L^+)$$

and

$$u_1 \ll u_2 \quad (u_1 \gg u_2) \quad \text{if} \quad u_2 - u_1 \in X_L^{++} \quad (u_1 - u_2 \in X_L^{++}).$$

Let

$$C^1_{\mathrm{per}}(\mathbb{R}^N) = \{u \in X_L | \frac{\partial u}{\partial x_i} \in X_L, i = 1, 2, \cdots, N\}.$$

It follows from [14] that $-\Delta$ is a sectorial operator on X_L, denote it by $-\Delta|_{X_L}$. Let X_L^α be the fractional power space of $-\Delta$ on X_L $(0 \leq \alpha \leq 1)$. Note that $X_L^0 = X_L$ and $X_L^1 = \mathcal{D}(-\Delta|_{X_L})$. Let $0 < \alpha_0 < 1$ be such that $X_L^{\alpha_0}$ is compactly imbedded into $C^1_{\mathrm{per}}(\mathbb{R}^N)$. Then by [14], for any $b \in Y$ and $u_0 \in X_L^{\alpha_0}$, there is a unique solution $u(t, \cdot; u_0, b) \in X_L^{\alpha_0}$ of (2.1)+(2.2) with initial condition $u(0, \cdot; u_0, b) = u_0(\cdot)$. Put $U(t, b)u_0 := u(t, \cdot; u_0, b)$ for $u_0 \in X_L^{\alpha_0}$. Following from the results in [14] and classical theory for parabolic equations, we have

Theorem 2.2. *(1)* (Joint continuity). *The map $[[0, \infty) \times X_L^{\alpha_0} \times Y \ni (t, u_0, b) \mapsto U(t, b)u_0 \in X_L^{\alpha_0}]$ is continuous.*

(2) (Norm continuity). *For any $t \geq 0$, the map $[Y \ni b \mapsto U(t, b) \in \mathcal{L}(X_L^{\alpha_0}, X_L^{\alpha_0})]$ is continuous.*

(3) (Strong monotonicity). *For any $t > 0$ and $b \in Y$, $U(t, b)$ is strongly monotone in the sense that if $u_1, u_2 \in X_L^{\alpha_0}$ and $u_1 \leq u_2$, $u_1 \neq u_2$, then $U(t, b)u_1 \ll U(t, b)u_2$ for any $t > 0$ and $b \in Y$.*

(4) (Compactness). *For any $t > 0$, $U(t, \cdot)$ is compact in the sense that for any bounded set $E \subset X_L^{\alpha_0}$, $\{U(t, b)u_0 | b \in Y, u_0 \in E\}$ is a relatively compact subset of $X_L^{\alpha_0}$.*

By Theorem 2.2, (2.1)+(2.2) generates a skew-product semiflow on $X_L^{\alpha_0} \times Y$:

$$\Sigma_t : X_L^{\alpha_0} \times Y \to X_L^{\alpha_0} \times Y, \quad t \geq 0,$$

$$\Sigma_t(u_0, b) = (U(t,b)u_0, b \cdot t).$$

The following Theorem follows from [38] (see also [28], [29]).

Theorem 2.3 (Exponential Separation)**.** *There are subspaces $X_L^{\alpha_0,1}(b)$, $X_L^{\alpha_0,2}(b) \subset X_L^{\alpha_0}$ for any $b \in Y$, $X_L^{\alpha_0,1}(b)$, $X_L^{\alpha_0,2}(b)$ are continuous in $b \in Y$, and satisfy the following properties:*

(1) $X_L^{\alpha_0} = X_L^{\alpha_0,1}(b) \oplus X_L^{\alpha_0,2}(b)$ *for any* $b \in Y$.

(2) $X_L^{\alpha_0,1}(b) = Span\{\tilde{w}(b)\}$, $\tilde{w}(b) \in \mathrm{Int}(X_L^{\alpha_0} \cap X_L^+)$ *and is continuous in* b, $\|\tilde{w}(b)\|_{X_L^{\alpha_0}} = 1$ *for any* $b \in Y$.

(3) $X_L^{\alpha_0,2}(b) \bigcap \mathrm{Int}(X_L^{\alpha_0} \cap X_L^+) = \emptyset$ *for any* $b \in Y$.

(4) $U(t,b)X_L^{\alpha_0,1}(b) = X_L^{\alpha_0,1}(\sigma_t b)$ *and* $U(t,b)X_L^{\alpha_0,2}(b) \subset X_L^{\alpha_0,2}(\sigma_t b)$ *for any* $b \in Y$ *and* $t > 0$.

(5) *There are* $M, \gamma > 0$ *such that*

$$\frac{\|U(t,b)w\|_{X_L^{\alpha_0}}}{\|U(t,b)\tilde{w}(b)\|_{X_L^{\alpha_0}}} \leq Me^{-\gamma t}$$

for any $t > 0$, $b \in Y$, $w \in X_L^{\alpha_0,2}(b)$ *with* $\|w\|_{X_L^{\alpha_0}} = 1$.

By classical theory for parabolic equations and the continuity of $\tilde{w}(b)$ in $b \in Y$ with respect to the $\|\cdot\|_{\alpha_0}$-norm, there are constants $C_1, C_2 > 0$ such that

$$\frac{1}{C_1}\|\tilde{w}(b)\|_{X_L^{\alpha_0}} \leq \|\tilde{w}(b)\|_{L_2(D)} \leq C_1\|\tilde{w}(b)\|_{X_L^{\alpha_0}}$$

and

$$\frac{1}{C_2}\|\tilde{w}(b)\|_{X_L^{\alpha_0}} \leq \|\tilde{w}(b)\|_{X_L} \leq C_2\|\tilde{w}(b)\|_{X_L^{\alpha_0}}$$

for any $b \in Y$.

Let

$$w(b) = \tilde{w}(b)/\|\tilde{w}(b)\|_{L_2(D)}. \tag{2.6}$$

Let $D = [0, p_1] \times [0, p_2] \times \cdots [0, p_N]$ and $\langle \cdot, \cdot \rangle$ be the inner product in $L_2(D)$. Let

$$\kappa(b) = \langle \Delta w(b) + \Sigma_{i=1}^N b_i(0,x)\frac{\partial w(b)}{\partial x_i} + b_0 w(b), w(b)\rangle. \tag{2.7}$$

Theorem 2.4. *$\kappa(b)$ is continuous in $b \in Y$.*

Proof. See [42, Theorem 2.3]. □

Let

$$\eta(t,b) = \|U(t,b)w(b)\|_{L_2(D)}.$$

Then we have

$$\eta_t(t,b) = \kappa(\sigma_t b)\, \eta(t,b).$$

Therefore

$$\eta(t,b)=e^{\int_0^t \kappa(\sigma_\tau b)d\tau}.$$

Let

$$v(t,x;b)=\frac{U(t,b)w(b)}{\|U(t,b)w(b)\|_{L_2}}\equiv w(\sigma_t b)(x).$$

We have that $v(t,x;b)$ satisfies

$$v_t=\Delta v+\Sigma_{i=1}^N b_i(t,x)\frac{\partial v}{\partial x_i}+b_0(t,x)v-\kappa(\sigma_t b)v,\quad x\in\mathbb{R}^N. \tag{2.8}$$

Theorem 2.5. *If $(Y,(\sigma_t)_{t\in\mathbb{R}})$ is unique ergodic, then*

$$\lim_{t-s\to\infty}\frac{1}{t-s}\ln\|U(t-s,\sigma_s b)\|_{\alpha_0}=\lim_{t-s\to\infty}\frac{1}{t-s}\int_s^t\kappa(\sigma_\tau b)d\tau$$

for all $b\in Y$ and the limit is independent of b and is uniform in $b\in Y$.

Proof. It follows from Theorems 2.3, 2.4, and the results in [21]. □

Definition 2.3. Assume that $(Y,(\sigma_t)_{t\in\mathbb{R}})$ is unique ergodic. $\lambda(Y)$ is called the *principal Lyapunov exponent* of (2.1)+(2.2), where

$$\lambda(Y)=\lim_{t-s\to\infty}\frac{1}{t-s}\ln\|U(t-s,\sigma_s b)\|_{\alpha_0}$$

for any $b\in Y$

We now consider a single linear parabolic equation

$$u_t=\Delta u+\sum_{i=1}^n a_i(t,x)\frac{\partial u}{\partial x_i}+a_0(t,x)u,\quad x\in\mathbb{R}^N \tag{2.9}$$

complemented with the periodic boundary condition (2.2), where $(\{a_i(t,x)\}_{i=1}^N,a_0(t,x))$ is recurrent and unique ergodic in t and periodic in x_j with period p_j $(j=1,2,\cdots,N)$, and are globally Hölder continuous in t, x.

