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EXISTENCE OF NONZERO POSITIVE SOLUTIONS OF SYSTEMS OF SECOND ORDER ELLIPTIC BOUNDARY VALUE PROBLEMS*

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Abstract Existence of nonzero positive solutions of systems of second order elliptic boundary value problems under sublinear conditions is obtained. The methodology is to establish a new result on existence of nonzero positive solutions of a fixed point equation in real Banach spaces by using the well-known theory of fixed point index for compact maps defined on cones, where the fixed point equation involves composition of a compact linear operator and a continuous nonlinear map. The conditions imposed on the nonlinear maps involve the spectral radii of the compact linear operators. Moreover, the nonlinear maps are not required to be increasing in ordered Banach spaces.

Keywords Systems of elliptic boundary value problems, fixed point equations, fixed point index.

MSC(2000) Primary:35J55, Secondary: 47H10, 47H30.

1. Introduction

We study existence of nonzero positive solutions of systems of second order elliptic equations

$$\mathbb{L}z_i(x) = g_i(x)f_i(x, \mathbf{z}(x)) \quad \text{on } \overline{\Omega},\ i \in I_n := \{1, \cdots, n\} \tag{1.1}$$

subject to boundary conditions involving first order boundary operators, where $\mathbb{L}$ is a strongly uniformly elliptic differential operator and Ω is a suitable bounded open set in $\mathbb{R}^m$.

When $n = 1$ and f_1 satisfies suitable monotonicity conditions, (1.1) was studied by Amann in [1, 2].

A special case of (1.1) with the Dirichlet boundary condition is the system of semilinear elliptic equations of the form

$$\begin{cases} -\triangle z_i(x) = \lambda f_i(\mathbf{z}(x)) \quad \text{on } \overline{\Omega},\ i \in I_n, \\ z_i(x) = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

An open question proposed by Lions in [19] is whether (1.2) with $\lambda = 1$ has a nonzero positive solution under sublinear or superlinear conditions which involve

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the principal eigenvalues of the corresponding linear systems (see [19, question (c) in section 4.2]).

There have been some results on the above question in more general settings under the sublinear cases. Hai and Wang [12] proved that (1.2) has a nonzero positive solution for each $\lambda \in (0,\infty)$ under the following sublinear condition:

(i) $\lim_{|\mathbf{z}|_1\to 0} \dfrac{f_{i_0}(\mathbf{z})}{|\mathbf{z}|_1} = \infty$ for some $i_0 \in I_n$ and $f_{i_0}(\mathbf{z}) > 0$ for $\mathbf{z} \in \mathbb{R}^n_+$ and

(ii) $\lim_{|\mathbf{z}|_1\to \infty} \dfrac{f_i(\mathbf{z})}{|\mathbf{z}|_1} = 0$ for each $i \in I_n$,

where $|\mathbf{z}|_1 = \sum_{i=1}^n |z_i|$ and p-Laplacian systems are considered (see [12, Theorem 1.2]). Recently, Lan [15] used the theory of fixed point index for compact maps defined on cones [1] to prove that (1.1) with $g_i \equiv 1$ has a nonzero positive solution under a sublinear condition which contains, as a special case, the following condition:

(i)$'$ $\lim_{|\mathbf{z}|\to 0} \dfrac{f_{i_0}(\mathbf{z})}{|\mathbf{z}|} > \mu_1$ for some $i_0 \in I_n$ and

(ii)$'$ $\lim_{|\mathbf{z}|\to \infty} \dfrac{f_i(\mathbf{z})}{|\mathbf{z}|} < \mu_1$ for each $i \in I_n$,

where μ_1 is the largest characteristic value of the linear system corresponding to (1.1) with $g_i \equiv 1$ and $|\mathbf{z}| = \max\{|z_i| : i \in I_n\}$. The above result improves the results in [12]. However, it is essential in [15] to require $g_i \equiv 1$ and thus, the approach used in [15] can not be used to treat the case when $g_i \not\equiv 1$. We refer to [9, 10, 11] for the existence and uniqueness of elliptic systems related to (1.2) under some sublinear conditions and to [4, 5, 6, 10, 16, 17, 18, 20, 23, 24] for the study of such systems under the superlinear cases and some related problems for similar systems.

