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ANALYTIC CONJUGATION, GLOBAL ATTRACTOR, AND THE JACOBIAN CONJECTURE*

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Abstract It is proved that the dilation λf of an analytic map f on $\mathbf{C}^n$ with $f(0) = 0, f'(0) = I, |\lambda| > 1$ has an analytic conjugation to its linear part λx if and only if f is an analytic automorphism on $\mathbf{C}^n$ and $x = 0$ is a global attractor for the inverse $(\lambda f)^{-1}$. This result is used to show that the dilation of the Jacobian polynomial of [12] is analyticly conjugate to its linear part.

Keywords Jacobian Conjecture, Analytic Conjugation, Global Stability, Polynomial Automorphism, Jacobian Polynomial, Analytic Linearization.

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1.

This paper was motivated by several recent results ([4], [9], [12], and [6]) on the following conjecture which first appeared in [7].
The Jacobian Conjecture: *If a polynomial map* $f : \mathbf{C}^n \to \mathbf{C}^n$ *has a nonzero constant Jacobian determinant* $\det f'(x) \equiv \text{constant} \neq 0$*, then* f *is a polynomial automorphism of* $\mathbf{C}^n$*, i.e., it is a bijective map with polynomial inverse.*
Upon normalization, we shall always assume $f(0) = 0$ and $f'(0) = I$, the $n \times n$ identity matrix. For simplicity, we shall also refer to such maps as Jacobian polynomials.

2.

There are several important results relevant to our discussion here. The first one is due to [10] and [3], see also [11]. It states that the injectivity of a Jacobian polynomial implies the polynomial automorphism property. The second result is the reduction of degree theorem due to [13] and [2]. It reduces the injectivity of a Jacobian polynomial to the injectivity of a Jacobian polynomial of the cubic homogeneous linearity: $f(x) = x + g(x)$ with $g(tx) = t^3 g(x)$ for all $t \in \mathbf{C}$ and $x \in \mathbf{C}^n$. The new dimension n is not necessarily the same as but usually greater

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than the original one. Note that the assumption $\det f'(x) \equiv 1$ implies $g'(x)$ is nilpotent of degree $\le n$ for all $x \in \mathbf{C}^n$. This result was further improved in [5] by allowing each component of g to be the cubic power of a homogeneous linear form.

3.

A dynamical system approach to the Jacobian Conjecture was recently proposed in [4]. It is based on the observation that f is 1-1 if and only if any dilation λf, $|\lambda| > 1$, is 1-1. Thus, it only suffices to show the existence of a global diffeomorphism $h_\lambda : \mathbf{C}^n \to \mathbf{C}^n$ so that the following diagram commutes

$$\begin{array}{ccc} \mathbf{C}^n & \xrightarrow{\lambda f} & \mathbf{C}^n \\ \downarrow{\scriptstyle h_\lambda} & & \downarrow{\scriptstyle h_\lambda} \\ \mathbf{C}^n & \xrightarrow{\lambda x} & \mathbf{C}^n \end{array}$$

for $|\lambda| > 1$. That is λf is conjugate to its linear part λx via h_λ. If this is the case, the injectivity of λf follows from the injectivity of its linear part λx.

It is known by the Poincaré-Siegel Theorem (cf. e.g. [1]) that such a conjugation h_λ exists *locally* near the origin for each $|\lambda| \ne 1$. Moreover, h_λ is analytic at $x = 0$ and its power series is uniquely determined once the linearization $h'_\lambda(0)$ is given. We shall assume $h'_\lambda(0) = I$ throughout. In both [4] and [9], all the dilations of invertible Jacobian polynomials with the cubic homogeneous nonlinearity are shown to be conjugate to their linear part by *polynomial automorphisms.* This made the author of [9] to conjecture that perhaps every such Jacobian polynomial has the same property. This conjecture proved to be false by [12] in which the following counterexample in $\mathbf{C}^4$

$$f(x) = \begin{bmatrix} x_1 + p(x)x_4 \\ x_2 - p(x)x_3 \\ x_3 + x_4^3 \\ x_4 \end{bmatrix}, \tag{1}$$

where $x = (x_1, x_2, x_3, x_4)$ and $p(x) = x_3x_1 + x_4x_2$ was produced to show that if exits, any analytic conjugation h_λ cannot be a polynomial. The inverse of this map is given as

$$f^{-1}(x) = \begin{bmatrix} (1 + x_4^4 - x_3x_4)x_1 - x_4^2x_2 \\ (x_4^6 - 2x_3x_4^3 + x_3^2)x_1 + (1 - x_4^4 + x_3x_4)x_2 \\ x_3 - x_4^3 \\ x_4 \end{bmatrix}. \tag{2}$$

It was demonstrated recently in [6] that the local conjugation h_λ for the example above is indeed an analytic automorphism in $\mathbf{C}^n$. Yet, it is still open whether or not the dilation of a Jacobian polynomial always has an analytic conjugation to its linear part.