For any $\mu\in\mathbb{R}$ and $\xi\in S^{N-1}=\{\xi\in\mathbb{R}^N\mid\|\xi\|=1\}$, consider also

$$u_t=\Delta u+\Sigma_{i=1}^N a_i^{\mu,\xi}\frac{\partial u}{\partial x_i}+a_0^{\mu,\xi}(t,x)u,\quad x\in\mathbb{R}^N \tag{2.10}$$

complemented with the periodic boundary condition (2.2), where $a_i^{\mu,\xi}=a_i-2\mu\xi_i,\quad i=1,2,\cdots,N$ and $a_0^{\mu,\xi}=a_0-\mu\sum_{i=1}^N a_i\xi_i+\mu^2$. Note that if $\mu=0$, then $a_i^{\mu,\xi}=a_i$ and $a_0^{\mu,\xi}=a_0$ for any $\xi\in S^{N-1}$.

Let $a:=(a_i,a_0):=(\{a_i\}_{i=1}^N,a_0)$ and

$$Y(a)=\mathrm{cl}\{a\cdot s|s\in\mathbb{R}\},$$

where $a\cdot s(t,x):=\sigma_s a(t,x):=a(t+s,x)$ and the closure is taken under the open compact topology. Then (H-Y) is satisfied with $Y=Y(a)$. Hence for any $u_0\in X_L^{\alpha_0}$ and any $s\in\mathbb{R}$, (2.9)+(2.2) has a unique classical solution $u(t,x;s,u_0,a)$ with initial condition $u(s,x;s,u_0,a)=u_0(x)$.

Similarly, let

$$Y(a^{\mu,\xi})=\mathrm{cl}\{a^{\mu,\xi}\cdot s|s\in\mathbb{R}\},$$

where $a^{\mu,\xi} = (a_i^{\mu,\xi}, a_0^{\mu,\xi}) := (\{a_i^{\mu,\xi}\}_{i=1}^N, a_0^{\mu,\xi})$, $a^{\mu,\xi} \cdot s := \sigma_s a^{\mu,\xi} := a^{\mu,\xi}(\cdot + s, \cdot)$, and the closure is taken under open compact topology. Then $Y(a^{\mu,\xi})$ satisfies (H-Y) with Y being replaced by $Y(a^{\mu,\xi})$. Hence (2.10)+(2.2) has a unique classical solution $u(t,x;s,u_0,a^{\mu,\xi})$ with $u(s,x;\,s,u_0,a^{\mu,\xi}) = u_0(x)$ for any $u_0 \in X_L^{\alpha_0}$.

Note that $Y(a^{0,\xi})(= Y(a))$ and $Y(a^{\mu,\xi})$ are unique ergodic and minimal for any $\mu \in \mathbb{R}$ and $\xi \in S^{N-1}$. Theorems 2.3 to 2.5 can then be applied to $Y = Y(a^{\mu,\xi})$. Put

$$\lambda(\mu,\xi;a) := \lambda(a^{\mu,\xi}) := \lambda(Y(a^{\mu,\xi}))$$

and

$$w^{\mu,\xi}(t,\cdot;a) := w(\sigma_t a^{\mu,\xi})(\cdot).$$

Definition 2.4. We call $\lambda(\mu,\xi;a)$ the *principal Lyapunov exponent* of (2.10)+(2.2).

Observe that $\lambda(\mu,\xi;a)$ and $w^{\mu,\xi}(t,\cdot;a)$ are analogs of principal eigenvalues and principal eigenfunctions of elliptic and periodic parabolic problems, respectively. In literature, $\{\mathrm{span}\big(w(\sigma_t a^{\mu,\xi})\big)\}_{t\in\mathbb{R}}$ is call the *principal Floquet bundle* of (2.10) associated to the principal Lyapunov exponent.

Theorem 2.6. *(1) $\lambda(\mu,\xi;a)$ is continuous in $\mu \in \mathbb{R}$, $\xi \in S^{N-1}$, and a with respect to uniform convergence topology.*

(2) Fix a. There is $\beta > 0$ such that

$$\lambda(\mu,\xi;a) \geq \beta\mu^2$$

for any $\xi \in S^{N-1}$ and $\mu \gg 1$.

(3) Fix a and assume that $\lambda(0,\xi;a) > 0$ (note that $\lambda(0,\xi;a)$ is independent of $\xi \in S^{N-1}$). There are $\mu_0^- > 0$ and $\mu_0^+ > 0$ with $\mu_0^- < \mu_0^+$, and $\beta_0 > 0$ such that

$$\inf_{\mu>0} \frac{\lambda(\mu,\xi;a)}{\mu} = \inf_{\mu_0^- \leq \mu \leq \mu_0^+} \frac{\lambda(\mu,\xi;a)}{\mu} \leq \beta_0 \mu_0^+$$

for any $\xi \in S^{N-1}$.

Proof. See [42, Theorem 2.7]. □

Observe that for any $u_0 \in C(\mathbb{R}^N,\mathbb{R})$ with

$$|u_0(x)| \leq Ce^{\alpha\sum_{i=1}^N |x_i|}$$

for some $\alpha, C > 0$, (2.9) has also a unique solution $u(t,x;s,u_0,a)$ with $u(s,x;\,s,u_0,a) = u_0(x)$ (see [11]). Regarding solutions of (2.9) without the periodic boundary condition, we have

Theorem 2.7. *For any given $\xi \in S^{N-1}$ and $\mu > 0$,*

$$u(t,x) = e^{-\mu\left(x\cdot\xi - \frac{\int_s^t \kappa(\sigma_\tau a^{\mu,\xi})d\tau}{\mu}\right)} w(\sigma_t a^{\mu,\xi})(x)$$

is a solution of (2.9).

When a_i and a_0 are periodic in t with period T, let $\lambda^*(\mu,\xi;a)$ be the principal eigenvalue of the periodic parabolic eigenvalue problem (1.4) and $w^*(t,x;\mu,\xi,a)$ be an associated positive principal eigenfunction. Then

Theorem 2.8. *For any given* $\xi \in S^{N-1}$, $\mu > 0$,

$$u(t,x) = e^{-\mu\left(x\cdot\xi - \frac{\lambda^*(\mu,\xi;a)t}{\mu}\right)} w^*(t,x;\mu,\xi,a)$$

is a solution of (2.9).

Note that $w(\sigma_t a^{\mu,\xi}) = \frac{w^*(t,\cdot;\mu,\xi,a)}{\|w^*(t,\cdot;\mu,\xi,a)\|_{L_2(D)}}$.

3. Positive recurrent solutions and super-, sub-solutions

In this section, we first prove the existence, uniqueness, and stability of space periodic and time recurrent positive solutions of (1.1). Then we construct some important super- and sub-solutions of (1.1) to be used in the proofs of the main results in next section. Throughout this section, we assume (H1)-(H3).

First of all, let $a = (\{a_i\}_{i=1}^N)$ and let $H(a,f)$ be the hull of (a,f), i.e.

$$H(f) = \mathrm{cl}\{(a,f)\cdot t(\cdot,\cdot,\cdot) := (a(\cdot+t,\cdot), f(\cdot+t,\cdot,\cdot)) \,|\, t \in \mathbb{R}\}$$

with the open compact topology, where the closure is taken under the open compact topology. For any $(b,g) = (\{b_i\}_{i=1}^N, g) \in H(a,f)$, consider

$$\frac{\partial u}{\partial t} = \Delta u + \Sigma_{i=1}^N b_i(t,x)\frac{\partial u}{\partial x_i} + ug(t,x,u), \quad x \in \mathbb{R}^N. \tag{3.1}$$

For any $z \in \mathbb{R}^N$, consider also

$$\frac{\partial u}{\partial t} = \Delta u + \Sigma_{i=1}^N a_i(t,x+z)\frac{\partial u}{\partial x_i} + uf(t,x+z,u), \quad x \in \mathbb{R}^N. \tag{3.2}$$

Let

$$X = \{u : \mathbb{R}^N \to \mathbb{R} \,|\, u \quad \text{is uniformly continuous and} \quad \sup_{x\in\mathbb{R}^N} |u(x)| < \infty\} \tag{3.3}$$

equipped with uniform convergence topology. For given $u_1, u_2 \in X$, we write

$$u_1 \le u_2 \quad (u_1 < u_2) \quad \text{if} \quad u_1(x) \le u_2(x) \quad (u_1(x) < u_2(x)) \quad \text{for} \quad x \in \mathbb{R}^N.$$

Let

$$X^+ = \{u \in X \,|\, u \ge 0\}.$$

Let X_L and X_L^+ be as in (2.3) and (2.4), respectively.