In this paper, we improve results in [15] and allow $g_i \not\equiv 1$. The largest characteristic values involved in our results depend on $g = (g_1, \cdots, g_n)$. Our approach is to change (1.1) into a special form of the following general fixed point equation

$$z = \mathcal{L}Fz := Az, \tag{1.3}$$

where $\mathcal{L}$ is a compact linear operator and F is a continuous nonlinear map defined on cones in real Banach spaces. We shall establish a new result on the existence of nonzero positive solutions of (1.3) by utilizing the well-known theory of fixed point index for compact maps. The existence of one or several solutions of (1.3) was studied by Amann [2], where F is an increasing map defined on an order interval. Our result does not require F to be increasing and we impose suitable conditions on the nonlinear map F which involve the spectral radius and the principle eigenvalue of the compact linear operator $\mathcal{L}$. These conditions imposed on F correspond to the sublinear conditions in applications. As illustrations, we apply our result to (1.1) with some specific nonlinearities.

2. Nonzero positive solutions of fixed point equations in ordered Banach spaces

In this section, we consider existence of nonzero positive solutions of fixed point equations of the form

$$z = \mathcal{L}Fz := Az, \tag{2.1}$$

where $\mathcal{L}$ is a linear operator and F is a nonlinear map defined in real Banach spaces.

The existence of one or several solutions of (2.1) was studied in ordered Banach spaces by Amman [2], where F was assumed to be an increasing map defined on order intervals. In the following we do not assume that F is increasing, but impose suitable conditions on F. In the applications given in the following sections, these conditions on F become the sublinear conditions involving the principal eigenvalues of the linear operators $\mathcal{L}$.

Recall that a nonempty closed convex subset P in a real Banach space X is called a cone if $\delta P \subset P$ for each $\delta \geq 0$ and $P \cap (-P) = \{0\}$. A cone P defines a partial order $\leq$ in X by

$$x \leq y \quad \text{if and only if } y - x \geq 0.$$

A cone P is said to be *reproducing* if $X = P - P$, to be *total* if $X = \overline{P - P}$ and to be *normal* if there exists $\sigma > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq \sigma\|y\|$. σ is called the normality constant of P. We refer to [1] for other cones.

Recall that a real number λ is called an eigenvalue of the linear operator $L : X \to X$ if there exists $\varphi \in X \setminus \{0\}$ such that $\lambda\varphi = L\varphi$. The reciprocals of eigenvalues are called *characteristic values* of L. The radius of the spectrum of L in X, denoted by $r(L)$, is given by the well-known spectral radius formula

$$r(L) = \lim_{m\to\infty} \sqrt[m]{\|L\|^m}, \tag{2.2}$$

where $\|L\|$ is the norm of L.

Let $\rho > 0$ and let $P_\rho = \{x \in P : \|x\| < \rho\}$, $\overline{P}_\rho = \{x \in P : \|x\| \leq \rho\}$ and $\partial P_\rho = \{x \in P : \|x\| = \rho\}$.

We need some results from the theory of fixed point index for compact maps defined on cones in X (see [1, 8]). Recall that a map $A : D \subset X \to X$ is said to be compact if A is continuous and $\overline{A(S)}$ is compact for each subset $S \subset D$.

Lemma 2.1. *Assume that $A : \overline{P}_\rho \to P$ is a compact map. Then the following results hold.*

(1) *If there exists $x_0 \in P \setminus \{0\}$ such that $z \neq Az + \nu x_0$ for $z \in \partial P_\rho$ and $\nu \geq 0$, then $i_P(A, P_\rho) = 0$.*

(2) *If there exists $z_0 \in P_\rho$ such that $z \neq \varrho Az + (1-\varrho)z_0$ for $x \in \partial P_\rho$ and $\varrho \in (0, 1]$, then $i_P(A, P_\rho) = 1$.*

(3) *Let $\rho_0 \in (0, \rho)$. If $i_P(A, P_\rho) = 1$ and $i_P(A, P_{\rho_0}) = 0$, then A has a fixed point in $P_\rho \setminus \overline{P}_{\rho_0}$.*

Let X, Y be real Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|_Y$ and with cones P and P_Y, respectively. We denote by $\preceq$ the partial order in Y induced by P_Y.