4.

The following theorem and its corollaries provide a partial answer to the question above.

Theorem. *Let $F : \mathbf{C}^n \to \mathbf{C}^n$ be an analytic map on $\mathbf{C}^n$ with $F(0) = 0$ and $A = F'(0)$ be the linearization of F at the fixed point $x = 0$. Suppose that A is invertible and all the eigenvalues of A are nonresonant and in the unit open disk on the complex plane. Then F has an analytic conjugation to its linear part Ax if and only if F is an analytic automorphism on $\mathbf{C}^n$ and the fixed point $x = 0$ is a global attractor, i.e., $F^k(x) := F \circ F^{k-1}(x) \to 0$ as $k \to \infty$.*

Proof. We first note that there is a constant $0 < \gamma < 1$ and $C > 0$ such that all eigenvalues of A are inside $|z| \le \gamma$ and $\|A^k\| \le C\gamma^k$ for all $k \ge 0$. Also, if H is an analytic conjugation of F to Ax, then $F^k(x) = H^{-1} \circ A^k \circ H(x)$ for all $k \ge 0$ and subsequently for all $k \le 0$. Hence, F is an analytic automorphism and $x = 0$ is a global attractor. This shows the conditions of the theorem are necessary.

Conversely, there exists a local analytic conjugation H of F to Ax near $x = 0$ by the Poincaré-Siegel Theorem [1] since the eigenvalues of A are nonresonant. Because $x = 0$ is a global attractor, one can extend H to be an analytic conjugation. To be precise, let B_1 be a small ball centered at the origin so that the local conjugation holds in B_1:

$$H \circ F(x) = A \circ H(x), \quad \text{equivalently,} \quad H \circ F^{-1}(x) = A^{-1} \circ H(x).$$

We now extend H to $\mathbf{C}^n$ as a conjugation by the only possible way via iteration. Specifically, for each $x \in \mathbf{C}^n$, there exists a first integer $k \ge 0$ so that $F^m(x) \in B_1$ for all $m \ge k$ because $x = 0$ attracts every point in $\mathbf{C}^n$ for F. Define

$$H(x) := A^{-m} \circ H \circ F^m(x), \text{ with } m \ge k.$$

Note that by local conjugacy this definition is independent of integers $m \ge k$ so long as $F^m(x) \in B_1$. Moreover, for every point x, there is a sufficiently large m and a neighborhood N of x so that H is defined on N as above for the same integer m. This implies that the extended H is analytic everywhere. Moreover, H is 1-1 because H is locally 1-1 in B_1, F is 1-1 by assumption, and matrix A is nonsingular. H is also onto. In fact, the inverse is defined as

$$H^{-1}(x) := F^{-k} \circ H^{-1} \circ A^k(x),$$

where $k \ge 0$ is the first integer so that $A^k x \in H(B_1)$. Finally, H is a global conjugation because by definition we have

$$H(F(x)) := A^{-(k-1)} \circ H \circ F^{(k-1)}(F(x)) = AH(x).$$

This completes the proof. □

Corollary A. *Let f be a Jacobian polynomial in $\mathbf{C}^n$. For $|\lambda| > 1$, the dilation λf has an analytic conjugation to its linear part λx if and only if f is an analytic automorphism and $x = 0$ is a global attractor for the inverse of λf.*

Proof. That the conditions are necessary is obvious. To show they are sufficient, we apply the Theorem to the inverse $(\lambda \circ f)^{-1} = f^{-1} \circ \lambda^{-1}$. Thus, $(\lambda \circ f)^{-1}$ has an analytic conjugation to its linear part $\lambda^{-1}x$. The same map also conjugates λf to its linear part λx. □

Corollary B. *Let f be a Jacobian polynomial in $\mathbf{C}^n$. For $|\lambda| < 1$, $f \circ \lambda$ has an analytic conjugation to its linear part λx if and only if f is an analytic automorphism and $x = 0$ is a global attractor for $f \circ \lambda$.*