By the classical theory for parabolic equations (see [11], [14]), for any $u_0 \in X$, (3.1) has a unique (local) solution $u(t,\cdot;u_0,b,g)$ with initial condition $u(0,\cdot;u_0,b,g) = u_0(\cdot)$. A function $u(t,x)$ is called an *entire solution* of (3.1) if it is a solution of (3.1) for $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$. Note that if $u_0 \in X^+$, then $u(t,\cdot;u_0,b,g)$ exists for all $t > 0$ and if $u_0 \in X_L$, then $u(t,\cdot;u_0,b,g) \in X_L$ for $t \ge 0$ at which $u(t,\cdot;u_0,b,g)$ exists.

Proposition 3.1 (Positive recurrent and unique ergodic solution)**.** *There is a continuous function* $q : H(a,f) \to X_L^+ \setminus \{0\}$ *such that* $u(t,\cdot;q(b,g),b,g) = q((b,g)\cdot t)$ *and* $u(t,\cdot;q(b,g),b,g)$ *is globally stable in the sense that for any* $u_0 \in X_L^+ \setminus \{0\}$,

$$\|u(t,\cdot;u_0,b,g) - q((b,g)\cdot t))\|_X \to 0 \quad \text{as} \quad t \to \infty$$

uniformly in $(b,g) \in H(b,f)$. *Let* $u^+(t,x) = u(t,x;q(a,f),a,f)$. *Then* $u^+(t,x)$ *is recurrent and unique ergodic in* t.

Proof. It follows from the arguments in [30, Theorem A] and [44, Theorem 3.5] (see also [29, Theorem 7.1.12]). □

In the following, we write $u(t,x;q(a,f),a,f)$ as $u^+(t,x;a,f)$ or simply as $u^+(t,x)$ if no confusion occurs. Let

$$u^+_{\sup} = \sup_{t\in\mathbb{R},x\in\mathbb{R}^N} u^+(t,x), \tag{3.4}$$

$$u^+_{\inf} = \inf_{t\in\mathbb{R},x\in\mathbb{R}^N} u^+(t,x). \tag{3.5}$$

Observe that $u^+_{\inf} > 0$.

Similarly, for any $u_0 \in X$, $s \in \mathbb{R}$, and $z \in \mathbb{R}^N$, (3.2) has a unique (local) solution $u(t,\cdot;s,z,u_0)$ with $u(s,x;s,z,u_0) = u_0(x)$. If $u_0 \in X^+$, $u(t,\cdot;s,z,u_0)$ exists for all $t > s$. The following proposition follows easily.

Proposition 3.2. *Assume that $u_n, u_0 \in X$, $\|u_n\| \le M$ for $n = 1,2,\cdots$ and some $M > 0$, and $u_n(x) \to u_0(x)$ as $n \to \infty$ uniformly for x in bounded subsets of $\mathbb{R}^N$. Then for any $t > 0$, $u(t,x;s,z,u_n) \to u(t,x;s,z,u_0)$ as $n \to \infty$ uniformly for $s \in \mathbb{R}$ and x in bounded subsets of $\mathbb{R}^N$.*

Let

$$S^{N-1} = \{\xi \in \mathbb{R}^N \mid \|\xi\| = 1\}. \tag{3.6}$$

Consider the linearization of (1.1) at 0, i.e.,

$$u_t = \Delta u + \sum_{i=1}^n a_i(t,x)\frac{\partial u}{\partial x_i} + a_0(t,x)u, \quad x \in \mathbb{R}^N, \tag{3.7}$$

where $a_0(t,x) = f(t,x,0)$. For any $z \in \mathbb{R}^N$, consider

$$u_t = \Delta u + \sum_{i=1}^n a_i(t,x+z)\frac{\partial u}{\partial x_i} + a_0(t,x+z)u, \quad x \in \mathbb{R}^N. \tag{3.8}$$

For any $\mu \in \mathbb{R}$ and $\xi \in S^{N-1}$, consider

$$u_t = \Delta u + \Sigma_{i=1}^N a_i^{\mu,\xi}\frac{\partial u}{\partial x_i} + a_0^{\mu,\xi}(t,x)u, \quad x \in \mathbb{R}^N \tag{3.9}$$

complemented with the periodic boundary condition

$$u(t,\cdot+p_i\mathbf{e_i}) = u(t,\cdot),\ \nabla u(t,\cdot+p_i\mathbf{e_i}) = \nabla u(t,\cdot), \quad i = 1,2,\cdots,N, \tag{3.10}$$

where $a_i^{\mu,\xi} = a_i - 2\mu\xi_i$, $\quad i = 1,2,\cdots,N$ and $a_0^{\mu,\xi} = a_0 - \mu\sum_{i=1}^N a_i\xi_i + \mu^2$. Note that if $\mu = 0$, then $a_i^{\mu,\xi} = a_i$ and $a_0^{\mu,\xi} = a_0$ for any $\xi \in S^{N-1}$.

Let $\sigma_t a^{\mu,\xi}(\cdot,\cdot) = a^{\mu,\xi}(\cdot+t,\cdot) = \big(\{a_i^{\mu,\xi}(\cdot+t,\cdot)\}_{i=1}^N, a_0^{\mu,\xi}(\cdot+t,\cdot)\big)$. Let $\lambda(\mu,\xi) := \lambda(\mu,\xi;a)$ be the principal Lyapunov exponent of (3.9)+(3.10) (see Definition 2.4 for the definition of $\lambda(\mu,\xi;a)$). Let $\kappa(\sigma_t a^{\mu,\xi})$ be as in (2.7). Let $w(\sigma_t a^{\mu,\xi})$ be as in (2.6). Then

$$\lambda(\mu,\xi) = \lim_{t\to\infty}\frac{1}{t}\int_s^{s+t}\kappa(\sigma_\tau a^{\mu,\xi})d\tau$$

uniformly in $s \in \mathbb{R}$ and

$$u(t,x;z) = e^{-\mu(x\cdot\xi - \frac{\int_s^t \kappa(\sigma_\tau a^{\mu,\xi})d\tau}{\mu})} w(\sigma_t a^{\mu,\xi})(x+z)$$

is an entire solution of (3.8).

By Theorem 2.6, for any $\xi \in S^{N-1}$, there is $\mu^*(\xi) > 0$ such that

$$\frac{\lambda(\mu^*(\xi),\xi)}{\mu^*(\xi)} = \inf_{\mu>0} \frac{\lambda(\mu,\xi)}{\mu} \tag{3.11}$$

and

$$\frac{\lambda(\mu,\xi)}{\mu} > \frac{\lambda(\mu^*(\xi),\xi)}{\mu^*(\xi)} \quad \text{for} \quad 0 < \mu < \mu^*(\xi). \tag{3.12}$$

Fix $\xi \in S^{N-1}$ and $c > \frac{\lambda(\mu^*(\xi),\xi)}{\mu^*(\xi)}$. There is $0 < \mu < \mu^*(\xi)$ such that

$$c = \frac{\lambda(\mu,\xi)}{\mu} \quad \text{and} \quad c > \frac{\lambda(\mu^{'},\xi)}{\mu^{'}} \quad \text{for} \quad \mu < \mu^{'} < \mu^*(\xi).$$