We always assume that the following conditions hold.

(H_0) $X \subset Y$, $P \subset P_Y$ and there exists $\xi > 0$ such that $\|x\|_Y \leq \xi\|x\|$ for $x \in X$.

(H_1) $\mathcal{L} : Y \to X$ is a compact linear operator such that $\mathcal{L}(P_Y) \subset P$.

(H_2) $F : P \to P_Y$ is a continuous map.

We write

$$L := \mathcal{L}|_X, \tag{2.3}$$

where $\mathcal{L}|_X$ is the restriction of $\mathcal{L}$ on X. By (H_0) and (H_1), $L : X \to X$ is a compact linear operator such that $L(P) \subset P$. We write

$$\mu_1 = \frac{1}{r(L)}. \tag{2.4}$$

The following result gives conditions which ensure that the fixed point index of A is zero.

Lemma 2.2. *Assume that the following conditions hold.*

(E) $r(L) \in (0, \infty)$ *and there exists* $\varphi \in P \setminus \{0\}$ *such that* $\varphi = \mu_1 L\varphi$.

$(F_0)_{\rho_0}$ *There exist* $\varepsilon > 0$ *and* $\rho_0 > 0$ *such that*

$$F(z) \succeq (\mu_1 + \varepsilon)z \quad \text{for } z \in \partial P_{\rho_0}.$$

Then if $z \neq Az$ *for* $z \in \partial P_{\rho_0}$, *then* $i_P(A, P_{\rho_0}) = 0$.

Proof. It is clear that under the hypotheses (H_1)-(H_2) the map A defined in (2.1) maps P into P and is compact. We prove that

$$z \neq Az + \nu\varphi \quad \text{for all } z \in \partial P_{\rho_0} \text{ and } \nu \geq 0. \tag{2.5}$$

In fact, if not, there exist $z \in \partial P_{\rho_0}$ and $\nu > 0$ such that

$$z = Az + \nu\varphi = \mathcal{L}F(z) + \nu\varphi. \tag{2.6}$$

Then $z \geq \nu\varphi$. Let $\tau_1 = \sup\{\tau > 0 : z \geq \tau\varphi\}$. Then $0 < \nu \leq \tau_1 < \infty$ and $z \geq \tau_1\varphi$. This, together with (2.6), $(F_0)_{\rho_0}$ and (H_1) implies that

$$z \geq \mathcal{L}F(z) \geq \mathcal{L}((\mu_1 + \varepsilon)z) = L((\mu_1 + \varepsilon)z) \geq (\mu_1 + \varepsilon)\tau_1 L\varphi = (\mu_1 + \varepsilon)\tau_1(\varphi/\mu_1).$$

Hence, we have $\tau_1 \geq (\mu_1 + \varepsilon)\tau_1/\mu_1 > \tau_1$, a contradiction. It follows from (2.5) and Lemma 2.1 (1) that $i_P(A, P_{\rho_0}) = 0$. □

Condition (E) of Lemma 2.2 requires $r(L)$ to be a positive eigenvalue of L with a positive eigenvector.

The following result is the well-known Krein-Rutman theorem which requires P to be total and shows that if $r(L) > 0$, then (E) holds (see [1, Theorem 3.1] or [14, 22]).

Lemma 2.3. *Assume that* P *is a total cone in* X *and* $L : X \to X$ *is a compact linear operator such that* $L(P) \subset P$ *and* $r(L) > 0$. *Then there exists an eigenvector* $\varphi \in P \setminus \{0\}$ *such that* $\varphi = \mu_1 L\varphi$.

In some applications, it is not easy to show $r(L) > 0$ by using the spectral radius formula (2.2) directly. The following result provides sufficient conditions which ensure that $r(L) > 0$ and will be used in section 3.