Proof. This is simply a special case of the Theorem. □

We end this section by a remark that for a Jacobian polynomial, the conjugation maps for $\lambda \circ f$ and $f \circ \lambda$ are related to each other in the following way. Let h_λ and k_λ be the conjugation maps for $\lambda \circ f$ and $f \circ \lambda$ respectively, with $h'_\lambda(0) = k'_\lambda(0) = I$. Substitute λx for x in $h_\lambda(\lambda f(x)) = \lambda h_\lambda(x)$, we have $(h_\lambda \circ \lambda)(f(\lambda x)) = \lambda(h_\lambda \circ \lambda)(x)$. Because the formal power series for h_λ, k_λ are uniquely determined by the normalization $h'_\lambda(0) = k'_\lambda(0) = I$ for $|\lambda| \neq 1$, we have

$$k_\lambda(x) = \frac{1}{\lambda} \circ h_\lambda \circ \lambda(x), \quad \text{equivalently,} \quad h_\lambda(x) = \lambda \circ k_\lambda \circ \frac{1}{\lambda}(x), \tag{3}$$

formally for $|\lambda| \neq 1$.

5.

We now use Corollary A to give an alternate proof to the result of [6] that concludes the local analytic conjugation h_λ with $h_\lambda(0) = I$ for map (1) is indeed global. By our approach, we only need to show $x = 0$ is a global attractor for $(\lambda \circ f)^{-1} = f^{-1} \circ \lambda^{-1}$. To this end, let $\mu = 1/\lambda$, $x_{k+1} := (f^{-1} \circ \mu)(x_k) = (f^{-1} \circ \mu)^{k+1}(x_0)$, the $(k+1)$st iterate of x_0, $x_k = (x_{1,k}, x_{2,k}, x_{3,k}, x_{4,k})$ for a abuse of notation. The dynamics of $x_{3,k}, x_{4,k}$ are decoupled from the first two components. The iterates can be explicitly expressed as

$$\begin{aligned}
x_{4,k+1} &= \mu x_{4,k} = \mu^{k+1} x_{4,0} \\
x_{3,k+1} &= \mu x_{3,k} - (\mu x_{4,k})^3 = \mu x_{3,k} - \mu^{3(k+1)} x_{4,0}^3 \\
&= \mu^{k+1} x_{3,0} - x_{4,0}^3 \textstyle\sum_{i=1}^{k+1} \mu^{k+1-i} \mu^{3i} \\
&= \mu^{k+1} x_{3,0} - x_{4,0}^3 \mu^3 \tfrac{\mu^{k+1} - \mu^{3(k+1)}}{\mu - \mu^3}.
\end{aligned}$$

Hence, $(x_{3,k}, x_{4,k}) \to (0,0)$ as $k \to \infty$ follows. To show $(x_{1,k}, x_{2,k}) \to (0,0)$ as $k \to \infty$, we use the fact that $f^{-1} \circ \mu$ is linear in the first two variables and that $(x_{3,k}, x_{4,k}) \to (0,0)$ as $k \to \infty$. More specifically, we first find $K > 0$ such that for $k \geq K$, we have

$$\begin{aligned}
|\mu| \max\{ \; & |1 + \mu^4 x_{4,k}^4 - \mu^2 x_{3,k} x_{4,k}|, |\mu^2 x_{4,k}^2|, \\
& |\mu^6 x_{4,k}^6 - 2\mu^4 x_{3,k} x_{4,k}^3 + \mu^2 x_{3,}^2|, |1 - \mu^4 x_{4,k}^4 + \mu^2 x_{3,k} x_{4,k}|\} \\
& \leq r < 1
\end{aligned}$$

for some constant $0 < r < 1$. Now, let $\Gamma_k := \max\{|x_{1,k}|, |x_{2,k}|\}$. Then

$$|x_{1,k+1}| \leq \max\{|(1 + \mu^4 x_{4,k}^4 - \mu^2 x_{3,k} x_{4,k})\mu|, |\mu^3 x_{4,k}^2|\}\Gamma_k \leq r\Gamma_k.$$

Similarly, we have $|x_{2,k+1}| \leq r\Gamma_k$ and

$$\Gamma_{k+1} \leq r\Gamma_k \leq r^{k+1-K}\Gamma_K \to 0$$

follows as $k \to \infty$. Therefore, $x = 0$ is a global attractor for $(\lambda f)^{-1}$ and by Corollary A, λf is conjugate to λx for $|\lambda| > 1$ by an analytic automorphism h_λ.