Put

$$\kappa_\mu(t) = \kappa(\sigma_t a^{\mu,\xi}).$$

For any given $T > 0$ and $s \in \mathbb{R}$, let

$$\lambda_T(\mu; s) = \frac{1}{T}\int_s^{s+T} \kappa_\mu(t)dt.$$

By the unique ergodicity of $a^{\mu,\xi}(t,x)$ in t,

$$\lim_{T\to\infty} \lambda_T(\mu; s) = \lambda(\mu,\xi)$$

uniformly for $s \in \mathbb{R}$ and μ in bounded subsets of $\mathbb{R}$. Hence there is $T > 0$ such that for any $s \in \mathbb{R}$, there is $\mu^*_{T,s} > 0$ ($s \in \mathbb{R}$) such that

$$\frac{\lambda_T(\mu^*_{T,s}; s)}{\mu^*_{T,s}} = \inf_{\tilde{\mu}>0} \frac{\lambda_T(\tilde{\mu}; s)}{\tilde{\mu}}, \tag{3.13}$$

$$\inf_{s\in\mathbb{R}} \mu^*_{T,s} - \mu > 0 \tag{3.14}$$

and

$$\frac{\lambda_T(\mu; s)}{\mu} - \sup_{s\in\mathbb{R}} \frac{\lambda_T(\mu^*_{T,s}; s)}{\mu^*_{T,s}} > 0, \tag{3.15}$$

$$\inf_{s\in\mathbb{R}} \Big(\frac{\lambda_T(\mu; s)}{\mu} - \frac{\lambda_T(\mu_1; s)}{\mu_1}\Big) > 0 \tag{3.16}$$

for some μ_1 with $\mu < \mu_1 < \min\{\inf_{s\in\mathbb{R}} \mu^*_{T,s}, 2\mu\}$.

In the following, we fix a $T > 0$ and μ_1 such that (3.13)-(3.16) hold.

Let

$$c^n_{\mu,s} = \frac{1}{T}\frac{\int_{s+(n-1)T}^{s+nT} \kappa_\mu(t)dt}{\mu}, \quad c^n_{\mu_1,s} = \frac{1}{T}\frac{\int_{s+(n-1)T}^{s+nT} \kappa_{\mu_1}(t)dt}{\mu_1},$$

and

$$\eta^n_\mu(t,x;s,z) = e^{\int_{s+(n-1)T}^t (\kappa_\mu(\tau)-\mu c^n_{\mu,s})d\tau} w(\sigma_t a^{\mu,\xi})(x+z), \tag{3.17}$$

$$\eta^n_{\mu_1}(t,x;s,z) = e^{\int_{s+(n-1)T}^t (\kappa_{\mu_1}(\tau)-\mu_1 c^n_{\mu_1,s})d\tau} w(\sigma_t a^{\mu_1,\xi})(x+z) \tag{3.18}$$

for $s+(n-1)T \le t \le s+nT$ and $n = 1, 2, \cdots$. Then

$$\inf_{s\in\mathbb{R}, n\ge 1} \left(c^n_{\mu,s} - c^n_{\mu_1,s}\right) > 0, \tag{3.19}$$

and for any $n \ge 1$,

$$\eta^n_\mu(s+nT, x; s, z) = \eta^{n+1}_\mu(s+nT, x; s) = w(\sigma_{s+nT} a^{\mu,\xi})(x+z),$$

$$\eta^n_{\mu_1}(s+nT, x; s, z) = \eta^{n+1}_{\mu_1}(s+nT, x; s) = w(\sigma_{s+nT} a^{\mu_1,\xi})(x+z).$$

Let

$$\phi(t, x; s, z) = e^{-\mu(x\cdot\xi - \sum_{k=1}^{n-1} c^k_{\mu,s}\cdot T - c^n_{\mu,s}\cdot(t-(n-1)T-s))} \eta^n_\mu(t, x; s, z),$$

$$\phi_1(t, x; s, z) = e^{-\mu_1(x\cdot\xi - \sum_{k=1}^{n-1} c^k_{\mu,s}\cdot T - c^n_{\mu,s}\cdot(t-(n-1)T-s))} \eta^n_{\mu_1}(t, x; s, z),$$

and

$$\psi_1(t, x; s, z) = d\phi(t, x; s, z), \tag{3.20}$$

$$\psi_2(t, x; s, z) = d\phi(t, x; s, z) - d_1\phi_1(t, x; s, z) \tag{3.21}$$

for $s+(n-1)T \le t < s+nT$ and $n = 1, 2, \cdots$, where d and d_1 are some positive constants.

Let

$$K_0 = 2 \sup_{0\le u\le u^+_{\sup}+1, t\in\mathbb{R}, x\in\mathbb{R}^N} |f_u(t, x, u)|, \tag{3.22}$$

where $u^+_{\sup}$ is as in (3.4). Let $\delta_0 > 0$ be such that

$$\delta_0 K_0 \le \inf_{s\in\mathbb{R}, n\ge 1} \mu_1(c^n_{\mu,s} - c^n_{\mu_1,s}). \tag{3.23}$$

It is not difficult to see that there is a C^1 function $\tilde{f}(t, x, u)$ such that

$$\tilde{f}(t, u) = \begin{cases} f(t, x, u) & \text{for} \quad u \ge 0 \\ f(t, x, 0) & \text{for} \quad u \le -\delta_0 \end{cases} \tag{3.24}$$

and

$$\tilde{f}_u(t, x, u) \ge -K_0/2 \quad \text{for} \quad u \le 0. \tag{3.25}$$

Proposition 3.3. *(i) For any $t \ge s$ and $x \in \mathbb{R}^N$,*

$$\phi(t, x; s, z) = e^{-\mu(x\cdot\xi - \frac{\int_s^t \kappa_\mu(\tau)d\tau}{\mu})} w(\sigma_t a^{\mu,\xi})(x+z)$$

and $\phi(t, x; s, z)$ (hence also $\psi_1(t, x; s, z)$) is a solution of (3.8) for $t > s$.

(ii) For any $d, d_1 > 0$, $\psi_2(t, x; s, z)$ is continuous in t and is a sub-solution of (3.8) on $s+(n-1)T < t < s+nT$ for $n = 1, 2, \cdots$.

Proof. (i) It follows from a direct calculation.

(ii) The continuity also follows easily. For $s+(n-1)T < t < s+nT$, we have

$$\begin{aligned} &\frac{\partial\psi_2(t, x; s, z)}{\partial t} - \Delta\psi_2(t, x; s, z) - \sum_{i=1}^N a_i(t, x+z)\frac{\partial\psi_2(t, x; s, z)}{\partial x_i} \\ &- a_0(t, x+z)\psi_2(t, x; s, z) \\ &= -d_1\mu_1(c^n_{\mu,s} - c^n_{\mu_1,s}) e^{-\mu_1(x\cdot\xi - \sum_{k=1}^{n-1} c^k_{\mu,s}T - c^n_{\mu,s}(t-(n-1)T-s))} \eta^n_{\mu_1}(t, x; s, z) \\ &< 0, \quad x \in \mathbb{R}^N. \end{aligned}$$

Hence $\psi_2(t,x;s,z)$ is a sub-solution of (3.8) on $s+(n-1)T < t < s+nT$ for $n = 1, 2, \cdots$. □

Proposition 3.4. *(i) For any $d > 0$ and $n \geq 1$, $\psi_1(t,x;s,z)$ is a super-solution of* (3.2) *on $s+(n-1)T < t < s+nT$.*

(ii) For any $d > 0$ and any $u_0 \in X^+$ with $u_0 \leq \min\{\psi_1(s,x;s,z), u^+(s,x+z)\}$,

$$u(t,x;s,z,u_0) \leq \min\{\psi_1(t,x;s,z), u^+(t,x+z)\} \quad \text{for} \quad t \geq s,\ x \in \mathbb{R}^N.$$

Proof. (i) For any $d > 0$, $\psi_1(t,x;s,z) > 0$ for $t \geq s$ and $x \in \mathbb{R}^N$. By (H2), $uf(t,x+z,u) \leq uf(t,x+z,0) = a_0(t,x+z)u$ for $u \geq 0$. It then follows from Proposition 3.3 (i) that $\psi_1(t,x;s,z)$ is a super-solution of (3.2) on $s+(n-1)T < t < s+nT$ for $n = 1, 2\cdots$.