Proposition 2.1. *Let* P *be a total cone in* X *and let* $L : X \to X$ *be a compact linear operator such that* $L(P) \subset P$. *Assume that there exist* $u \in P - P$ *with* $-u \notin P$, $m \in \mathbb{N}$ *and* $\alpha > 0$ *such that*

$$L^m u \geq \alpha u.$$

Then Lemma 2.2 (E) *holds.*

Proof. By [13, Theorem 2.5], there exist $\lambda_0 \geq \alpha^{\frac{1}{m}} > 0$ and $x_0 \in P \setminus \{0\}$ such that $Lx_0 = \lambda_0 x_0$. Since $r(L) \geq \lambda_0$, it follows that $r(L) > 0$. The result follows from Lemma 2.3. □

The following result provides conditions which ensure that the fixed point index of A is 1, where $r(L)$ is not required to be an eigenvalue of L, but P needs to be normal.

Lemma 2.4. *Assume that P is a normal cone, $r(L) \in (0,\infty)$ and the following condition holds.*

(F^∞) *There exist $y_0 \in Y$ and $\varepsilon > 0$ such that*

$$F(z) \preceq y_0 + (\mu_1 - \varepsilon)z \quad \text{for } z \in P.$$

Then there exists $\rho^* > 0$ such that $i_P(A, P_\rho) = 1$ for $\rho > \rho^*$.

Proof. Since $r((\mu_1-\varepsilon)L) = (\mu_1-\varepsilon)r(L) < 1$, $(I-(\mu_1-\varepsilon)L)^{-1}$ exists and is a bounded linear operator such that $(I-(\mu_1-\varepsilon)L)^{-1}(P) \subset P$. Let $z_0 \in P$ and let σ be the normality constant of P and

$$\rho^* = \max\{\|z_0\|, \sigma\|(I-(\mu_1-\varepsilon)L)^{-1}(\mathcal{L}y_0 + z_0)\|\}.$$

Then $\rho^* \in (0,\infty)$. Let $\rho > \rho^*$. Then $z_0 \in P_\rho$. We prove that

$$z \neq \varrho Az + (1-\varrho)z_0 \quad \text{for } z \in \partial P_\rho \text{ and } \varrho \in (0,1]. \tag{2.7}$$

In fact, if not, there exist $z \in \partial P_\rho$ and $\varrho \in (0,1]$ such that $z = \varrho Az + (1-\varrho)z_0$. This, together with (F^∞) and (H_1) implies

$$z \leq Az + z_0 = \mathcal{L}Fz + z_0 \leq \mathcal{L}y_0 + (\mu_1-\varepsilon)Lz + z_0.$$

and

$$(I-(\mu_1-\varepsilon)L)z \leq \mathcal{L}y_0 + z_0.$$

This, together with $(I-(\mu_1-\varepsilon)L)^{-1}(P) \subset P$, implies

$$z \leq (I-(\mu_1-\varepsilon)L)^{-1}(\mathcal{L}y_0 + z_0).$$

Since P is normal,

$$\|z\| \leq \sigma\|(I-(\mu_1-\varepsilon)L)^{-1}(\mathcal{L}y_0 + z_0)\| \leq \rho^*.$$

Hence, we have $\rho = \|z\| \leq \rho^* < \rho$, a contradiction. The result follows from (2.7) and Lemma 2.1 (2). □

Now, we are in a position to give our main result on existence of nonzero positive solutions of (2.1).

Theorem 2.1. *Assume that P is a normal cone and the following conditions hold.*

(i) (E) and $(F_0)_{\rho_0}$ of Lemma 2.2 hold.

(ii) (F^∞) of Lemma 2.4 holds.