Exactly the same argument can be used to show that $f \circ \lambda$ is conjugate to λx for $|\lambda| < 1$ by an analytic automorphism k_λ. By identities (3) we conclude that both h_λ and k_λ are analytic automorphisms for $|\lambda| \neq 1$. Hence, λf is conjugate to λx for $|\lambda| \neq 1$ by the same analytic automorphism h_λ.

6.

Because of identity (3) and the relation $(\lambda \circ f)^{-1} = f^{-1} \circ \lambda^{-1}$, we now only consider the global conjugation problem for $f \circ \lambda$. We also restrict our attention to Jacobian polynomials with the cubic homogeneous nonlinearity g, including the special cubic-linear homogeneous ones. The question is whether or not for such maps f, $x = 0$ is always a global attractor for $f \circ \lambda$ with $|\lambda| < 1$. Note that the eigenvalues of the linearization $(f \circ \lambda)'(x)$ are λ everywhere. This also implies the volume of any set is contracted by a constant rate of $|\lambda|^{nk}$ under the kth iterate of $f \circ \lambda$ for all $k \geq 1$. This naturally leads to the following conjecture.

Conjecture A. *If f is a Jacobian polynomial with the cubic homogeneous nonlinearity, then the origin $x = 0$ is a global attractor for $f \circ \lambda$.*

In light of Corollary B, this conjecture implies the following conjecture.

Conjecture B. *The Jacobian Conjecture is true if and only if for any Jacobian polynomial f of the cubic homogeneous nonlinearity, $f \circ \lambda$, $|\lambda| < 1$, has an analytic conjugation to its linear part λx.*

We remark that Conjecture A is analogous to the Markus-Yamabe Conjecture for ordinary differential equations. However, unlike the Markus-Yamabe Conjecture which implies the Jacobian Conjecture (c.f., e.g. [8]), Conjecture A simply implies the equivalence between the Jacobian Conjecture and the analytic conjugation of $f \circ \lambda$ to its linear part.

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Appendix

The Markus-Yamabe Conjecture (MYC) was made 50 years ago and it has been solved for 15 years. It will be taught as long as the global stability of a dynamical system is the concern. Therefore it is important to have a comprehensive documentation on its solution for the benefit of future students as well as for the integrity of its literature. However, after examined the keystone papers ([14,15]), the authoritative biography by Gary Meisters ([19]), and the comprehensive review by Arno van den Essen ([16]), I have come to the conclusion that a hole was left in the literature 15 years ago and it has remained unfilled since. Thanks to the understanding of this journal's editor, the missing work, which was referred to as “inspired” ([18]), “formed the starting point of” ([15]), and “led to” ([19]) the final solution of MYC, appears at last in this issue as the text above.

I will restrict my comments only to events complementary to [16,19] by which we knew that the solution of MYC began unexpectedly with the attempt as a new approach to Keller's Jacobian Conjecture (JC) by Gaetano Zampieri, Gary Meisters, and myself ([4]) in the spring of 1994 to find global conjugation h_λ for the scalar multiple of Keller's polynomials f to their linear part: $h_\lambda \circ (\lambda f) \circ h_\lambda^{-1}(x) = \lambda x$, for which $f'(x)$ has 1 as its only eigenvalue everywhere. It was Gaetano who got me interested in the JC problem when he was spending his sabbatical in Lincoln to work with Gary. I suggested this approach to Gaetano and then with Gaetano to Gary mainly because of my familiarity with the Poincaré-Siegel theory (Demonstrating its proof from [1] was my first seminar presentation suggested by my PhD thesis advisor Shui-Nee Chow at Michigan State University).

From [14,15,16,18,19] we also know that there is a big class of counterexamples to MYC and to its discrete version (DMYC) also known as the LaSalle problem, and that Arno and his student, Englebert Hubbers, discovered the first example of this class. It was this map $F_2(x) = (x_1 + d(x)^2 x_4, x_2 - d(x)^2 x_3, x_3 + x_4^3, x_4)$ with $x = (x_1, x_2, x_3, x_4)$ and $d(x) = x_3 x_1 + x_4 x_2$. Here the subscript of F_k corresponds to the powers $d(x)^k$ in F_k. We need the $k = 1$ case to show how it played the pivotal role in the unraveling of the conjectures.