(ii) First, we have $u_0(\cdot) \leq u^+(s,\cdot+z)$ and $u_0(\cdot) \leq \psi_1(s,\cdot;s,z)$. By (i) and comparison principle for parabolic equations again, $u(t,x;s,z,u_0) \leq u^+(t,x+z)$ and $u(t,x;s,z,u_0) \leq \psi_1(t,x;s,z)$ for $t \geq s$ and $x \in \mathbb{R}^N$. (ii) then follows. □

Proposition 3.5. *(i) There is $d_0 > 0$ such that for any d, d_1 with $0 < \frac{d}{2} \leq d_1$ and $d < d_0$, $\psi_2(t,x;s,z)$ is a sub-solution of* (3.2) *with f being replaced by $\tilde{f}$ on $s+(n-1)T < t < s+nT$ for $n = 1, 2, \cdots$.*

(ii) For any d, d_1 with $0 < \frac{d}{2} \leq d_1$ and $d < d_0$ and any $u_0 \in X^+$ with $u_0(x) \geq \max\{\psi_2(s,x;s,z), 0\}$,

$$u(t,x;s,z,u_0) \geq \max\{\psi_2(t,x;s,z), 0\} \quad \text{for} \quad t > s,\ x \in \mathbb{R}^N.$$

Proof. The proposition can be proved by the arguments similar to those in [43, Lemma 3.3]. For the completeness and the reader's convenience, we provide a proof in the following.

(i) First of all, it is clear that for d sufficiently small, we have $\psi_2(t,x;s,z) \leq u^+_{\inf}(\leq u^+_{\sup}+1)$ for $t \geq s$ and $x \in \mathbb{R}^N$, where $u^+_{\sup}$ and $u^+_{\inf}$ are as in (3.4) and (3.5), respectively.

Observe that for $s+(n-1)T < t < s+nT$ and $n \geq 1$,

$$\begin{aligned}
&\frac{\partial \psi_2}{\partial t} - \Delta\psi_2 - \sum_{i=1}^N a_i(t,x+z)\frac{\partial \psi_2}{\partial x_i} - \psi_2\tilde{f}(t,x+z,\psi_2)\\
=&\frac{\partial \psi_2}{\partial t} - \Delta\psi_2 - \sum_{i=1}^N a_i(t,x+z)\frac{\partial \psi_2}{\partial x_i} - a_0(t,x+z)\psi_2\\
&- \psi_2(\tilde{f}(t,x+z,\psi_2) - \tilde{f}(t,x+z,0))\\
=&- d_1\mu_1(c^n_{\mu,s} - c^n_{\mu_1,s})\phi_1(t,x;s,z) - \psi_2(\tilde{f}(t,x+z,\psi_2) - \tilde{f}(t,x+z,0)), \quad x \in \mathbb{R}^N.
\end{aligned}$$

If $\psi_2(t,x;s,z) \leq -\delta_0$, then by (3.24), $\tilde{f}(t,x+z,\psi(t,x;s,z)) = \tilde{f}(t,x+z,0) = f(t,x+z,0)$ and hence

$$\begin{aligned}
&\frac{\partial \psi_2}{\partial t} - \Delta\psi_2 - \sum_{i=1}^N a_i(t,x+z)\frac{\partial \psi_2}{\partial x_i} - \psi_2\tilde{f}(t,x+z,\psi_2)\\
=&- d_1\mu_1(c^n_{\mu,s} - c^n_{\mu_1,s})\phi_1(t,x;s,z) < 0.
\end{aligned}$$

If $-\delta_0 < \psi_2(t,x;s,z) \leq 0$, then

$$0 \leq d_1\phi_1(t,x;s,z) - d\phi(t,x;s,z) \leq \delta_0.$$

Hence

$$
\begin{aligned}
&\frac{\partial\psi_2}{\partial t}-\Delta\psi_2-\sum_{i=1}^N a_i(t,x+z)\frac{\partial\psi_2}{\partial x_i}-\psi_2\tilde f(t,x+z,\psi_2)\\
=&-d_1\mu_1(c^n_{\mu,s}-c^n_{\mu_1,s})\phi_1(t,x;s,z)-\psi_2(t,x;s,z)(\tilde f(t,x+z,\psi_2(t,x;s,z))\\
&-\tilde f(t,x+z,0))\\
\le&-d_1\mu_1(c^n_{\mu,s}-c^n_{\mu_1,s})\phi_1(t,x;s,z)+K_0\psi_2^2(t,x;s,z)\quad(\text{by}\quad(3.25))\\
\le&-d_1\mu_1(c^n_{\mu,s}-c^n_{\mu_1,s})\phi_1(t,x;s,z)+K_0d_1\phi_1(t,x;s,z)\delta_0\\
\le&-d_1\phi_1(t,x;s,z)\big(\mu_1(c^n_{\mu,s}-c^n_{\mu_1,s})-K_0\delta_0\big)\quad(\text{by}\quad(3.23))\\
\le&0.
\end{aligned}
$$

If $\psi(t,x;s,z)>0$, then $d\phi(t,x;s,z)>d_1\phi_1(t,x;s,z)$. This together with $d_1\ge\frac{d}{2}$ implies that

$$
e^{(\mu_1-\mu)(x\cdot\xi-\sum_{k=1}^{n-1}c^k_{\mu,s}T-c^n_{\mu,s}(t-(n-1)T-s))}>\frac{\eta^n_{\mu_1}(t,x;s,z)}{2\eta^n_\mu(t,x;s,z)}
$$

and then

$$
x\cdot\xi-\sum_{k=1}^{n-1}c^k_{\mu,s}T-c^n_{\mu,s}(t-(n-1)T-s)\ge\frac{1}{\mu_1-\mu}\ln\frac{\eta^n_{\mu_1}(t,x;s,z)}{2\eta^n_\mu(t,x;s,z)}.
$$

Hence

$$
\begin{aligned}
&\frac{\partial\psi_2}{\partial t}-\Delta\psi_2-\sum_{i=1}^N a_i(t,x+z)\frac{\partial\psi_2}{\partial x_i}-\psi_2\tilde f(t,x+z,\psi_2)\\
&\le -d_1\mu_1(c^n_{\mu,s}-c^n_{\mu_1,s})\phi_1(t,x;s,z)+K_0\psi_2^2\\
&\le -d_1\mu_1(c^n_{\mu,s}-c^n_{\mu_1,s})\phi_1(t,x;s,z)+K_0d^2\phi^2(t,x;s,z)\\
&\le -d_1e^{-\mu_1(x\cdot\xi-\sum_{k=1}^{n-1}c^k_{\mu,s}T-c^n_{\mu,s}(t-(n-1)T-s))}\\
&\quad\cdot\Big[\mu_1(c^n_{\mu,s}-c^n_{\mu_1,s})\eta^n_{\mu_1}(t,x;s,z)-2dK_0e^{(\mu_1-2\mu)\frac{1}{\mu_1-\mu}\ln\frac{\eta^n_{\mu_1}(t,x;s,z)}{2\eta^n_\mu(t,x;s,z)}}(\eta^n_\mu(t,x;s,z))^2\Big]\\
&<0
\end{aligned}
$$

for $0<\frac{d}{2}\le d_1$ and $d\ll 1$. Therefore, there is $d_0>0$ such that for $0<\frac{d}{2}\le d_1$ and $d<d_0$, $\psi_2(t,x;s,z)$ is a sub-solution of (3.2) on $s+(n-1)T<t<s+nT$ for $n=1,2,\cdots$.

(ii) If $u_0\ge\max\{\psi_2(s,\cdot;s,z),0\}$, then $u_0\ge 0$ and $u_0\ge\psi_2(s,\cdot;s,z)$. By comparison principle for parabolic equations, $u(t,\cdot;s,z,u_0)\ge 0$ and $u(t,\cdot;s,z,u_0)\ge\psi_2(t,\cdot;s,z)$ for $s>t$. (ii) then follows. □

Proposition 3.6. *Assume that $u_0\in X^+$ and $u_0\ge 0$ and*

$$
\begin{aligned}
&de^{-\mu x\cdot\xi}w(\sigma_s a^{\mu,\xi})(x+z)-d_1e^{-\mu_1x\cdot\xi}w(\sigma_s a^{\mu_1,\xi})(x+z)\\
&\le u_0(x)\\
&\le de^{-\mu x\cdot\xi}w(\sigma_s a^{\mu,\xi})(x+z)
\end{aligned}
$$

for $x \in \mathbb{R}^N$ and some $d, d_1 > 0$ with $0 < \frac{d}{2} \le d_1$ and $d < d_0$, where d_0 is as in Proposition 3.5. Then

$$\lim_{x\cdot\xi\to\infty} \frac{e^{\mu x\cdot\xi} u(t+s, x+\frac{\xi}{\mu}\int_s^{t+s}\kappa_\mu(\tau)d\tau; s, z, u_0)}{w(\sigma_{t+s}^{\mu,\xi})(x+z+\frac{\xi}{\mu}\int_s^{t+s}\kappa_\mu(\tau)d\tau)} \to d \quad \text{uniformly in} \quad s\in\mathbb{R},\, t\ge 0.$$

Proof. It follows from the similar arguments as [43, Lemma 3.4]. Again, for the completeness and the reader's convenience, we provide a proof in the following.