Then (2.1) *has a nonzero positive solution in P.*

Proof. We may assume that $x \neq Ax$ for $x \in \partial P_{\rho_0}$. By (i), (ii) and Lemmas 2.2 and 2.4, $i_P(A, P_{\rho_0}) = 0$ and there exists $\rho > \rho_0$ such that $i_P(A, P_\rho) = 1$. The result follows from Lemma 2.1 (3). □

3. Nonzero positive solutions of systems of elliptic boundary value problems

In this section we study existence of nonzero positive solutions of systems of second order elliptic boundary value problems of the form

$$\mathbb{L}z_i(x) = g_i(x)f_i(x, \mathbf{z}(x)) \text{ on } \overline{\Omega},\ i \in I_n \tag{3.1}$$

subject to the following boundary condition

$$Bz_i(x) = 0 \text{ on } \partial\Omega, \tag{3.2}$$

where $\mathbf{z}(x) = (z_1(x), \cdots, z_n(x))$, $\mathbb{L}$ is a strongly uniformly elliptic differential operator and B is a first order boundary operator.

When $n = 1$, (3.1) was studied in [1, 2] and the references therein, where $g_i \equiv 1$ and f_i satisfies suitable monotonicity conditions. We refer to [3, 7] and the references therein for the study of systems similar to (3.1).

Recently, Lan [15] studied existence of nonzero positive solutions of system (1.1) with $g_i \equiv 1$ under a sublinear condition using the theory of fixed point index for compact maps defined on cone [1]. However, the approach used in [15] can not be used to treat the case when $g_i \neq 1$. We shall apply results obtained in the above section to obtain results on existence of nonzero positive solutions of system (1.1), where g_i is not required to be 1.

Following [1, section 4 of Chapter 1], if $m = 1$, let $\Omega = (x_0, x_1)$, where $x_0, x_1 \in \mathbb{R}$ with $x_0 < x_1$. If $m \geq 2$, the boundary $\partial\Omega$ of Ω is assumed to be an $(m-1)$-dimensional $C^{2+\mu}$-manifold for some $\mu \in (0, 1)$ such that Ω lies locally on one side of $\partial\Omega$.

Let $\hat{\mu} = 0$ if $m = 1$ and $\hat{\mu} = \mu$ if $m \geq 2$. Recall that a second order elliptic differential operator $\mathcal{L}$ defined by

$$\mathbb{L}u = -\sum_{k,j=1}^{m} a_{kj}(x)\frac{\partial^2 u}{\partial x_k \partial x_j} + \sum_{k=1}^{m} b_k(x)\frac{\partial u}{\partial x_k} + c(x)u \tag{3.3}$$

is called to be strongly uniformly elliptic if $a_{kj}, b_k, c \in C^{\hat{\mu}}(\overline{\Omega})$ for $k, j \in I_m$ and $c(x) \geq 0$ for $x \in \overline{\Omega}$; $a_{kj}(x) = a_{jk}(x)$ for $x \in \overline{\Omega}$ and $k, j \in I_m$, and there exists $\mu_0 > 0$ such that

$$\sum_{k,j=1}^{m} a_{kj}(x)\xi_k\xi_j \geq \mu_0|\xi|^2 \quad \text{for } x \in \overline{\Omega} \text{ and } \xi = (\xi_1, ..., \xi_m) \in \mathbb{R}^m.$$

If $m = 1$, the first order boundary operator B is

$$Bu(x) = \begin{cases} \alpha_0 u(x_0) - \beta_0 u'(x_0) & \text{if } x = x_0, \\ \alpha_1 u(x_1) - \beta_1 u'(x_1) & \text{if } x = x_1, \end{cases} \tag{3.4}$$

where $\alpha_0, \alpha_1, \beta_0, \beta_1 \in [0, \infty)$ satisfy $(\alpha_0 + \beta_0)(\alpha_1 + \beta_1) > 0$. If $m \geq 2$, then

$$Bu = bu + \delta\frac{\partial u}{\partial v}, \tag{3.5}$$

where v is an outward pointing, nowhere tangent vector field on $\partial\Omega$ of $C^{1+\mu}$, $\partial u/\partial v$ denotes the directional derivative of u with respective to v, and δ and b satisfy one of the following conditions: (i) $\delta = 0$ and $b \equiv 1$ (Dirichlet boundary operator); (ii) $\delta = 1$, $b \equiv 0$ and $c \not\equiv 0$ on $\overline{\Omega}$ (Neumann boundary operator) or (iii) $\delta = 1$, $b \in C^{1+\mu}(\partial\Omega)$, $b(x) \geq 0$ and $b \not\equiv 0$ on $\partial\Omega$ (Regular oblique derivative boundary operator).