By July of 1995 we knew from Arno that $\lambda F_1(x)$ does not have a polynomial conjugacy h_λ for $\lambda \neq 1$ but from Gianluca Gorni and Gaetano ([17]) that nonetheless the local conjugacy guaranteed by the Poincaré-Siegel theory can be extendedly globally. This paper gave an alternative proof to Gianluca and Gaetano's result. We know from Arno's account [16] that after he received my preprint he was motivated first by the *necessary* condition of the Theorem above to look for counterexamples to the global conjugation problem with a preliminary but mixed success. What led Arno to squaring the expression $d(x)$ to give his first counterexample $\lambda F_2(x)$ which eventually settled both MYC and DMYC? The answer is clear when one puts this paper and [15] side-by-side for a comparison — the two proofs are two sides of a coin. My proof shows $x = 0$ is a global attractor of λF_1 with $\lambda < 1$ since both x_1, x_2 are essentially linear and thus decay exponentially. Their proof shows $x = 0$ is not a global attractor for λF_2 because both x_1, x_2 are quadratic and thus will grow exponentially for some large enough initial points, analogous to the case that the dynamics of the quadratic map $y = x^2$ will out grow that of any linear map $y = \mu x$ for initial points $x > 1$. The lines of argument for both are essentially parallel except that wherever I wanted exponential decay Arno and Englebert made exponential growth by reverting corresponding inequalities. (Since $F_k'(x)$ has 1 as the only eigenvalue, λF_2 became a counterexample to DMYC, and the parallel case for MYC followed as well for the equation of $x' = -F_2(x)$ ([16]).) This critical difference between the squaring and not squaring $d(x)$ also holds for the final prototypical counterexample to DMYC with $F(x) = (\frac{1}{2}x_1 + x_3 d(x)^2, \frac{1}{2}x_2 - d(x)^2, \frac{1}{2}x_3, \ldots, \frac{1}{2}]x_n), x = (x_1, x_2, \ldots, x_n)$ (having a diverging orbit $x^{(n)} = (\frac{147}{32} \cdot 2^n, -6332 \cdot 2^{2n}, \frac{1}{2^n}, 0, \ldots, 0))$ and the prototypical counterexample to MYC with $x' = (-x_1 + x_3 d(x)^2, -x_2 - d(x)^2, -x_3, \ldots, -x_n)$ (having a diverging solution $x(t) = (18e^t, -12e^{2t}, e^{-t}, 0, \ldots, 0)$ by the method of undetermined coefficients), both from [2] with $d(x) = x_1 + x_3 x_2$ instead. That is, by not squaring $d(x)$ the origin $x = 0$ will become a global attractor by the argument of this paper for both the discrete and continuous examples, no counterexample to DMYC or MYC respectively.

Thus it can be argued that it was my verification of the *sufficient* condition for the global conjugation of λF_1 that in fact "formed the starting point" of Arno and Englebert's work [15]. Had Arno also included this part of my preprint in his comprehensive exposition of MYC ([16]), there would have been no need to publish it here 15 years later. Without the benefit of putting the two works together a vital link is missing from the chain of events leading to the solution of MYC.

When Gary and I learned in January 1996 from Franc Forstneric that the Theorem above was a rediscovery of Rosay and Rudin's result ([20]) unrelated to JC or

MYC, I withdrew the preprint from *Advances in Mathematics* for publication even though its alternative proof of Gaetano and Gianluca's result and its pivotal role in leading Arno and Englebert to their crucial discovery were sufficient justifications. My thinking at that time was, the MYC problem was over, I had done more than I hoped since my foray into JC was only ancillary, and it was the time for me to focus my attention on my overdue NSF project which gave me considerable stress. With the publication of this paper here I hope that my youthful neglect to the integrity of the literature for MYC will at last be mitigated.

One anecdote. As soon as Gary got my preprint in late August of 1995, he came straight to my house unannounced which he never did nor has done since. He was extremely excited about the new approach, but both of us missed its direct link to DMYC. He promised me to work on the global linearization problem anew but told me that he could not do so until later in the fall because his son was hospitalized after a bad car accident, and he and his wife had to go right away to visit and stay with his son's family in Colorado for sometime. We now know that he never had the time before nor reason after Arno and Englebert completed their breakthrough work ([15]) in October. I agree with Gary ([19]) that serendipity played a big role in the final solution of MYC. I may add luck as well, as an opinion. And last, one speculation. Without the contribution from Gary, Gaetano, or Gianluca, the MYC may still stand today.

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