By Propositions 3.3-3.5,

$$\begin{aligned}
&dw(\sigma_{t+s}^{\mu,\xi})(x+z+\frac{\xi}{\mu}\int_s^{t+s}\kappa_\mu(\tau)d\tau)\\
&- d_1 e^{(\mu-\mu_1)x\cdot\xi} e^{-\mu_1(\frac{1}{\mu}\int_s^{t+s}\kappa_\mu(\tau)d\tau - \sum_{k=1}^{n-1} c_{\mu,s}^k T - c_{\mu,s}^n (t-(n-1)T))}\\
&\times \eta_{\mu_1}^n (t+s, x+z+\frac{\xi}{\mu}\int_s^{t+s}\kappa_\mu(\tau)d\tau; s)\\
&\le e^{\mu x\cdot\xi} u(t+s, x+\frac{\xi}{\mu}\int_s^{t+s}\kappa_\mu(\tau)d\tau; s, z, u_0)\\
&\le dw(\sigma_{t+s}^{\mu,\xi})(x+z+\frac{\xi}{\mu}\int_s^{t+s}\kappa_\mu(\tau)d\tau)
\end{aligned}$$

for $x \in \mathbb{R}^N$, $(n-1)T \le t < nT$ and $n = 1, 2, \cdots$. Note that there is $M > 0$ such that

$$\begin{aligned}
&|\frac{1}{\mu}\int_s^{t+s}\kappa_\mu(\tau)d\tau - \sum_{k=1}^{n-1} c_{\mu,s}^k T - c_{\mu,s}^n (t-(n-1)T)|\\
&= |\frac{1}{\mu}\int_{(s+n-1)T}^{t+s}\kappa_\mu(\tau)d\tau - c_{\mu,s}^n (t-(n-1)T)|\\
&\le M
\end{aligned}$$

for $s \in \mathbb{R}$, $(n-1)T \le t < nT$, and $n = 1, 2, \cdots$. It then follows that

$$\lim_{x\cdot\xi\to\infty} \frac{e^{\mu x\cdot\xi} u(t+s, x+z+\frac{\xi}{\mu}\int_s^{t+s}\kappa_\mu(\tau)d\tau; s, u_0)}{w(\sigma_{s+t} a^{\mu,\xi})(x+z+\frac{\xi}{\mu}\int_s^{s+t}\kappa_\mu(\tau)d\tau)} = d$$

uniformly in $s \in \mathbb{R}$ and $t \ge 0$. □

4. Generalized traveling wave solutions

In this section, we investigate the existence of generalized traveling wave solutions of (1.1). First, we introduce the notion of generalized wave solutions.

Definition 4.1 (Generalized traveling wave solution)**.** Let $\xi \in S^{N-1}$ be given.

(1) An entire solution $u(t, x)$ of (1.1) is called a generalized traveling wave solution of (1.1) in the direction of ξ (connecting $u^+(\cdot)$ and 0) with average propagating speed c (or averaged wave speed c) if there are $\zeta : \mathbb{R} \to \mathbb{R}$ ($\zeta(0) = 0$) and $\Phi : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^+$ such that

(i)

$$u(t,x) = \Phi(x-\zeta(t)\xi, t, \zeta(t)\xi) \quad \text{for} \quad t \in \mathbb{R},\, x \in \mathbb{R}^N; \tag{4.1}$$

(ii) $\lim_{x\cdot\xi\to-\infty}(\Phi(x,t,z)-u^+(t,z+x)) = 0$ and $\lim_{x\cdot\xi\to\infty}\Phi(x,t,z) = 0$ uniformly in $t \in \mathbb{R}$, $z \in \mathbb{R}^N$;

(iii) $\Phi(x,t,z+p_i\mathbf{e_i}) = \Phi(x,t,z), \quad i = 1,2,\cdots,N$;

(iv) $\Phi(x,t,z-x) = \Phi(x^{'},t,z-x^{'}) \quad \forall x,x^{'} \in \mathbb{R}^N$ with $x\cdot\xi = x^{'}\cdot\xi$;

(v) $\lim_{t\to\infty}\frac{\zeta(t+s)-\zeta(s)}{t} = c \quad$ uniformly in $\quad s \in \mathbb{R}$.

(2) We say that $\Phi : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^+$ generates a generalized traveling wave solution of (1.1) in the direction of ξ with average propagating speed c if there is $\zeta : \mathbb{R} \to \mathbb{R}$ such that (1)(ii)-(v) are satisfied and $u(t,x) = \Phi(x-\zeta(t)\xi, t, \zeta(t)\xi)$ is an entire solution of (1.1).

(3) A generalized traveling wave solution $u = \Phi(x-\zeta(t)\xi,t,\zeta(t)\xi)$ of (1.1) is said to have uniform exponential decay rate μ at ∞ if there is $y_0 \in \mathbb{R}$ such that

$$\lim_{x\cdot\xi\to\infty}\frac{e^{\mu x\cdot\xi}\Phi(x+x_0,t,z)}{w(\sigma_t a^{\mu,\xi})(x+x_0+z)} = 1$$

uniformly in $t \in \mathbb{R}$ and $z \in \mathbb{R}^N$.

(4) When (1.1) is periodic in t with period q, a generalized traveling wave solution $u = \Phi(x-\zeta(t)\xi,t,\zeta(t)\xi)$ of (1.1) is said to be periodic if $\Phi(x,t+q,z) = \Phi(x,t,z)$ and $\zeta^{'}(\cdot)$ exists and $\zeta^{'}(t+q) = \zeta^{'}(q)$.

Remark 4.1. (1) Suppose that $u(t,x) = \Phi(x-\zeta(t)\xi,t,\zeta(t)\xi)$ is a generalized traveling wave solution of (1.1) in the direction of ξ. Let

$$\Psi(r,t,z) = \Phi(x,t,z-x),$$

where $x \in \mathbb{R}^N$ is such that $x\cdot\xi = r$. Then $\Psi(r,t,z)$ is well defined,

$$u(t,x) = \Psi(x\cdot\xi-\zeta(t),t,x) \quad \forall t \in \mathbb{R},\ x \in \mathbb{R}^N, \tag{4.2}$$

$$\lim_{r\to-\infty}\Psi(r,t,z) = u^+(t,z), \quad \lim_{r\to\infty}\Psi(r,t,z) = 0,$$

and

$$\Psi(r,t,z+p_i\mathbf{e_i}) = \Psi(r,t,z).$$

(2) If (1.1) is periodic in t with period q and $u(t,x) = \Phi(x-\zeta(t)\xi,t,\zeta(t)\xi)$ or $u(t,x) = \Psi(x\cdot\xi-\zeta(t),t,x)$ is a periodic traveling wave solution with average propagating speed c, then it can be written as

$$u(t,x) = \tilde{\Phi}(x-ct\xi,t,ct\xi) \tag{4.3}$$

or

$$u(t,x) = \tilde{\Psi}(x\cdot\xi-ct,t,x), \tag{4.4}$$

where

$$\tilde{\Phi}(x,t,z) = \Phi(x+ct\xi-\zeta(t)\xi,t,z+\zeta(t)\xi-ct\xi)$$

and

$$\tilde{\Psi}(r,t,z) = \Psi(r+ct-\zeta(t),t,z).$$

Note that $\tilde{\Phi}(x,t+q,z) = \tilde{\Phi}(x,t,z+p_i\mathbf{e_i}) = \tilde{\Phi}(x,t,z)$ and $\tilde{\Psi}(r,t+q,z) = \tilde{\Psi}(r,t,z+p_i\mathbf{e_i}) = \tilde{\Psi}(r,t,z)$. The form (4.4) for traveling wave solutions in space and/or time periodic cases is used in [31], [47].

The main results of the paper state as follows.