Lemma 3.1. *[1, 21] For every $v \in C^{\hat{\mu}}(\overline{\Omega})$, the linear boundary value problem*

$$\begin{cases} \mathbb{L}u(x) = v(x) \ \ on\ \overline{\Omega}, \\ Bu(x) = 0 \ on\ \partial\Omega \end{cases} \tag{3.6}$$

has a unique solution $u \in C^{2+\hat{\mu}}(\overline{\Omega})$.

For every $v \in C^{\hat{\mu}}(\overline{\Omega})$, we denote by T^*v the unique solution of (3.6). It is known that $T^* : C^{\hat{\mu}}(\overline{\Omega}) \to C^{2+\hat{\mu}}(\overline{\Omega})$ is a bounded and surjective linear operator and has a unique extension, denote by T, to $C(\overline{\Omega})$. We write

$$e = T^*v_0, \quad \text{ where } v_0(x) \equiv 1. \tag{3.7}$$

It is known that e is an interior point of the positive cone in $C(\overline{\Omega})$:

$$C_+(\overline{\Omega}) = \{z \in C(\overline{\Omega}) : z(x) \geq 0 \quad \text{for } x \in \overline{\Omega}\}.$$

The following result gives the properties of T which are contained in [1, Theorem 4.2] and [2, Lemma 5.3].

Lemma 3.2. *$T : C(\overline{\Omega}) \to C^1(\overline{\Omega}) \subset C(\overline{\Omega})$ is a compact linear operator such that $T(C_+(\overline{\Omega})) \subset C_+(\overline{\Omega})$ and for each $v \in C_+(\overline{\Omega}) \setminus \{0\}$, there exists $\alpha_v > 0$ such that $Tv \geq \alpha_v e$.*

We always assume the following conditions on g_i and f_i: For each $i \in I_n$,

(C_1) $g_i \in C^{\hat{\mu}}(\overline{\Omega})$ with $g_i(x) > 0$ for $x \in \Omega$.

(C_2) $f_i : \overline{\Omega} \times \mathbb{R}^n_+ \to \mathbb{R}_+$ is continuous.

We denote by $C(\overline{\Omega};\mathbb{R}^n)$ the Banach space of continuous functions from $\overline{\Omega}$ into $\mathbb{R}^n$ with norm $\|\mathbf{z}\| = \max\{\|z_i\| : i \in I_n\}$, where

$$\mathbf{z}(x) = (z_1(x), \cdots, z_n(x)) \quad \text{for } x \in \overline{\Omega}.$$

We shall use the standard positive cone in $C(\overline{\Omega};\mathbb{R}^n)$ defined by

$$P = C(\overline{\Omega};\mathbb{R}^n_+). \tag{3.8}$$

It is well known that P is a normal and reproducing cone in $C(\overline{\Omega};\mathbb{R}^n)$.

We define $L : C(\overline{\Omega},\mathbb{R}^n) \to C(\overline{\Omega},\mathbb{R}^n)$ by

$$(L\mathbf{z})(x) = ((L_1z_1)(x), \cdots, (L_nz_n)(x)), \tag{3.9}$$

where $L_i : C(\overline{\Omega}) \to C^{2+\hat{\mu}}(\overline{\Omega})$ is defined by $(L_iu)(x) = (Tg_iu)(x)$.

By (C_1), $g_iu \in C^{\hat{\mu}}(\overline{\Omega})$ for $u \in C(\overline{\Omega})$. Hence, we have for $u \in C(\overline{\Omega})$,

$$L_iu = T(g_iu) = T^*(g_iu) \in C^{2+\hat{\mu}}(\overline{\Omega}).$$

The following result gives the properties of L defined in (3.9).

Lemma 3.3. *The linear operator L defined in* (3.9) *has the following properties:*

(*i*) $L : C(\overline{\Omega}, \mathbb{R}^n) \to C(\overline{\Omega}, \mathbb{R}^n)$ *is compact and satisfies* $L(P) \subset P$.

(*ii*) $r(L) \in (0, \infty)$ *and there exists* $\varphi \in P \setminus \{0\}$ *such that* $\varphi = \mu_g L\varphi$, *where* $\mu_g = 1/r(L)$.

Proof. (*i*) By Lemma 3.2, for each $i \in I_n$, $L_i : C(\overline{\Omega}) \to C(\overline{\Omega})$ is compact and $L_i(C_+(\overline{\Omega})) \subset C_+(\overline{\Omega})$. The results follows.

(*ii*) By Lemma 3.2, for each $i \in I_n$, there exists $\alpha_i > 0$ such that $L_i e \geq \alpha_i e$. Let $\alpha = \min\{\alpha_i : i \in I_n\}$. Then $\alpha > 0$ and

$$L\mathbf{e} = (L_1 e, \cdots, L_n e) \geq (\alpha_1 e, \cdots, \alpha_n e) \geq \alpha \mathbf{e}.$$

This, together with Lemma 2.1 implies that (*ii*) holds. □

We define a Nemytskii operator $F : P \to P$ by

$$(F\mathbf{z})(x) = (f_1(x, \mathbf{z}(x)), \cdots, f_n(x, \mathbf{z}(x))). \tag{3.10}$$

It is easy to see that (3.1) is equivalent to the following fixed point equation:

$$\mathbf{z}(x) = (LF\mathbf{z})(x) := A\mathbf{z}(x) \quad \text{for } x \in \overline{\Omega}. \tag{3.11}$$

Now, we give our main result of this section.

Theorem 3.1. *Assume that the following conditions hold.*

$((f_i)_0)_{\rho_0}$ *There exist $\varepsilon > 0$ and $\rho_0 > 0$ such that for each $i \in I_n$,*

$$f_i(x, \boldsymbol{z}) \geq (\mu_g + \varepsilon) z_i \quad \textit{for } x \in \overline{\Omega} \textit{ and all } \boldsymbol{z} \in \mathbb{R}^n_+ \textit{ with } |\boldsymbol{z}| \in [0, \rho_0].$$

$(f_i^\infty)_{\rho_1}$ *There exist $\varepsilon > 0$ and $\rho_1 > 0$ such that for each $i \in I_n$,*

$$f_i(x, \boldsymbol{z}) \leq (\mu_g - \varepsilon) z_i \quad \textit{for } x \in \overline{\Omega} \textit{ and all } \boldsymbol{z} \in \mathbb{R}^n_+ \textit{ with } |\boldsymbol{z}| \geq \rho_1.$$

Then (3.1) *has a nonzero positive solution in P.*

Proof. Let $X = Y = C(\overline{\Omega}; \mathbb{R}^n)$. Then (H_0) holds. By Lemma 3.3 (*i*) and the continuity of f_i, (H_1) and (H_2) hold. By Lemma 3.3 (*ii*), Lemma 2.4 (E) holds. It is easy to verify that $((f_i)_0)_{\rho_0}$ implies that Lemma 2.4 $(F_0)_{\rho_0}$ with F defined in (3.10) holds. By (C_2), there exists $b > 0$ such that for each $i \in I_n$,

$$f_i(x, \mathbf{z}) \leq b \quad \text{for } x \in \overline{\Omega} \text{ and } \mathbf{z} \in \mathbb{R}^n_+ \text{ with } |\mathbf{z}| \leq \rho_1.$$

This, together with $(f_i^\infty)_{\rho_1}$ implies for each $i \in I_n$,

$$f_i(x, \mathbf{z}) \leq b + (\mu_g - \varepsilon) z_i \quad \text{for } x \in \overline{\Omega} \text{ and all } \mathbf{z} \in \mathbb{R}^n_+.$$

Hence, F defined in (3.10) satisfies Lemma 2.4 (F^∞) with $y_0 = (b, \cdots, b)$. By Theorem 2.1, (3.11) has a nonzero solutions in P. The result follows from the equivalence between (3.1) and (3.11). □

When $n = 1$, (3.1) can be written as follows.

$$\begin{cases} \mathbb{L}z(x) = g(x) f(x, z(x)) & \text{on } \overline{\Omega}, \\ Bz(x) = 0 \text{ on } \partial\Omega \end{cases} \tag{3.12}$$

and g and f satisfy (C_1) and (C_2).