Theorem 4.1. *For any given* $\xi \in S^{N-1}$ *and* $c > c^*(\xi) := \frac{\lambda(\mu^*(\xi),\xi)}{\mu^*(\xi)}$, *let* $0 < \mu < \mu^*(\xi)$ *be such that* $c = \frac{\lambda(\mu,\xi)}{\mu}$. *There is* $\Phi : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^+$ *such that*

(1) $\Phi(\cdot,\cdot,\cdot)$ *generates a generalized traveling wave solution* $u = \Phi(x-\zeta(t)\xi, t, \zeta(t)\xi)$ *of* (1.1) *with average propagating speed* c. *Moreover,*

$$\zeta(t) = \frac{1}{\mu}\int_0^t \kappa_\mu(\tau)d\tau.$$

(2) The generalized traveling wave solution $u = \Phi(x-\zeta(t)\xi, t, \zeta(t)\xi)$ *has uniform exponential decay rate* μ *at* ∞.

(3) If $a_i(t,x)$ $(i = 1, 2, \cdots, N)$ *and* $f(t,x,u)$ *are periodic in* t *with period* q, *then*

$$\Phi(x, t+q, z) = \Phi(x, t, z)$$

and $u = \Phi(x-\zeta(t)\xi, t, \zeta(t)\xi)$ *is a periodic traveling wave solution of* (1.1).

Proof. First of all, let $d_1 = \frac{d}{2}$ and $d < d_0$, where d_0 is as in Proposition 3.5. For any given $s < t$, let

$$\zeta(t;s) = \frac{1}{\mu}\int_s^t \kappa_\mu(\tau)d\tau.$$

For each $m \in \mathbb{N}$, $s \in \mathbb{R}$, and $z \in \mathbb{R}^N$, let

$$\begin{aligned} u^-_{m,s,z}(x) = &\max\{0, de^{-\mu x\cdot\xi} w(\sigma_{-mT+s}a^{\mu,\xi})(x + z - \zeta(s;-mT+s)\xi) \\ &- d_1 e^{-\mu_1 x\cdot\xi} w(\sigma_{-mT+s}a^{\mu_1,\xi})(x+z-\zeta(s;-mT+s)\xi)\} \end{aligned}$$

$$\begin{aligned} u^+_{m,s,z}(x) = \min\{&u^+(-mT+s, x+z-\zeta(s;-mT+s)\xi), \\ &de^{-\mu x\cdot\xi} w(\sigma_{-mT+s}a^{\mu,\xi})(x+z-\zeta(s;-mT+s)\xi)\}. \end{aligned}$$

Then

$$u^-_{m,s,z}(\cdot) \le u^+_{m,s,z}(\cdot).$$

By Propositions 3.4 and 3.5,

$$\begin{aligned} &de^{-\mu x\cdot\xi} w(\sigma_{t+s}a^{\mu,\xi})(x+z-\zeta(s;t+s)\xi) \\ &- d_1 e^{-\mu_1(x\cdot\xi+\zeta(t+s;-(n+1)T+s)\xi - c^{m-n}_{\mu,-mT+s}\cdot(t+(n+1)T))} \\ &\times \eta^{m-n}_{\mu_1}(t, x+z-\zeta(s;t+s)\xi); -mT+s, 0) \\ \le\, &u(t+s, x+\zeta(t+s;-mT+s)\xi; -mT+s, z-\zeta(s;-mT+s)\xi, u^+_{m,s,z}) \\ \le\, &\min\{u^+(t+s, x+z-\zeta(s;t+s)\xi), de^{-\mu x\cdot\xi} w(\sigma_{t+s}a^{\mu,\xi})(x+z-\zeta(s;t+s)\xi)\} \end{aligned} \tag{4.5}$$

for $x \in \mathbb{R}^N$, $(-n-1)T \le t \le -nT$ and $0 \le n \le m-1$.

For given $m \in \mathbb{N}$, let

$$u_{m,s,z}(x) = u(s, x+\zeta(s;-mT+s)\xi; -mT+s, z-\zeta(s;-mT+s)\xi, u^+_{m,s,z}).$$

Observe that

$$\eta_{\mu_1}^m(s,x;-mT+s,z) = w(\sigma_s a^{\mu_1,\xi})(z+x).$$

This together with (4.5) implies that

$$\begin{aligned}
&de^{-\mu x\cdot\xi}w(\sigma_s a^{\mu,\xi})(x+z) - d_1 e^{-\mu_1 x\cdot\xi}w(\sigma_s a^{\mu_1,\xi})(x+z)\\
&\le u_{m,s,z}(x)\\
&\le \min\{u^+(s,x+z), de^{-\mu x\cdot\xi}w(\sigma_s a^{\mu,\xi})(x+z)\} \qquad (4.6)
\end{aligned}$$

for $x \in \mathbb{R}^N$. Hence there is $\delta_0 > 0$ and $M_1 < M_2$ such that

$$u_{m,s,z}(x) \ge \delta_0 \quad \forall s \in \mathbb{R},\ z \in \mathbb{R}^N,\ x \in \mathbb{R}^N \text{ with } M_1 \le x\cdot\xi \le M_2. \qquad (4.7)$$

Note that for any $r > 0$,

$$\begin{aligned}
&u_{m,s,z}(x-r\xi)\\
&= u\big(s, x - r\xi + \zeta(s;-mT+s)\xi; -mT+s, z - \zeta(s;-mT+s)\xi, u^+_{m,s,z}\big)\\
&= u\big(s, x + \zeta(s;-mT+s)\xi; -mT+s, z - r\xi - \zeta(s;-mT+s)\xi, u^+_{m,s,z}(\cdot - r\xi)\big)\\
&\ge u\big(s, x + \zeta(s;-mT+s)\xi; -mT+s, z - r\xi - \zeta(-mT+s)\xi, u^+_{m,s,z-r\xi}(\cdot)\big).
\end{aligned}$$

This together with (4.7) implies that

$$u_{m,s,z}(x) \ge \delta_0 \quad \forall s \in \mathbb{R},\ z \in \mathbb{R}^N,\ x \in \mathbb{R}^N \text{ with } x\cdot\xi \le M_2. \qquad (4.8)$$

Observe also that

$$u_{m+1,s,z}(x) \le u_{m,s,z}(x)$$

for $m = 1, 2, \cdots$. Hence there are $u^*_{s,z}(x) \ge 0$ such that

$$\lim_{m\to\infty} u_{m,s,z} = u^*_{s,z}(x),$$

uniformly for x in bounded subsets of $\mathbb{R}^N$. By (4.5)

$$\begin{aligned}
&de^{-\mu x\cdot\xi}w(\sigma_s a^{\mu,\xi})(x+z) - d_1 e^{-\mu_1 x\cdot\xi}w(\sigma_s a^{\mu_1,\xi})(x+z)\\
&\le u^*_{s,z}(x)\\
&\le \min\{u^+(s,x+z), de^{-\mu x\cdot\xi}w(\sigma_s a^{\mu,\xi})(x+z)\} \qquad (4.9)
\end{aligned}$$

for $x \in \mathbb{R}^N$ and $s \in \mathbb{R}$. By (4.8),

$$u^*_{s,z}(x) \ge \delta_0 \quad \forall s \in \mathbb{R},\ z \in \mathbb{R}^N,\ x \in \mathbb{R}^N \text{ with } x\cdot\xi \le M_2. \qquad (4.10)$$

Let

$$\Phi(x,s,z) = u^*_{s,z}(x).$$

We prove that $\Phi(\cdot,\cdot,\cdot)$ satisfies the conclusions in the theorem.

(1) We prove that $\Phi(\cdot,\cdot,\cdot)$ generates a generalized traveling wave solution in the direction of ξ with average propagating speed $c = \frac{\lambda(\mu,\xi)}{\mu}$.