Let

$$\underline{f}(z) = \inf_{x\in\overline{\Omega}} f(x,z), \quad \overline{f}(z) = \sup_{x\in\overline{\Omega}} f(x,z);$$
$$f_0 = \liminf_{z\to 0+} \underline{f}(z)/z, \quad f^{\infty} = \limsup_{z\to\infty} \overline{f}(z)/z.$$

The following result shows that when $n = 1$, the conditions: $((f_i)_0)_{\rho_0}$ and $(f_i^{\infty})_{\rho_1}$ in Theorem 3.1 can be replaced by suitable stronger limit conditions.

Corollary 3.1. *Assume that the following condition holds.*
(H) $f^{\infty} < \mu_g < f_0$.
Then (3.12) *has a nonzero positive solution in* P.

When $\mathbb{L}u = -\triangle u$ and B is the Dirichlet boundary operator, Corollary 3.1 improves [19, Theorem 1.3], where $g \equiv 1$, $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous satisfying $f(0) = 0$, and [5, Corollary II.1], where f satisfies the Carathéodory conditions, but the positive solutions are in $W^{2,p}(\Omega)$ for every $1 < p < \infty$.

When $m = 1$, $\mathbb{L}z = -z''$ and $[x_0, x_1] = [0, 1]$, the boundary operator B in (3.12) becomes (3.4) and Corollary 3.1 can be obtained by Theorem 4.1 (H_2) in [25].

As illustration, we consider existence of nonzero positive solutions of systems of second order elliptic boundary value problems of the form

$$\mathbb{L}z_i(x) = g_i(x)\frac{a_i}{1+|\mathbf{z}(x)|^{\alpha_i}} z_i(x) \quad \text{for } x \in \overline{\Omega} \text{ and } i \in I_n \tag{3.13}$$

subject to the boundary condition (3.2).

Theorem 3.2. *Assume that the following conditions hold.*
(i) For each $i \in I_n$, g_i *satisfies* (C_1).
(ii) For each $i \in I_n$, $\alpha_i > 0$ *and* $a_i > \mu_g$.
Then (3.13)-(3.2) *has a nonzero positive solution in* P.

Proof. For each $i \in I_n$, we define a function $f_i : \overline{\Omega} \times \mathbb{R}_+^n \to \mathbb{R}_+$ by

$$f_i(x,\mathbf{z}) = \frac{a_i}{1+|\mathbf{z}|^{\alpha_i}} z_i.$$

Let $\varepsilon > 0$ be such that $a_i > (\mu_g + \varepsilon)$ for $i \in I_n$. Let $\rho_0 > 0$ satisfy

$$\rho_0 \le \min\{\big(\frac{a_i}{\mu_g+\varepsilon} - 1\big)^{\frac{1}{\alpha_i}} : i \in I_n\}.$$

Then for $x \in \overline{\Omega}$ and $\mathbf{z} \in \mathbb{R}_+^n$ with $|\mathbf{z}| \in [0, \rho_0]$,

$$f_i(x,\mathbf{z}) \ge \frac{a_i}{1+\rho_0^{\alpha_i}} z_i \ge (\mu_g+\varepsilon) z_i.$$

Hence, $((f_i)_0)_{\rho_0}$ holds. Let $\rho_1 > \rho_1$ and $\rho_1 \ge \max\{\big(\frac{a_i}{\mu_g-\varepsilon} - 1\big)^{\frac{1}{\alpha_i}} : i \in I_n\}$. Then for $x \in \overline{\Omega}$ and $\mathbf{z} \in \mathbb{R}_+^n$ with $|\mathbf{z}| \in [\rho_1, \infty)$,

$$f_i(x,\mathbf{z}) \le \frac{a_i}{1+\rho_1^{\alpha_i}} z_i \le (\mu_g-\varepsilon) z_i.$$

The result follows from Theorem 3.1. $\square$

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