(i) We first prove that for any $s < t$,

$$u(t,x;s,z,\Phi(\cdot,s,z)) = \Phi(x - \zeta(t;s)\xi, t, z + \zeta(t;s)\xi). \qquad (4.11)$$

Fix $s < t$. We have

$$
\begin{aligned}
& u\big(t, x+\zeta(t;s)\xi; s, z, u^*_{s,z}(\cdot)\big) \\
=& \lim_{m\to\infty} u\big(t, x+\zeta(t;s)\xi; s, z, u(s, \cdot+\zeta(s;-mT+s)\xi; -mT+s, \\
& z-\zeta(s;-mT+s)\xi, u^+_{m,s,z})\big) \\
=& \lim_{m\to\infty} u\big(t, x+\zeta(t;s)\xi; s, z, u(s, \cdot+\zeta(s;-mT+s)\xi; -mT+t, z-\zeta(s;-mT+s)\xi, \\
& \quad u(-mT+t, \cdot; -mT+s, z-\zeta(s;-mT+s)\xi, u^+_{m,s,z}))\big).
\end{aligned}
$$

Observe that for $m \gg 1$,

$$
\begin{aligned}
& u\big(s, \cdot+\zeta(s;-mT+s)\xi; -mT+t, z-\zeta(s;-mT+s)\xi, \\
& \quad u(-mT+t, \cdot; -mT+s, z-\zeta(s;-mT+s)\xi, u^+_{m,s,z})\big) \\
& = u\big(s, x+\zeta(s;-mT+t)\xi; -mT+t, z-\zeta(s;-mT+t)\xi, \\
& \quad u(-mT+t, \cdot+\zeta(-mT+t;-mT+s)\xi; -mT+s, z-\zeta(s;-mT+s)\xi, u^+_{m,s,z})\big)
\end{aligned}
$$

and

$$
\begin{aligned}
& u(-mT+t, \cdot+\zeta(-mT+t;-mT+s)\xi; -mT+s, z-\zeta(s;-mT+s)\xi, u^+_{m,s,z}) \\
& \le u^+_{m,t,z+\zeta(t;s)\xi}(\cdot).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& u\big(t, x+\zeta(t;s)\xi; s, z, u(s, \cdot+\zeta(s;-mT+s)\xi; \\
& -mT+s, z-\zeta(s;-mT+s)\xi, u^+_{m,s,z}))\big) \\
& \le u\big(t, x+\zeta(t;s)\xi; s, z, u(s, \cdot+\zeta(s;-mT+t)\xi; -mT+t, z-\zeta(s;-mT+t)\xi, \\
& \quad u^+_{m,t,z+\zeta(t;s)\xi}(\cdot))\big) \\
& = u\big(t; x+\zeta(t;-mT+t)\xi; s, z-\zeta(s;-mT+t)\xi, \\
& u(s, \cdot; -mT+t, z-\zeta(s;-mT+t)\xi, \\
& \quad u^+_{m,t,z+\zeta(t;s)\xi}(\cdot))\big) \\
& = u\big(t, x+\zeta(t;-mT+t)\xi; -mT+t, z-\zeta(s;-mT+t)\xi, u_{m,t,z+\zeta(t;s)\xi}(\cdot)\big) \\
& = u\big(t, x+\zeta(t;-mT+t)\xi; -mT+t, z+\zeta(t;s)\xi-\zeta(t;-mT+t)\xi, u^+_{m,t,z+\zeta(t;s)\xi}(\cdot)\big) \\
& \to u^*_{t,z+\zeta(t;s)\xi}(x) \quad \text{as} \quad m\to\infty.
\end{aligned}
$$

This implies that

$$
u(t, x+\zeta(t;s)\xi; s, z, u^*_{s,z}) \le u^*_{t,z+\zeta(t;s)\xi}(x). \tag{4.12}
$$

Conversely, for any $n \in \mathbb{N}$, we have

$$
\begin{aligned}
& u^*_{t,z+\zeta(t;s)\xi}(x) \\
=& \lim_{m\to\infty} u\big(t, x+\zeta(t;-mT+t)\xi; -mT+t, \\
& \quad z+\zeta(t;s)\xi-\zeta(t;-mT+t)\xi, u^+_{m,t,z+\zeta(t;s)\xi}\big) \\
=& \lim_{m\to\infty} u\big(t, x+\zeta(t;-mT+t)\xi; -nT+s, z-\zeta(s;-mT+t)\xi, \\
& \quad u(-nT+s, \cdot; -mT+t, z-\zeta(s;-mT+t)\xi, u^+_{m,t,z+\zeta(t;s)\xi})\big).
\end{aligned}
$$

Observe that for $m > n$ with $-nT + s > -mT + t$,

$$\begin{aligned}
&u\big(t, x + \zeta(t; -mT + t)\xi; -nT + s, z - \zeta(s; -mT + t)\xi,\\
&\quad u(-nT + s, \cdot; -mT + t, z - \zeta(s; -mT + t)\xi, u^+_{m,t,z+\zeta(t;s)\xi})\big)\\
&= u\big(t, x + \zeta(t; -nT + s)\xi + \zeta(-nT + s; -mT + t)\xi; -nT + s, z - \zeta(s; -mT + t)\xi,\\
&\quad u(-nT + s, \cdot; -mT + t, z - \zeta(s; -mT + t)\xi, u^+_{m,t,z+\zeta(t;s)\xi})\big)\\
&= u\big(t, x + \zeta(t; -nT + s)\xi; -nT + s, z - \zeta(s; -nT + s)\xi,\\
&\quad u(-nT + s, \cdot + \zeta(-nT + s; -mT + t)\xi, z - \zeta(s; -mT + t)\xi, u^+_{m,t,z+\zeta(t;s)\xi})\big)\\
&\le u\big(t, x + \zeta(t; -nT + s)\xi; -nT + s, z - \zeta(s; -nT + s)\xi, u^+_{n,s,z}\big)\\
&\to u(t, x + \zeta(t; s)\xi; s, z, u^*_{s,z}) \quad \text{as} \quad m \to \infty.
\end{aligned}$$

Therefore,

$$u^*_{t,z+\zeta(t;s)\xi}(x) \le u(t, x + \zeta(t; s)\xi; s, z, u^*_{s,z}) \tag{4.13}$$

It then follows from (4.12) and (4.13) that

$$u(t, x; s, z, u^*_{s,z}) = u^*_{t,z+\zeta(t;s)\xi}(x - \zeta(t; s)\xi)$$

and

$$u(s, x; t, u^*_{t,z}) = u_{s,z-\zeta(t;s)\xi}(x + \zeta(t; s)\xi)$$

for all $s < t$.

(ii) We prove that

$$\lim_{x\cdot\xi\to-\infty}(\Phi(x, t, z) - u^+(t, x + z)) = 0, \quad \lim_{x\cdot\xi\to\infty}\Phi(x, t, z) = 0$$

uniformly in $t \in \mathbb{R}$ and $z \in \mathbb{R}^N$. By (4.10) and Propositions 3.1 and 3.2, for any $\epsilon > 0$, there are $T > 0$ and $M > 0$ such that

$$|u(s + T, x; s, z, \Phi(\cdot, s, z)) - u^+(s + T, x + z)| < \epsilon \quad \forall s \in \mathbb{R},\ z \in \mathbb{R}^N,\ x \in \mathbb{R}^N$$

with $x \cdot \xi \le -M$.

Note that

$$\Phi(x, t, z + \zeta(t; t - T)\xi) = u(t, x + \zeta(t; t - T)\xi; t - T, z, \Phi(\cdot, t - T, z))$$

and there is $\tilde{M} > 0$ such that

$$|\zeta(t; t - T)| < \tilde{M} \quad \forall t \in \mathbb{R}.$$

It then follows that

$$|\Phi(x, t, z) - u^+(t, x + x)| < \epsilon \quad \forall t \in \mathbb{R},\ z \in \mathbb{R}^N,\ x \in \mathbb{R}^N \text{ with } x \cdot \xi \le -(M + \tilde{M}).$$

Hence

$$\lim_{x\cdot\xi\to-\infty}|\Phi(x, t, z) - u^+(t, x + z)| = 0$$

uniformly in $t \in \mathbb{R}$ and $z \in \mathbb{R}^N$. By (4.9), $\lim_{x\cdot\xi\to0}\Phi(x, t, z) = 0$ uniformly in $t \in \mathbb{R}$ and $z \in \mathbb{R}^N$.

(iii) It follows directly that $\Phi(x, t, z + p_i\mathbf{e_i}) = \Phi(x, t, z), \quad i = 1, 2, \cdots, N$.

(iv) It also follows directly that $\Phi(x, t, z - x) = \Phi(x', t, z - x') \quad \forall x, x' \in \mathbb{R}^N$ with $x \cdot \xi = x' \cdot \xi$.

(v) By Theorem 2.5, $\lim_{t\to\infty}\frac{\zeta(t+s)-\zeta(s)}{t} = c \quad \text{uniformly in} \quad s \in \mathbb{R}$.

(i)-(v) implies that (1) holds.

(2) It follows from (4.9).

(3) It follows from the construction. □

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